THE SIGNIFICANCE OF NUMBER OF DESIGN VARIABLES ON OPTIMAL SHELL SOLUTIONS

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Abstract

This paper describes the significance of the effect of the number of design variables on the optimal solution. An 18-node hexahedral shell is adopted as the basic element formulation with the inclusion of proposed methodologies to alleviate various locking phenomena. An automatic mesh generation scheme using Bezier surface method is employed to avoid re-meshing structures after each iteration of optimization process. Various benchmarks are performed to demonstrate the dependence of the optimal solutions on the number of design variables.

Keywords: Bezier surface, Design variable, Locking, Shell, Thickness optimization

Introduction

Structural optimization has drawn the attention to many researchers for a few decades. Mathematical programming methods used in structural optimization are generally based on two categories. First method is based on gradient methods which normally use objective function, derivatives of objective function and constraint functions. The other method based on non-gradient methods which use only objective function and constraint functions. A gradient-based method is expected to be more efficient since it requires more information in each analysis. Therefore it needs fewer iterations of structural analysis. Researches in structural optimization may be categorized into three main areas namely shape optimization, thickness optimization and topology optimization.

One of the earlier research on optimization of structures was Zienkiewicz and Campbell[1]. They have discussed the problem of finding the optimum shape of two-dimensional structures known as shape optimization. Since then several researchers have contributed in this area such as Choi and Haug[2] and Haftka and Grandhi[3]. General methods to obtain the optimum structure are varying the shape of an initial structure. Therefore solutions obtained from shape optimization methods maintain the same thickness of structures. In sizing optimization problems, design variables are cross-sectional areas (beams and trusses) or the thicknesses (plate and shell structures) while the geometric shape of structures remains unchanged. Optimality criterion are generally employed for this area[4].

The objective of the present work is to incorporate fully automated mesh generation scheme and analytical sensitivity analysis into thickness optimization of geometrically linear shell structures. First the fully automated mesh generation scheme for the model generation is discussed. Three main objectives in the model generation phase are to implement the Bezier surface concept, to choose the design variables which are thicknesses of structures and to smooth the geometry of the structures. Next the proposed analytical design sensitivities are discussed. These sensitivities are obtained from differentiation of the finite element equations with respect to design variables. Finally, examples validating the technique are presented.

Element Formulation

To evaluate sensitivities efficiently, it is important to use an efficient element formulation which can be applied to various shell structures with no sign of locking phenomena. An element formulation is proposed by augmenting the basic 18-node hexahedral solid-shell element with methodologies to eliminate lockings. The geometry of 18-node hexahedral solid-shell element is defined by nine nodes at each of the top ($\zeta = 1$) and bottom ($\zeta = -1$) surfaces (Figure 1).

With three degrees of freedom at each node the deformation of the element is described by 54 d.o.f.. The reference geometry X is defined as

$$X(\xi,\eta,\zeta) = [x\ y\ z]^T = \sum N_i(\xi,\eta,\zeta)X_i \quad (1)$$

Where the N_i are expressed as,

$$N_{2i-1} = ½\, N_i^9(1+\zeta)\ N_{2i}$$
$$= ½\, N_i^9(1-\zeta),\ i = 1\text{-}9 \quad (2)$$

In which the N_i^9 are nine-node Lagrangian shape functions. The displacements at any point within the element are written as,

$$U(\xi,\eta,\zeta) = [u\ v\ w]^T = \sum N_i(\xi,\eta,\zeta)U_i \quad (3)$$

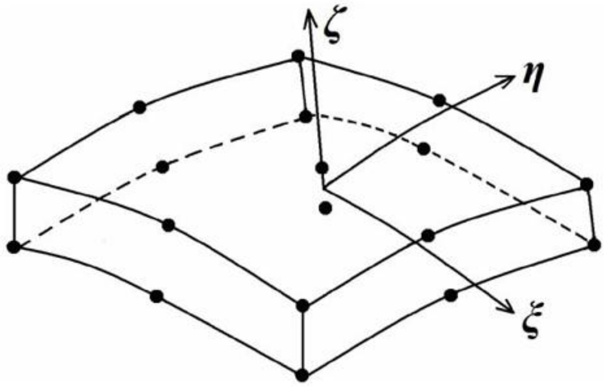

Figure 1. Geometry of eighteen node hexahedral shell element

Locking Alleviation

In its standard form, this element formulation exhibits transverse shear, membrane, thickness and trapezoidal locking. A number of approaches to eliminate locking phenomena have been proposed for completeness and consistency. A further detail of proposed element formulation can be found in Petchsasithon and Gosling[5].

Sensitivity Analysis

The major difficulty in shape optimization problem is that the shape of structures changes in every iteration during the optimization process. This means that using the same finite element mesh may cause distortion which may lead to inaccurate results from finite element analysis. Therefore structures need to be re-meshed each iteration of optimization process. In order to save computational time, automatic mesh generation scheme using Bezier surface technique is employed.

Sensitivity Analysis Using Bezier Surface Technique

Analytical design sensitivities are also derived employing Bezier surface concept at the equilibrium configuration. The equilibrium equation of shell finite element discretization is

$$\{\psi\}=\varphi(U,D)-\{P\}=05 \quad (4)$$

Where $\{\psi\}$, φ, D and $\{P\}$ are out-of-balance force, internal force vector, design variables and external force vectors, respectively. Differentiate equilibrium equation with respect to design variable, d_k gives

$$\{\partial\varphi(U,D)/\partial d_k\}+\{\partial\varphi(U,D)/\partial U\}\{\partial U)/\partial d_k\}-\{\partial P/\partial d_k\}=0$$
$$\text{or } \{\partial U/\partial d_k\}=\{\partial\varphi/\partial U\}^{-1}\{\partial P/\partial d_k-\partial\varphi/\partial d_k\}$$
$$\text{and finally, } \{\partial U/\partial d_k\}= [K_T]^{-1}\{\hat{R}\} \tag{5}$$

Where $[K_T]$ and $\hat{R}$ are tangent stiffness evaluated at equilibrium state and pseudo load vector respectively. $[K_T]$ may be expressed as

$$[K_T] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [T]^T\{[B_l + B_{nl}]^T[D]\ [B_l + B_{nl}]+[G]^T[\sigma][G\}[T]|J|d\xi d\eta d\zeta \tag{6}$$

Where $[T]$, $[B_l]$, $[B_{nl}]$, $[D]$, $[G]$, $[\sigma]$ and $|J|$ are transformation, linear and nonlinear strain-displacement, constitutive, derivative of shape function, stress, and determinant of Jacobian matrices, respectively. It can be seen from Equation (5) that to obtain the derivative of U with respect d_k, the derivative of the internal force and load vector with respect to design variables have to be evaluated.

The derivative of an internal force vector can be obtained by assembling the derivatives of element internal force vector. Element internal force vector can be written as

$$[\varphi_e] = \quad [T]^T\{[B_l + B_{nl}]^T[D]\ [B_l + B_{nl}][T]|J|d\xi d\eta d\zeta \tag{7}$$

It is clear that the constitutive matrix does not depend on the design variables of thickness therefore derivative of element stiffness matrix may be defined as

$$\begin{aligned}[\partial\varphi_e/\partial d_k] = \quad & [\partial T/\partial d_k]^T[B]^T[D][B][T]|J|d\xi d\eta d\zeta \\ + \quad & [T]^T[\partial B/\partial d_k]^T[D][B][T]|J|d\xi d\eta d\zeta \\ + \quad & [T]^T[B]^T[D][\partial B/\partial d_k][T]|J|d\xi d\eta d\zeta \\ + \quad & [T]^T[B]^T[D][B][\partial T/\partial d_k]|J|d\xi d\eta d\zeta \\ + \quad & [T]^T[B]^T[D][B][T]\{\partial|J|/\partial d_k\}d\xi d\eta d\zeta \end{aligned} \tag{8}$$

Where $[B]$ is $[B_l+B_{nl}]$. Integration of Equation (8) is performed by numerical integration with 3x3x2 Gauss integration points.

Optimization

Generally, optimization problems can be mathematically stated as

$$\begin{aligned}&\text{Minimize } F(d)\\ &\text{Subjected to } g_i(d)\le 0,\ i = 1,2,\ldots,n\\ &d_j^l \le d_j \le d_j^u,\ j = 1,2,\ldots,m\end{aligned} \tag{9}$$

where $F(d)$ and $g_i(d)$ are objective function and constraint functions which can be equality or inequality constraints, respectively. d is the vector of design variables. d_j^l and d_j^u are lower and upper bounds on design variables, respectively. n is number of constraints and m is number of design variables. There are generally three types of objective functions for structural problems which are displacements, stresses and masses of structures.

Numerical Examples

Numerical examples have been performed to demonstrate the behavior of the element formulation with respect to transverse shear, membrane, thickness and trapezoidal locking in the analysis of geometrically linear shell structures.

Clamped Square Plate

The analysis of a clamped square plate subjected to a central point load with length/thickness ratio (L/t) in the range 100 to 10000 tests the accuracy of predicting the behavior of thin pates. These normally suffer from transverse shear and thickness locking when represented by solid shell elements. The geometry of the generic clamped square plate is shown in Figure 2 and discretized by 2x2 and 4x4 meshes. Assuming symmetry, only 1x1 (viz 2x2) and 2x2 (viz 4x4) meshes are analyzed. Normalized maximum deflections for proposed formulation are illustrated in Figure 3.

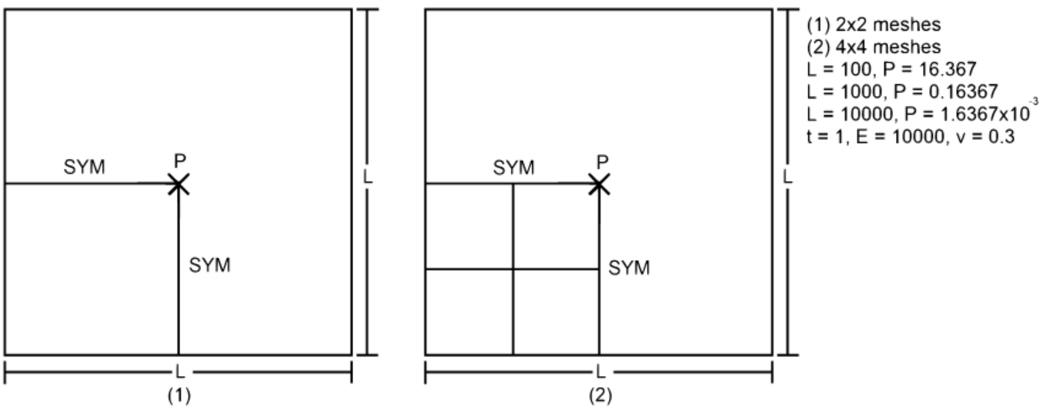

Figure 2. Geometry, loading and material properties of clamped square plate

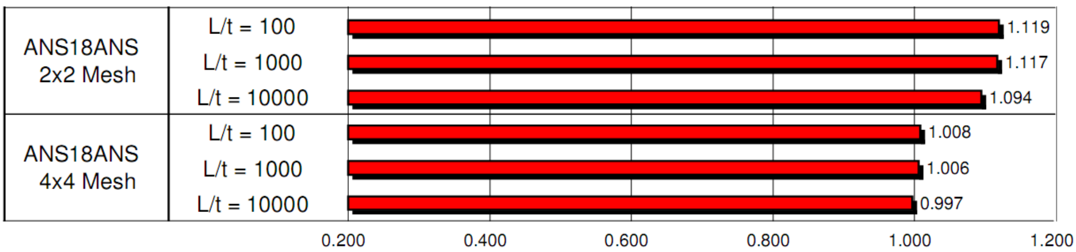

Figure 3. Normalized maximum deflections of clamped square plate

From Figure 3, it is clear that the proposed formulation converge rapidly to the exact solution with mesh refinement. For example, the normalized central deflections for all L/t ratio obtained from 4x4 meshes are very close to unity (less than 1% error). Furthermore, when the length to thickness ratio (L/t) becomes large (the plate becomes thin), the accuracy of the results is upheld, even for L/t of 10000, which is far beyond the practical range.

Hemispherical shell with 18^o aperture

The analysis of the hemispherical 'open' shell test problems couple membrane and bending modes. They test if the element formulations exhibit membrane locking, on the basis that the membrane strain energy should not dominate the total energy for thin curved shells. To investigate if the element formulation is free from membrane locking, the proposed formulation is compared with the exact solution.

The hemispherical shell with an 18^o aperture is represented by a quadrant (assuming symmetry) and subjected to two equal and opposite concentrated loads as shown in Figure 4. The structure is discretized by 2x2, 4x4, 6x6, and 8x8 meshes. This problem investigates the ability of an element to represent inextensional bending modes, given that membrane strains within the shell are small. The exact (non-dimensional) radial displacement at the points of load application is 0.094 (Macneal and Harder[8]).

Normalized results, shown in Figure 5, indicate that the proposed formulation performs well, even for a very coarse mesh (e.g. 4x4), and are demonstrably free of membrane locking. Solutions from the proposed formulations converge with mesh refinement, such that an 8x8 element discretization is generally sufficient.

Examples

A computer program, Fortran 90 programming language is used to develop proposed method for thickness optimization problems. Various classes of structures including beams and plates are analyzed to study the effect of the number of design variables on the optimal solutions.

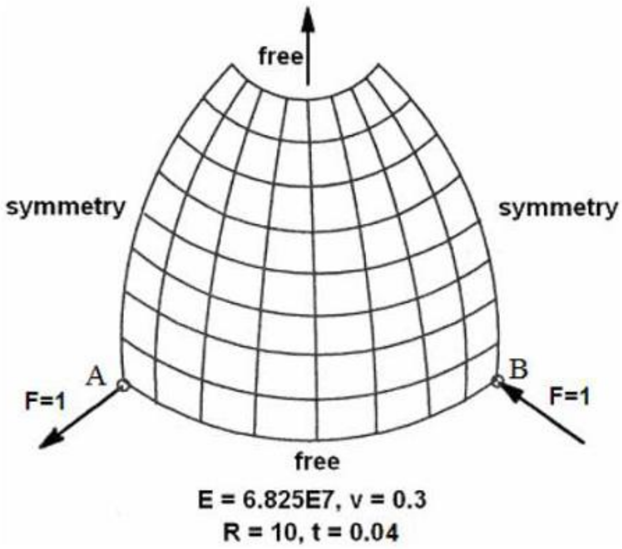

Figure 4. Geometry of a hemispherical shell with 18^o aperture

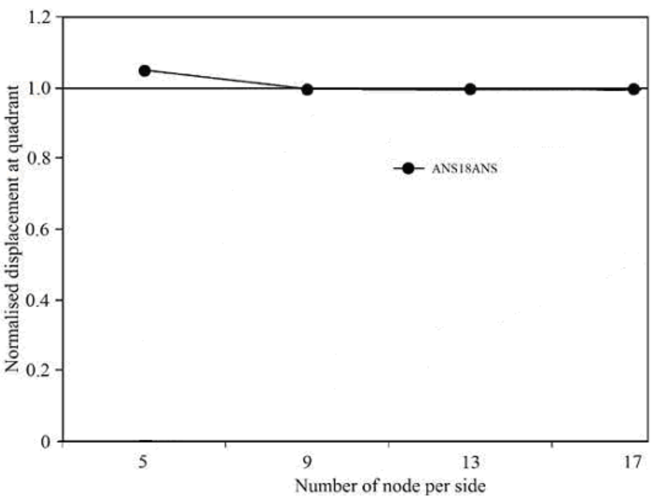

Figure 5. Normalized displacement at quadrant of hemispherical shell with 18^o cut

Geometrically non-linear numerical examples

Hemispherical shell with 18^o aperture

A hemispherical shell with an 18^o circular opening at its pole subjected to alternating point loads *P*s along its equator, illustrated in Figure 2, is again used for geometrically non-linear problem. This problem has been widely used as a benchmark in many papers [7,8,9,10]. Only one quadrant of the shell is performed by utilizing the symmetry condition and is modeled with 8x8, 12x12 and 16x16 meshes. A total of 10 equal load steps are employed. The maximum alternating point loads applied are 400 units. The deformed shape of hemispherical shells under the maximum load is shown in Figure 6. The relationship between displacements at load points, *A* and *B* and magnitude of loads is compared with those in Sze *et al* [10] and depicted in Figure 7.

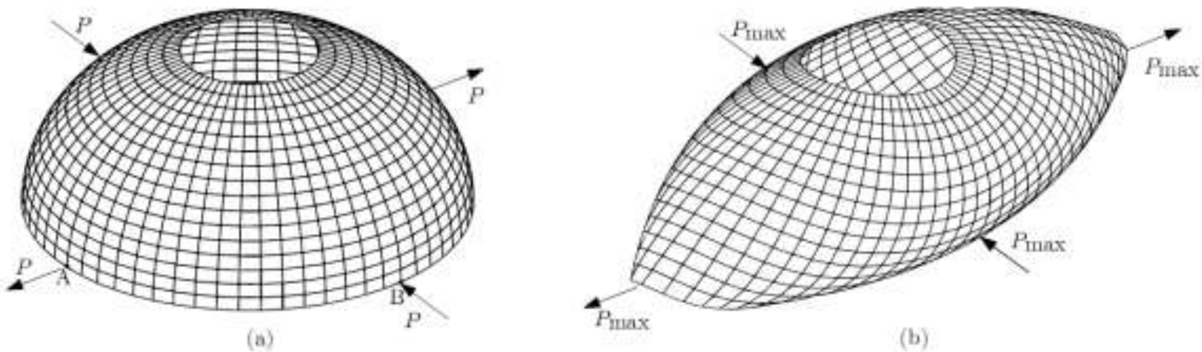

Figure 6. (a) Initial shape of hemispherical shell with 18^o opening (b) Deformed shape of hemispherical shell with 18^o opening subjected to 400-unit load

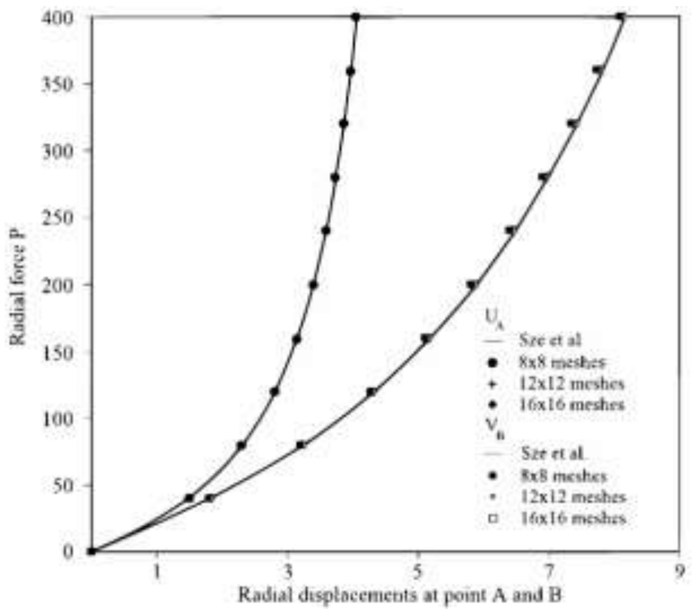

Figure 7. Relationship between radial force and radial displacement at point *A* and *B* of hemispherical shell with 18^o opening

Results from 8x8 meshes are slightly different from Sze *et al* 's [10] (Figure7). Nevertheless results from finer meshes, 12x12 and 16x16 meshes, are coincident with results from Sze *et al* [10]. This may be summarized that reasonable accurate results are achieved by using 12x12 meshes. For sensitivity purpose, 8x8 meshes are used to modeled hemispherical shell since it produces reasonable accurate result.

Sensitivity Analysis

The major difficulty in shape optimization problem is that the shape of structures changes in every iteration during the optimization process. This means that using the same finite element mesh may cause distortion which may lead to inaccurate results from finite element analysis. Therefore structures need to be re-meshed each iteration of optimization process. In order to save computational time, automatic mesh generation scheme [11] using Bezier surface technique is employed.

Sensitivity analysis using Bezier surface technique

Analytical design sensitivities are also derived employing Bezier surface concept at the equilibrium configuration. The equilibrium equation of shell finite element discretization is

$$\{\psi\} = \phi(U, D) - \{P\} = 0, \tag{4}$$

Where $\{\psi\}$, ϕ , D and $\{P\}$ are out-of-balance force, internal force vector, design variables and external force vectors, respectively. Differentiate equilibrium equation with respect to design variable, d_k gives

$$\left\{\frac{\partial \phi(U, D)}{\partial d_k}\right\} + \left\{\frac{\partial \phi(U, D)}{\partial U}\right\}\left\{\frac{\partial U}{\partial d_k}\right\} - \left\{\frac{\partial P}{\partial d_k}\right\} = 0$$

$$\text{or } \left\{\frac{\partial U}{\partial d_k}\right\} = \left\{\frac{\partial \phi}{\partial U}\right\}^{-1}\left\{\frac{\partial P}{\partial d_k} - \frac{\partial \phi}{\partial d_k}\right\}$$

$$\text{and finally, } \left\{\frac{\partial U}{\partial d_k}\right\} = \left[K_T\right]^{-1}\{\hat{R}\} \tag{5}$$

Where $[K_T]$ and $\hat{R}$ are tangent stiffness evaluated at equilibrium state and pseudo load vector respectively. $[K_T]$ may be expressed as

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It can be seen from Equation (5) that to obtain the derivative of U with respect D_k, the derivative of the internal force and load vector with respect to design variables have to be evaluated.

The derivative of an internal force vector can be obtained by assembling the derivatives of element internal force vector. Element internal force vector can be written as

$$[\phi_e] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [T]^T [B_l + B_{nl}]^T [D][B_l + B_{nl}][T]\,|J|\,d\xi d\eta d\zeta \{U\} \tag{7}$$

Where $[T]$, $[B]$, $[D]$ and $|J|$ are transformation, strain-displacement, constitutive and determinant of Jacobian matrices, respectively. It is clear that the constitutive matrix does not depend on the design variables of thickness therefore derivative of element stiffness matrix may be defined as

$$\begin{aligned}\left[\frac{\partial \phi_e}{\partial d_k}\right] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} &\left[\frac{\partial T}{\partial d_k}\right]^T [B]^T[D][B][T]\,|J| + [T]^T\left[\frac{\partial B}{\partial d_k}\right]^T [B]^T[D][B][T]\,|J| \\ &+ [T]^T[B]^T[D]\left[\frac{\partial B}{\partial d_k}\right][T]\,|J| + [T]^T[B]^T[D][B]\left[\frac{\partial T}{\partial d_k}\right]|J| \\ &+ [T]^T[B]^T[D][B][T]\frac{\partial|J|}{\partial d_k}\end{aligned} \tag{8}$$

Where $[B]$ is $[B_l+B_{nl}]$. Integration of Equation (8) is performed by numerical integration with 3x3x2 Gauss integration points.

Optimization

Generally, optimization problems can be mathematically stated as

$$\begin{aligned} &\text{Minimize } F(d) \\ &\text{Subjected to } g_i(d) \leq 0,\ i = 1,2,\ldots,n \\ &d_j^l \leq d_j \leq d_j^u,\ j = 1,2,\ldots,m \end{aligned} \qquad (9)$$

where $F(d)$ and $g_i(d)$ are objective function and constraint functions which can be equality or inequality constraints, respectively. d is the vector of design variables. d_j^l and d_j^u are lower and upper bounds on design variables, respectively. n is number of constraints and m is number of design variables. There are generally three types of objective functions for structural problems which are displacements, stresses and masses of structures.

Examples

A computer program, Fortran 90 programming language is used to develop proposed method for thickness optimization problems. Various classes of structures including beams and plates are analyzed to study the effect of the number of design variables on the optimal solutions.

Cantilevered beam subjected to bending moment

Cantilevered beam with different number of design variables used (13 design variables shown in Figure 8a. Beam is modeled by 6x1 mesh elements. Solutions (Figure 8b) are compared with those using 19 design variables (Figure 8c)

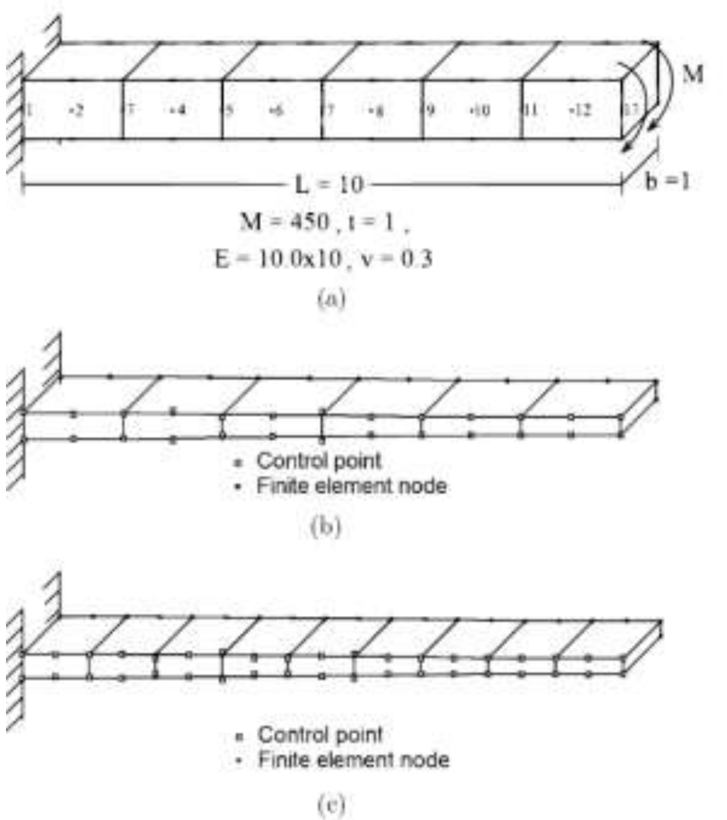

Figure 8. (a) Geometry, material properties and loading of cantilever beam subjected to bending. Optimum design of cantilever beam subjected to bending employing Bezier technique (b) 13 design variables (c) 19 variables

Cantilevered Beam Subjected to Bending Moment

Cantilevered beam with different number of design variables used (13 design variables shown in Figure 8a. Beam is modeled by 6x1 mesh elements. Solutions (Figure 8b) are compared with those using 19 design variables (Figure 8c)

Optimal geometry obtained from using 13 design variables (Figure 8b) is similar to that using 19 design variables (Figure 8c) with the optimum mass for both number of design variables being 1.073 (exact solution for optimum mass is 1.077 [9]). This implies that using different number of design variables does not affect the optimal solution for the beam problems.

Having illustrated the effect of number of design variables on optimal solutions for beam problems, plate structure is now analyzed using various numbers of design variables to model the structures.

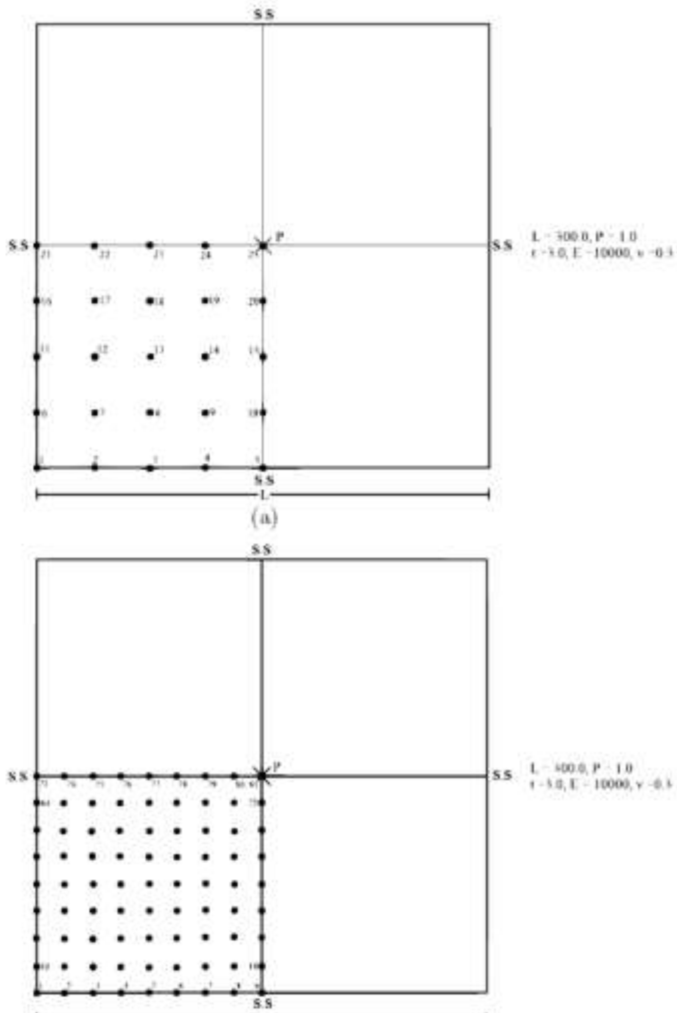

Figure 9. Geometry, material properties and loading of simply supported plate subjected to central point load modeled with (a) 25 design variables (b) 81 design variables

Simply Supported Plate Subjected to Central Point Load

A square plate displayed in Figure 9 is simply supported on all sides and subjected to central point load. Length and thickness of the plate are 300 and 3, respectively. Young's modulus and Poisson's ratio of this structure are 1000 and 0.3, respectively. Owing to symmetry, only a quarter of this structure is analyzed using 8x8 mesh elements. Geometry of the structure is defined by 25 and 81 design variables which control thickness of each design node (Figure 9a and 9b). This example is to minimize the total strain energy of the structure by performing thickness optimization. The maximum displacement and total strain energy of the initial model are 4.2391x10^{-2} and 2.1195x10^{-2}, respectively. The allowable minimum and maximum thickness are 1 and 20 mm, respectively. Lam *et al*[4] has also investigated this problem with much finer finite element meshes (2912 linear membrane triangular 3-noded plate elements)

Optimum geometries of simply supported plate subjected to central point load using 25 and 81 design variables are illustrated in Figure 10a and 10b, Table 1 and 2 and are compared with that from Lam *et al*[4] (Figure 11). The total strain energy of the optimum designs for 25 and 81 design variables are 6.4466x10^{-3} and 4.2615x10^{-3} which are 69.6%and 79.9% reduction from initial design, respectively. In the optimum designs, ribs are formed connecting the mid sides of the plate. The optimum design obtained using 81 design variables shows good agreement with that from Lam *et al*[4] with the maximum thickness being 17.36 and 17.29, respectively. While result using 25 design variables is poorer than that utilizing 81 design variables since total strain energy is 33.9% higher than that for the latter case.

Clamped Square Plate Subjected to Central Point Load

In this example, we seek to minimize the total strain energy of the square plate shown in Figure 12. The square plate is clamped at all edges and subjected to point load of 100 unit at the centre of the plate. Length and thickness of the plate are 10 and 0.1, respectively. Material with Young's modulus, E, of 10.92x10^{5} and Poisson's ratio of 0.3 is used for this problem. The design domain is discretized into 16x16 meshes. Taking advantage of the symmetry, only one quarter of the structure is analyzed employing Bezier methods. This test is employed as an alternative example to demonstrate the effect of the number of design variable on the optimal solutions. Thickness distribution of the plate is controlled by 9, 16 and 25 design variables as illustrated in Figure 12.

Table 1. Optimum Thicknesses and Maximum Displacement of Simply Supported Plate Subjected to Central Point Load (25 Design Variables)

Variable	Thickness	Variable	Thickness
1	1.000	14	1.000
2	1.000	15	6.107
3	1.000	16	1.000
4	1.000	17	1.000
5	1.000	18	1.000
6	1.000	19	1.000
7	1.000	20	8.507
8	1.000	21	1.000
9	1.000	22	7.907
10	7.907	23	6.107
11	1.000	24	8.507
12	1.000	25	11.96
13	1.000	Total SE	6.4466x10^{-3}

Table 2. Optimum Thicknesses and Maximum Displacement of Simply Supported Plate Subjected to Central Point Load (81 Design Variables)

Variable	Thickness	Variable	Thickness
1-9	1.000	63	17.97
10-17	1.000	64-71	1.000
18	5.563	72	9.721
19-26	1.000	73	1.000
27	10.91	74	5.563
28-35	1.000	75	10.91
36	10.03	76	10.03
37-44	1.000	77	10.25
45	10.25	78	15.23
46-53	1.000	79	17.97
54	15.23	80	9.721
55-62	1.000	81	17.36
Total SE	4.2615×10^{-3}		

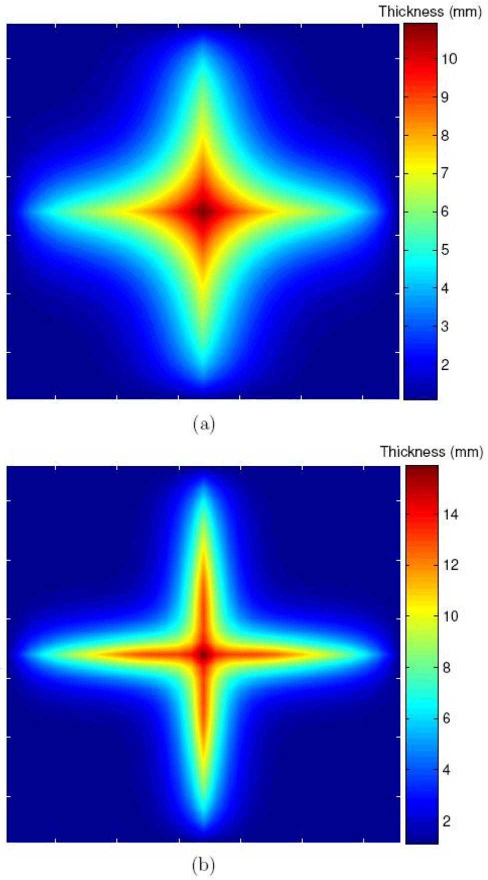

Figure 10. Optimum design of simply supported plate subjected to central load obtained from Bezier technique (a) 25 design variables (b) 81 design variables

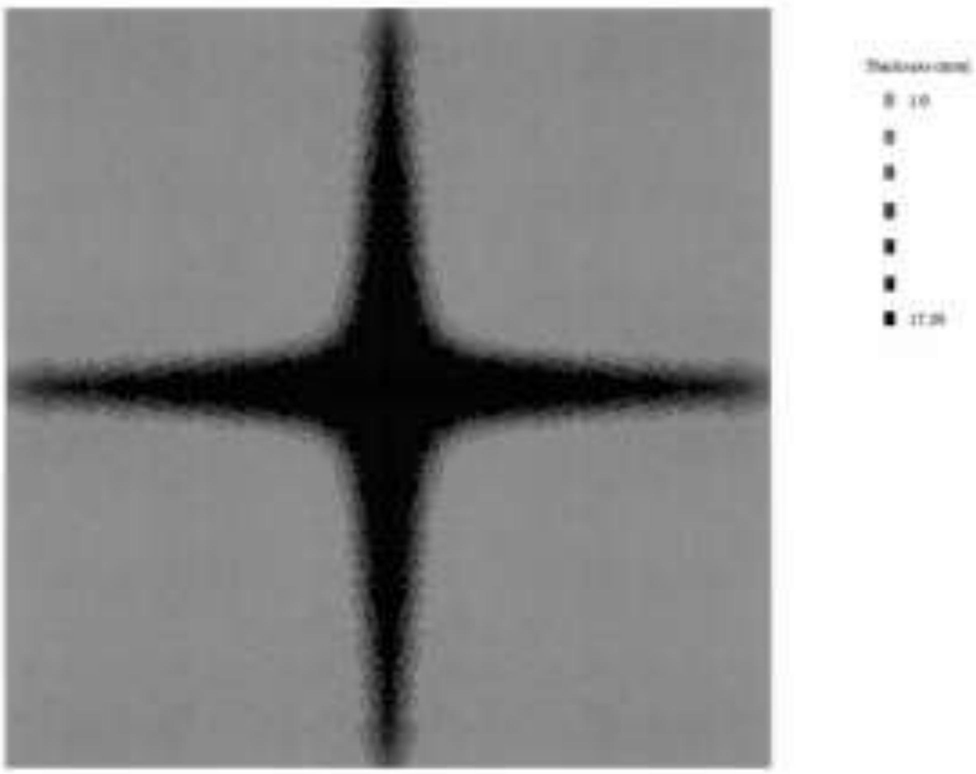

Figure 11. Optimum design of simply supported plate subjected to central load [4]

Optimum thickness distribution of the plate using 9, 16 and 25 design variables are displayed in Figure 13, 14 and 15, respectively. Total strain energy of initial and optimum designs are given in Table 3. Cappello and Mancuso [10] performed topology optimization of this clamped square plate problem using 400 four-node shell elements.

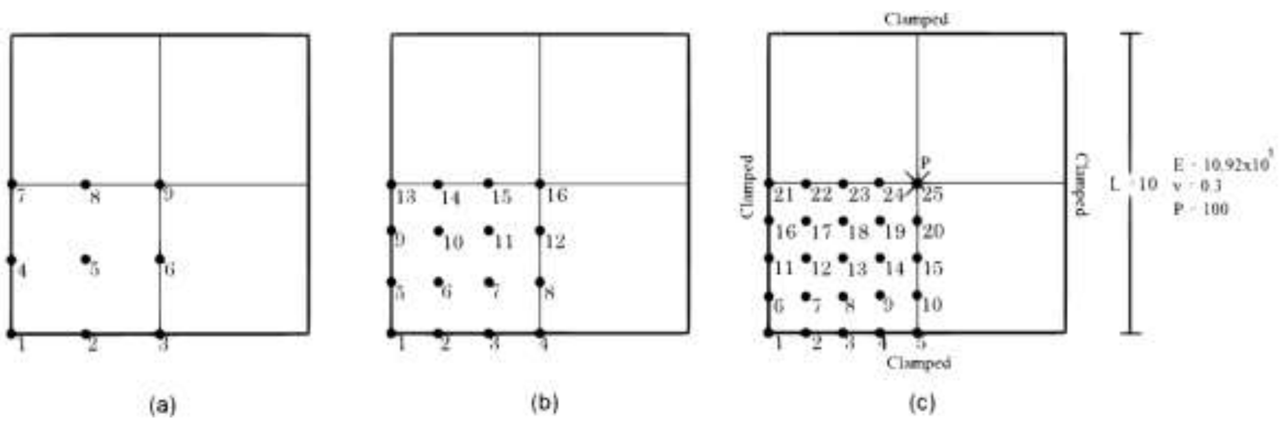

Figure 12. A clamped plate subjected to central point load (a) 9 design variables (b) 16 design variables (c) 25 design variables

Using alternative numbers of design variables exhibits similar optimal designs with the more reductions in strain energy being achieved when utilizing more design variables (e.g. see Table 3. This implies that the optimal design is independent on the number of design variables. Furthermore results from proposed method agree well with those from Cappello and Mancuso [10] (Figure 16) indicating that equivalent material distributions can be obtained from different methodologies (thickness and topology optimization). Results suggest that more material is required at the centre and mid-side of the plate.

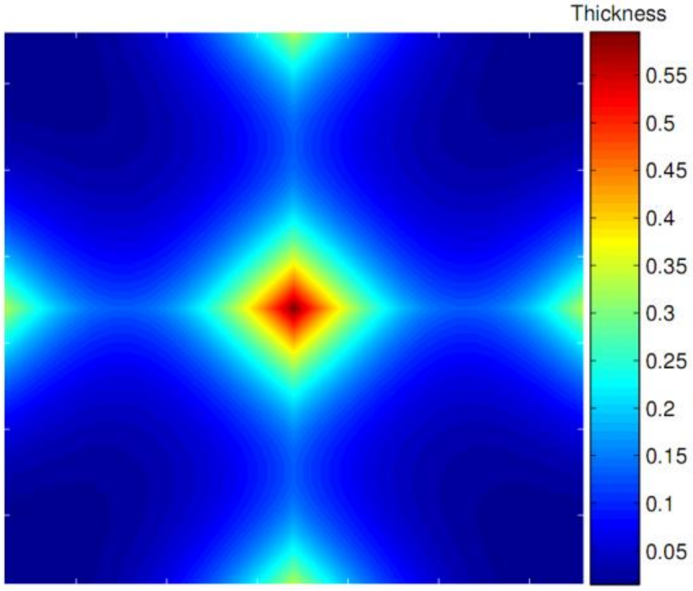

Figure 14. Thickness optimum design of clamped square plate subjected to central point load (16 design variables)

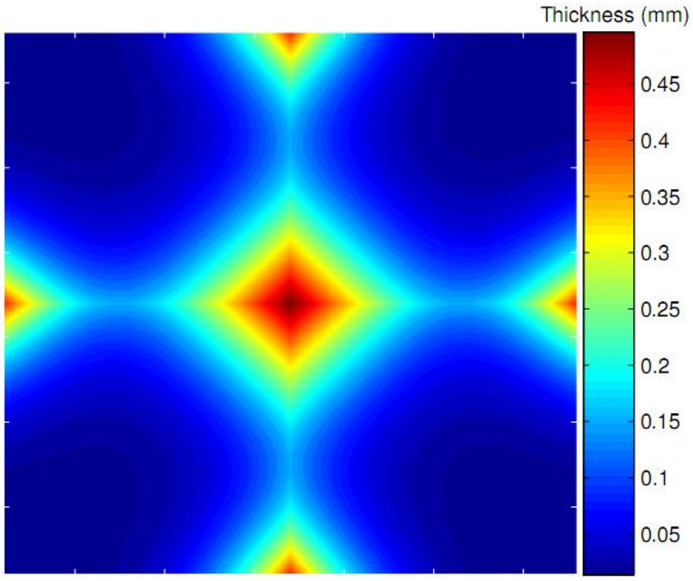

Figure 15. Thickness optimum design of clamped square plate subjected to central point load (25 design variables)

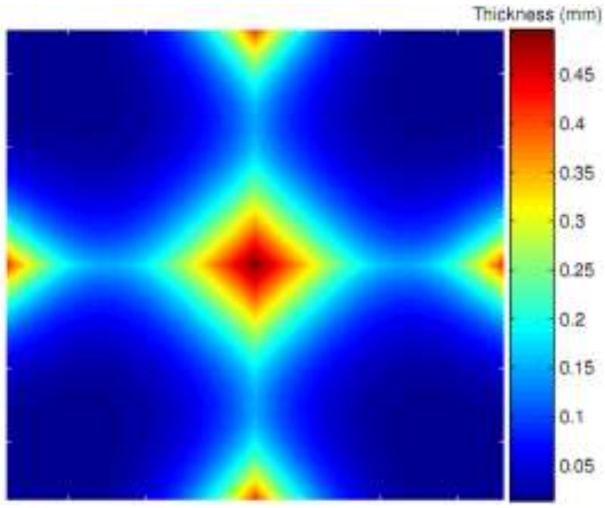

Figure 15. Thickness optimum design of clamped square plate subjected to central point load (25 design variables)

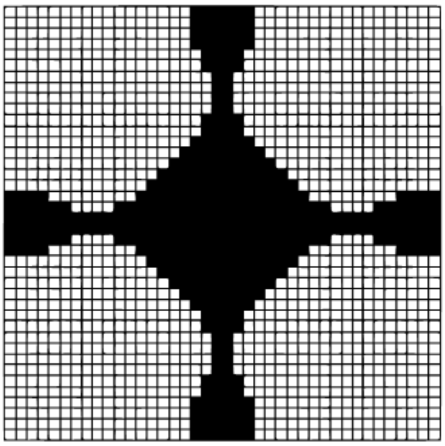

Figure 16. Optimal topology of the clamped square plate subjected to central point load [10]

Table 3. Total Strain Energy of Initial and Optimum Model of a Clamped Square Plate Subjected to Central Point Load Using Various Numbers of Design Variables

	Design Variables		
	3x3	4x4	5x5
Initial SE	28.16	28.16	28.16
Optimal SE	6.471	4.660	3.817
Percentage reduction	77.0	83.5	86.4

Conclusions

Thickness optimization of geometrically linear shells has been evaluated by using gradient-based method which can be applied to various shell structures. Proposed optimization technique has been extensively assessed against geometrically linear numerical examples. The optimal beam results demonstrate that the number of design variables has a marginal effect on the optimal solution. In contrast, the analyses of plate problems indicate that the optimal solutions are evidently dependent on the number of design variables. Therefore a structure with a large number of design variables defined by a uniform mesh may be required to produce the converged solutions with a corresponding computational cost.

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