

# DAMAGE IDENTIFICATION WITH SENSOR PLACEMENT AND REFERENCE FREQUENCY CHOSEN BASED ON THEORETICAL ERRORS

Aiko Furukawa<sup>1</sup> and Hisanori Otsuka<sup>2</sup>

<sup>1</sup>Department of Urban Management, Kyoto University, Kyoto, Japan, e-mail: furukawa.aiko.3w@kyoto-u.ac.jp

<sup>2</sup>Department of Civil Engineering, Kyushu University, Fukuoka, Japan, e-mail: otsuka@doc.kyushu-u.ac.jp

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## Abstract

This paper presents a vibration-based damage identification technique for structures using Fourier amplitudes of acceleration responses generated by ground shaking. The proposed damage identification technique estimates structural damage by solving a simultaneous equation. This simultaneous equation is derived from the equation of motion of the structure before and after damage and composed by structural parameters of the intact state, such as mass, damping and stiffness, and measured Fourier amplitudes of the damaged state. Questions as to where to put sensors and Fourier amplitudes at which reference frequency should be used are unsettled. Since there always exist modeling and measurement errors, and since identification accuracy depends on the sensor placement and reference frequency, determination method of the sensor placement and reference frequency giving better accuracy is important. Therefore, the determination method based on theoretical error of the simultaneous equation is also proposed in this study. In the proposed method, the sensor placement and reference frequency generating small condition number of the coefficient matrix of the simultaneous equation is recommended. The validity of the damage identification technique and determination method is confirmed through experiment on a cantilever beam.

**Keywords:** Damage identification, Fourier amplitude, Reference frequency, Sensor placement, Theoretical error

## Introduction

When a large earthquake occurs, important infrastructures must be assessed and repaired immediately to prevent the expansion of secondary damage. Damage assessment, particularly regarding the need for repair or reinforcement of a structure, must be made and adequate measures taken to avoid catastrophic situations. To meet these needs, damage identification techniques have been developed in this study.

Vibration-based damage identification is a technique that evaluates the global structural condition. This method uses the fact that structural damage usually causes a decrease in structural stiffness, thereby producing a change in vibration characteristics. Vibration-based damage identification techniques can be categorized according to various criteria. Sorting with regard to the vibration characteristics used for identification, the methods are divided into time-domain, frequency-domain, or time-frequency domain techniques.

The Kalman filter [1], Monte Carlo filter [2], and  $H^\infty$  filter [3] are well known time-domain techniques, sequentially updating the structural parameters step by step. These methods have the advantage of being able to identify damage on-line. The application of these techniques, however, are limited to systems with low degrees of freedom. Time-frequency-domain damage identification techniques [4][5][6] are based on time-frequency analysis which is capable of describing concurrent time and frequency

information. This analysis can determine when and where a particular event took place. Currently, many researchers have succeeded in detecting damage existence and damage localization, but studies regarding quantification of damage severity is lacking. Frequency-domain techniques based on the Fourier transform are frequently and commonly used, from a simple structure like cantilever beams [7] to complex structures such as long span bridges with high degrees of freedom [8].

Vibration data used for frequency-domain damage identification techniques fall into two categories; modal data and frequency response function (FRF) [9] [10] [11]. Modal data includes the natural frequency [7] [12] [13], mode shape [14] [15] [16], mode shape curvature [17], modal flexibility [18], and modal strain energy [19] [20]. As indicated by Banks et al. [21], Wang et al. [9], and Lee et al. [11], the use of modal data has disadvantages. Since test data are indirectly measured, they may be contaminated by measurement errors as well as modal extraction errors. In addition, the majority of these methods require complete modal data, which cannot be obtained in most cases because a large number of sensors usually are required. In contrast, FRF is less contaminated because they are measured directly from structures. Also, more damage information can be provided in a desired frequency range than with modal data because the latter is mainly extracted from very limited FRF related to resonance [9]. For these reasons, the use of FRF has greater potential than modal data. This study develops a damage identification technique using FRF.

With regard to external force used to vibrate the structure, damage identification techniques are divided into two types: one using artificial vibrations due to exciters or actuators, and the other using natural vibrations such as ground motion and wind force. Artificial vibration is advantageous as accurate input and output data can be used for identification, thereby identification accuracy is theoretically high. The artificial vibration, however, is usually expensive and impractical, and sometimes not feasible due to the condition and scale of the structure. In contrast, vibration measurement using the natural vibration is freely available and a useful alternative to the artificial excitation. In this study, it is assumed that a structure is excited by ground shaking, and Fourier amplitudes of structural acceleration responses are used for identification. As for the source of ground shaking, earthquake ground motion and ambient vibration such as traffic loads are considered. It is assumed that effect of other forces like wind force is negligibly small compared to the ground shaking or that the ground shaking is large enough that other forces are negligible. Since FRF is a ratio of the Fourier amplitude of the structural acceleration to that of the ground acceleration, the proposed technique can be categorized as the FRF-based damage identification technique. In the procedure, input acceleration and structural acceleration responses are recorded and the responses converted to Fourier amplitudes. Damage is assumed to be accompanied by changes in structural parameters, namely a decrease in stiffness and an increase in damping causing an alteration in the Fourier amplitudes. Comparison of the equation of motion of the structure before and after the damage provides a damage identification equation which is a simultaneous equation relating local changes in structural parameters to changes in the Fourier amplitudes. If Fourier amplitudes are obtained, then changes in the structural parameters can be determined by solving the damage identification equation. Change in a structural parameter directly pinpoints the location and magnitude of the damage. The proposed technique that uses Fourier amplitudes has several advantages. The major advantage is that data are easily accumulated, by solely changing measurement points and reference frequency of Fourier amplitudes. When the equivalent number of measurements as unknown parameters are completed, the damage identification equation presents a determined problem. Also, by

conducting a greater number of measurements than the number of unknown parameters, higher accuracy is expected.

Here, questions as to where to put sensors and Fourier amplitudes at which reference frequency should be used are unsettled. Since identification accuracy depends on the sensor placement and reference frequency, determination method of appropriate sensor placement and reference frequency is important. A predicament commonly encountered in practical cases is modeling and measurement noises. Many statistical approaches have been proposed, including the prediction error method [22], Kalman filter [1] and  $H^\infty$  filter [3]. Xia et al. [23] studied the influence of uncertainties in modeling and measurement errors and estimated the probability of damage existence. Stiffness parameter statistics were derived by the perturbation method and the probabilistic distribution was determined from Monte Carlo simulation results with the assumption of a normal distribution pattern. The majority of these approaches are based on the ideal assumption that errors are normally distributed and therefore these techniques do not always work effectively as errors are not normally distributed. Since identification accuracy depends on the sensor placement and reference frequency and the problem of modeling and measurement errors needs to be handled, this paper proposes a determination method of the sensor placement and reference frequency which give smaller theoretical error in the identification results. The combination of the sensor placement and reference frequency generating smaller condition number of the coefficient matrix of the damage identification equation is expected to give smaller error and considered to be one of the "appropriate" selections.

In the next section, the theoretical algorithm of the proposed damage identification technique is first modeled. The determination method of the sensor placement and reference frequency is then developed. The validity of the proposed damage identification technique and the determination method are shown through experimental study of a cantilever beam.

## Damage Identification Technique Using Fourier Amplitude

### Modeling of Damage

The total mass  $\mathbf{M}$ , damping  $\mathbf{C}$ , and stiffness  $\mathbf{K}$  matrices of the intact structure are the summation of the element matrices;

$$\mathbf{M} = \sum_{e=1}^n \mathbf{M}^e \quad \mathbf{C} = \sum_{e=1}^n \mathbf{C}^e \quad \mathbf{K} = \sum_{e=1}^n \mathbf{K}^e \quad (1)$$

where  $n$  is the number of elements, and  $\mathbf{M}^e$ ,  $\mathbf{C}^e$ , and  $\mathbf{K}^e$  ( $e = 1, \dots, n$ ) are the contribution of the  $e$ -th element to the total mass, damping, and stiffness matrices of the intact structure, respectively. Damage to the structure is assumed to cause changes of  $\delta\mathbf{C}$  in the damping matrix and  $\delta\mathbf{K}$  in the stiffness matrix. The mass is assumed to be constant before and after damage. Changes in the  $e$ -th element damping  $\delta\mathbf{C}^e$  and stiffness  $\delta\mathbf{K}^e$  matrices are assumed to be proportional to the element matrices;

$$\delta\mathbf{C}^e = \delta c_e \mathbf{C}^e \quad \delta\mathbf{K}^e = \delta k_e \mathbf{K}^e \quad (2)$$

where  $\delta c_e$  and  $\delta k_e$  are respectively proportional changes in damping and stiffness of the  $e$ -th element, and  $\delta c_e$  and  $\delta k_e$  are larger than -1. If  $\delta c_e$  and  $\delta k_e$  are larger than 0, this means that the damping and stiffness of the  $e$ -th element are increased. Variation in the total structural matrices due to damage are therefore expressed as;

$$\delta\mathbf{C} = \sum_{e=1}^n \delta c_e \mathbf{C}^e \quad \delta\mathbf{K} = \sum_{e=1}^n \delta k_e \mathbf{K}^e \quad (3)$$

A technique is proposed which obtains  $\delta c_e$  and  $\delta k_e$  from the vibration measurements. If the identified stiffness change  $\delta k_e$  is smaller than 0.0, such an element is considered to be damaged and the magnitude of  $\delta k_e$  indicates the severity of damage.

## Vibration Responses of Intact Structure

It is assumed that acceleration response of a structure under ground shaking is measured to identify structural damage. Let  $\mathbf{U}(\omega)$  be the Fourier spectra of acceleration vector of measured ground accelerations. First, the vibration response to ground shaking  $\mathbf{U}(\omega)$  of the structure in the intact state is predicted. The equation of motion for the intact structure in the frequency domain is;

$$[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]\mathbf{X}(\omega) = -\mathbf{M}\mathbf{U}(\omega) \quad (4)$$

where  $\mathbf{X}(\omega)$  is the Fourier spectra of displacement response vector. It is noted that dimensions of these matrices and vectors are equivalent to the number of freedoms of the model. The vibration response is obtained as;

$$\mathbf{X}(\omega) = -\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \quad (5)$$

where  $\mathbf{H}(\omega)$  is the transfer function for the intact structure;

$$\mathbf{H}(\omega) = [-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]^{-1} \quad (6)$$

## Vibration Responses of Damaged Structure

The equation of motion for a damaged structure in the frequency domain is predicted as;

$$[-\omega^2\mathbf{M} + i\omega(\mathbf{C} + \delta\mathbf{C}) + (\mathbf{K} + \delta\mathbf{K})](\mathbf{X}(\omega) + \delta\mathbf{X}(\omega)) = -\mathbf{M}\mathbf{U}(\omega) \quad (7)$$

where  $\delta\mathbf{C}$  and  $\delta\mathbf{K}$  are respectively variations in the damping and stiffness matrices, and  $\delta\mathbf{X}(\omega)$  is the increase in the displacement response.

Substituting Equation (4) into Equation (7) and neglecting higher terms yields the equation for  $\delta\mathbf{X}(\omega)$ ;

$$[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]\delta\mathbf{X}(\omega) = -[i\omega\delta\mathbf{C} + \delta\mathbf{K}]\mathbf{X}(\omega) \quad (8)$$

Substituting Equation (5) into Equation (8) then

$$[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]\delta\mathbf{X}(\omega) = [i\omega\delta\mathbf{C} + \delta\mathbf{K}]\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \quad (9)$$

The increase in displacement response,  $\delta\mathbf{X}(\omega)$ , is obtained by solving Equation (9);

$$\delta\mathbf{X}(\omega) = \mathbf{H}(\omega)[i\omega\delta\mathbf{C} + \delta\mathbf{K}]\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \quad (10)$$

Displacement response after damage,  $\tilde{\mathbf{X}}(\omega)$ , is obtained by adding  $\mathbf{X}(\omega)$  in Equation (5) to  $\delta\mathbf{X}(\omega)$  in Equation (10).

$$\begin{aligned} \tilde{\mathbf{X}}(\omega) &= \mathbf{X}(\omega) + \delta\mathbf{X}(\omega) \\ &= -\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) + \mathbf{H}(\omega)[i\omega\delta\mathbf{C} + \delta\mathbf{K}]\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \\ &= -\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) + i\omega\mathbf{H}(\omega)\delta\mathbf{C}\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) + \mathbf{H}(\omega)\delta\mathbf{K}\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \\ &= -\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) + i\omega \sum_{e=1}^n \mathbf{H}(\omega)\mathbf{C}^e \mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega)\delta c_e + \sum_{e=1}^n \mathbf{H}(\omega)\mathbf{K}^e \mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega)\delta k_e \end{aligned} \quad (11)$$

Here, let  $\mathbf{P}^e(\omega)$ ,  $\mathbf{Q}^e(\omega)$  and  $\mathbf{R}^e(\omega)$  be vectors as follows:

$$\mathbf{P}^e(\omega) = -\mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \quad \mathbf{Q}^e(\omega) = i\omega\mathbf{H}(\omega)\mathbf{C}^e \mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \quad \mathbf{R}^e(\omega) = \mathbf{H}(\omega)\mathbf{K}^e \mathbf{H}(\omega)\mathbf{M}\mathbf{U}(\omega) \quad (12)$$

From here, the Fourier amplitude of response after damage  $\tilde{\mathbf{X}}(\omega)$  is as follows.

$$\tilde{\mathbf{X}}(\omega) = \mathbf{P}(\omega) + \sum_{e=1}^n \mathbf{Q}^e(\omega)\delta c_e + \sum_{e=1}^n \mathbf{R}^e(\omega)\delta k_e \quad (13)$$

## Derivation of Damage Identification Equation

This study assumes that acceleration response in the damaged state is measured at several points, and their Fourier amplitudes at the reference frequency,  $\omega$ , is used to identify damage. The ground acceleration also has to be measured. Assuming that the acceleration is measured at point  $i$  and that the reference frequency is  $\omega$ , Fourier amplitude,  $a(i, \omega)$ , becomes

$$a(i, \omega) = -\omega^2 \tilde{\mathbf{X}}_i(\omega) = -\omega^2 \mathbf{P}_i^e(\omega) - \omega^2 \sum_{e=1}^n \mathbf{Q}_i^e(\omega)\delta c_e - \omega^2 \sum_{e=1}^n \mathbf{R}_i^e(\omega)\delta k_e \quad (14)$$

In Equation (14),  $P_i(\omega)$ ,  $Q_i(\omega)$ , and  $R_i(\omega)$  are known quantities obtained from the structural

parameters of the intact state,  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$ , and the reference frequency,  $\omega$ . The Fourier amplitude,  $a(i, \omega)$ , is obtained from measurements of the structure after damage. Transposing the unknown terms to the left side and the known ones to the right side, Equation (14) becomes

$$-\omega^2 \sum_{e=1}^n \mathbf{Q}_i^e(\omega) \delta c_e - \omega^2 \sum_{e=1}^n \mathbf{R}_i^e(\omega) \delta k_e = a(i, \omega) + \omega^2 \mathbf{P}_i^e(\omega) \quad (15)$$

In Equation (15), the measurement point  $i$  and reference frequency  $\omega$  are arbitrary values. Thus, choosing  $m$  different sets of  $i$  and  $\omega$ , this relationship can be written as a simultaneous equation

$$[\mathbf{S} \quad \mathbf{T}] \begin{Bmatrix} \delta \mathbf{c} \\ \delta \mathbf{k} \end{Bmatrix} = \{\delta \mathbf{a}\} \quad (16)$$

where  $\mathbf{S}$  and  $\mathbf{T}$ , respectively, are the  $m \times n$  known complex matrices whose  $(l, e)$  components are

$$S_{le} = -\omega^2 Q_i^e(\omega) \quad T_{le} = -\omega^2 R_i^e(\omega) \quad (17)$$

where,  $e$  indicates the element number ( $e = 1, \dots, n$ ), and  $l$  indicates the number of measured Fourier amplitudes ( $l = 1, \dots, m$ ),  $\delta \mathbf{c}$  is the  $n$ -dimensional real vector of unknown changes in damping  $\delta c_e$ , and  $\delta \mathbf{k}$  is the  $n$ -dimensional real vector of unknown changes in stiffness  $\delta k_e$

$$\{\delta \mathbf{c}\} = \{\delta c_1, \dots, \delta c_n\}^T \quad \{\delta \mathbf{k}\} = \{\delta k_1, \dots, \delta k_n\}^T \quad (18)$$

$\delta \mathbf{a}$  is  $m$ -dimensional complex vector of changes in the Fourier amplitudes of acceleration responses.

$$\delta a_l = a(i, \omega) + \omega^2 P_i^e(\omega) \quad (19)$$

Separating the complex parameters into real and imaginary parts, Equation (16) yields

$$\begin{bmatrix} \text{Re } \mathbf{S} & \text{Re } \mathbf{T} \\ \text{Im } \mathbf{S} & \text{Im } \mathbf{T} \end{bmatrix} \begin{Bmatrix} \delta \mathbf{c} \\ \delta \mathbf{k} \end{Bmatrix} = \begin{Bmatrix} \text{Re } \delta \mathbf{a} \\ \text{Im } \delta \mathbf{a} \end{Bmatrix} \quad (20)$$

Here  $\mathbf{A}$ ,  $\mathbf{x}$  and  $\mathbf{b}$  are defined as

$$\mathbf{A} = \begin{bmatrix} \text{Re } \mathbf{S} & \text{Re } \mathbf{T} \\ \text{Im } \mathbf{S} & \text{Im } \mathbf{T} \end{bmatrix} \quad \mathbf{x} = \begin{Bmatrix} \delta \mathbf{c} \\ \delta \mathbf{k} \end{Bmatrix} \quad \mathbf{b} = \begin{Bmatrix} \text{Re } \delta \mathbf{a} \\ \text{Im } \delta \mathbf{a} \end{Bmatrix} \quad (21)$$

Then the Equation (20) becomes

$$\mathbf{A} \mathbf{x} = \mathbf{b} \quad (22)$$

Thus,  $m$  different Fourier amplitudes may yield  $2m$  equations for  $2n$  unknowns. By solving Equation (22), changes in structural parameters can be obtained. Solving Equation (22) gives the location and magnitude of damage. The simultaneous equation (22) is named "damage identification equation" in this study.

The present damage identification technique requires the following two types of data:

- structural parameters of the intact state;  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$
- Fourier amplitudes of the damaged state;  $a(i, \omega)$

Changing the measurement point,  $i$ , and the reference frequency,  $\omega$ , gives various equations. By conducting as many measurements as there are elements, the damage identification equation gives a determined problem, and by conducting a greater number of measurements, the damage identification equation gives an overdetermined problem.

## Determination Method of Sensor Placement and Reference Frequency

### Definition of Appropriate Sensor Placement and Reference Frequency

As mentioned earlier, structural damage is identified by solving the simultaneous equation named the damage identification equation.

$$\mathbf{A} \mathbf{x} = \mathbf{b} \quad (23)$$

Here, matrix  $\mathbf{A}$  is obtained from the structural parameters of the intact structure,  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$ , and the reference frequency,  $\omega$ . Vector  $\mathbf{b}$  is the difference between the measured Fourier amplitudes in the damaged state,  $a(i, \omega)$ , and the calculated ones in the intact state from the intact structural parameters. Vector  $\mathbf{x}$  corresponds to change in damping  $\delta c_e$  and stiffness  $\delta k_e$ .

Accumulating equations as many as or larger than the number of unknowns by increasing measurement points and reference frequencies does not always give accurate solutions. The reasons for this are as follows;

- If there are measurement noises in the measured Fourier amplitudes in the damaged state, vector  $\mathbf{b}$  includes errors  $\delta \mathbf{b}$ , leading errors in the estimated value  $\mathbf{x}$ .
- If there are modeling errors of the intact state, matrix  $\mathbf{A}$  and vector  $\mathbf{b}$  both contain errors,  $\delta \mathbf{A}$  and  $\delta \mathbf{b}$ , resulting in errors in the estimated value  $\mathbf{x}$ .
- Inappropriate selection of measurement points and reference frequencies leads to errors in the estimates  $\mathbf{x}$  even though there are no measurement and modeling errors.

This study considers the combination of measurement points and reference frequencies with small expected estimation error is one of the appropriate selections. The expected error is evaluated based on the theoretical error of the damage identification equation.

### Theoretical Error of Determined Damage Identification Equation

Here, the theoretical error of the determined damage identification equation, the simultaneous equation which has the same number of equation as the number of solutions, is investigated [24]. Let error in matrix  $\mathbf{A}$  denoted by  $\varepsilon \mathbf{F}$ , and error in vector  $\mathbf{b}$  by  $\varepsilon \mathbf{f}$ . Estimate under the errors is denoted by  $\mathbf{x}(\varepsilon)$ . As already mentioned, the error in matrix  $\mathbf{A}$  corresponds to the modeling error, and the error in vector  $\mathbf{b}$  corresponds to both the modeling and measurement errors. Therefore, under the condition of modeling and measurement errors, damage identification equation solved is as follows.

$$(\mathbf{A} + \varepsilon \mathbf{F})\mathbf{x}(\varepsilon) = \mathbf{b} + \varepsilon \mathbf{f} \quad (24)$$

Let the estimate where there are no errors in both  $\mathbf{A}$  and  $\mathbf{b}$  ( $\varepsilon = 0$ ) be  $\mathbf{x}(0) = \mathbf{x}$ . By expanding Equation (24)

$$\dot{\mathbf{x}}(0) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{x}(\varepsilon) - \mathbf{x}(0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \mathbf{A}^{-1}(\mathbf{f} - \mathbf{F}\mathbf{x}(\varepsilon)) = \mathbf{A}^{-1}(\mathbf{f} - \mathbf{F}\mathbf{x}(0)) \quad (25)$$

Taylor expansion of the estimate  $\mathbf{x}(\varepsilon)$  is

$$\mathbf{x}(\varepsilon) = \mathbf{x}(0) + \dot{\mathbf{x}}(0)\varepsilon + \mathbf{O}(\varepsilon^2) = \mathbf{x}(0) + \mathbf{A}^{-1}(\mathbf{f} - \mathbf{F}\mathbf{x}(0))\varepsilon + \mathbf{O}(\varepsilon^2) \quad (26)$$

From this, the relative error in the estimate  $\mathbf{x}$  becomes

$$\frac{\|\mathbf{x}(\varepsilon) - \mathbf{x}\|}{\|\mathbf{x}\|} = \varepsilon \|\mathbf{A}^{-1}\| \left\{ \frac{\|\mathbf{f}\|}{\|\mathbf{x}\|} - \|\mathbf{F}\| \right\} + \mathbf{O}(\varepsilon^2) \quad (27)$$

where  $\|\cdot\|$  indicates the norm of matrix and vector, and  $|\cdot|$  indicates the absolute value of scalar. Using Cauchy-Schwarz inequality

$$\|\mathbf{b}\| = \|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \|\mathbf{x}\| \quad (28)$$

the relative error in the estimate becomes

$$\frac{\|\mathbf{x}(\varepsilon) - \mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \|\mathbf{x}\| \|\mathbf{A}^{-1}\| \left( |\varepsilon| \frac{\|\mathbf{f}\|}{\|\mathbf{x}\|} + |\varepsilon| \frac{\|\mathbf{F}\|}{\|\mathbf{A}\|} \right) = \kappa(\mathbf{A}) \left( |\varepsilon| \frac{\|\mathbf{f}\|}{\|\mathbf{x}\|} + |\varepsilon| \frac{\|\mathbf{F}\|}{\|\mathbf{A}\|} \right) \quad (29)$$

where  $\|\mathbf{A}\| \|\mathbf{A}^{-1}\|$  is called condition number of matrix  $\mathbf{A}$  and denoted by  $\kappa(\mathbf{A})$ .

$$\kappa(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\| \quad (30)$$

From this, the upper bound of the relative errors in estimate is controlled by the modeling and measurement errors,  $\mathbf{F}$  and  $\mathbf{f}$ , and the condition number,  $\kappa(\mathbf{A})$ . It is apparent that reducing the modeling and measurement errors leads to reducing the estimation error, but there is a limitation as mentioned earlier. To reduce the estimation error under the

circumstance of the modeling and measurement errors, the sensor placement and reference frequency giving the small condition number  $\kappa(\mathbf{A})$  should be selected. However, in the case where the measured Fourier amplitudes apparently contain large measurement errors even though a reference frequency with small condition number is used, another reference frequency with small condition number and reliable measurement data should be used. The condition number takes different values depending on the adopted norm. The condition number for Euclidean Norm,  $\kappa^2(\mathbf{A})$ , is

$$\kappa_2(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 = \frac{\sigma_1(\mathbf{A})}{\sigma_n(\mathbf{A})} \quad (31)$$

This type of condition number is a ratio of the largest singular value of  $\mathbf{A}$ ,  $\sigma_n(\mathbf{A})$ , to the smallest singular value,  $\sigma_1(\mathbf{A})$ . Singular values can be calculated by singular value decomposition of matrix  $\mathbf{A}$ .

### Theoretical Errors of Overdetermined Damage Identification Equation

Next, theoretical errors of the overdetermined equation, the simultaneous equation with more equations than unknowns, is investigated [24]. Least squares method is assumed to be used to solve this type of equation. The optimization problem to be solved is as follows;

$$\min \|\mathbf{Ax} - \mathbf{b}\|_2 \quad (32)$$

In the case where modeling and measurement errors exist, the optimization problem becomes

$$\min \|(\mathbf{A} + \varepsilon\mathbf{F})\mathbf{x}(\varepsilon) - (\mathbf{b} + \varepsilon\mathbf{f})\|_2 \quad (33)$$

Expanding Equation (33)

$$\begin{aligned} \dot{\mathbf{x}}(0) &= \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{x}(\varepsilon) - \mathbf{x}(0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} (\mathbf{A}^T \mathbf{A})^{-1} \{ \mathbf{A}^T (f - \mathbf{F}\mathbf{x}(\varepsilon)) + \mathbf{F}^T (\mathbf{b} - \mathbf{A}\mathbf{x}(\varepsilon)) \} \\ &= (\mathbf{A}^T \mathbf{A})^{-1} \{ \mathbf{A}^T (f - \mathbf{F}\mathbf{x}) + \mathbf{F}^T (\mathbf{b} - \mathbf{A}\mathbf{x}) \} \end{aligned} \quad (34)$$

Therefore, Taylor expansion of the estimate  $\mathbf{x}(\varepsilon)$  is as follows.

$$\mathbf{x}(\varepsilon) = \mathbf{x}(0) + \dot{\mathbf{x}}(0)\varepsilon + \mathbf{O}(\varepsilon^2) = \mathbf{x}(0) + (\mathbf{A}^T \mathbf{A})^{-1} \{ \mathbf{A}^T (f - \mathbf{F}\mathbf{x}) + \mathbf{F}^T (\mathbf{b} - \mathbf{A}\mathbf{x}) \} \varepsilon + \mathbf{O}(\varepsilon^2) \quad (35)$$

Assuming  $\|\mathbf{f}\|_2 < \|\mathbf{b}\|_2$  and  $\|\mathbf{F}\|_2 < \|\mathbf{A}\|_2$ , the relative error in the estimate  $\mathbf{x}$  becomes

$$\frac{\|\mathbf{x}(\varepsilon) - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq |\varepsilon| \left\{ \begin{aligned} &\|\mathbf{A}\|_2 \|(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T\|_2 \left( \frac{\|\mathbf{f}\|_2}{\|\mathbf{A}\|_2 \|\mathbf{x}\|_2} + \frac{\|\mathbf{F}\|_2}{\|\mathbf{A}\|_2} \right) \\ &+ \|\mathbf{A}\|_2^2 \|(\mathbf{A}^T \mathbf{A})^{-1}\|_2 \frac{\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2}{\|\mathbf{A}\|_2 \|\mathbf{x}\|_2} \frac{\|\mathbf{F}\|_2}{\|\mathbf{A}\|_2} \end{aligned} \right\} \quad (36)$$

Using the condition number for Euclidean Norm,  $\kappa_2(\mathbf{A})$ ,  $\|\mathbf{A}\|_2 / \|(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T\|_2$  and  $\|\mathbf{A}\|_2^2 / \|(\mathbf{A}^T \mathbf{A})^{-1}\|_2$  are transformed into

$$\kappa_2(\mathbf{A}) = \|\mathbf{A}\|_2 \|(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T\|_2 \quad \kappa_2^2(\mathbf{A}) = \|\mathbf{A}\|_2^2 \|(\mathbf{A}^T \mathbf{A})^{-1}\|_2 \quad (37)$$

Therefore, the upper bound of the relative error becomes

$$\frac{\|\mathbf{x}(\varepsilon) - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq |\varepsilon| \left\{ \kappa_2(\mathbf{A}) \left( \frac{\|\mathbf{f}\|_2}{\|\mathbf{A}\|_2 \|\mathbf{x}\|_2} + \frac{\|\mathbf{F}\|_2}{\|\mathbf{A}\|_2} \right) + \kappa_2^2(\mathbf{A}) \frac{\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2}{\|\mathbf{A}\|_2 \|\mathbf{x}\|_2} \frac{\|\mathbf{F}\|_2}{\|\mathbf{A}\|_2} \right\} \quad (38)$$

The relative error of the estimate when the number of data is larger than the number of unknowns is composed by two terms. The first term comes from the modeling and measurement errors,  $\mathbf{F}$  and  $\mathbf{f}$ , and the condition number of the coefficient matrix,  $\kappa_2(\mathbf{A})$ . This is a common term with the determined equation. The second term is composed by the modeling error,  $\mathbf{F}$ , least squares error,  $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2$ , and the square of the condition number,  $\kappa_2^2(\mathbf{A})$ . To reduce the first and the second terms under the modeling and measurement errors, the sensor placement and reference frequency giving the small condition number,  $\kappa_2(\mathbf{A})$ , should be selected. However, in the case where the measured Fourier amplitudes apparently contain large measurement errors, another reference frequency with small condition number and reliable measurement data should be used.

## How to Determine Sensor Placement and Reference Frequency

From the theoretical investigation of estimation error for both the determined and overdetermined equations, it is proposed to select measurement points and reference frequencies with the small condition number of the damage identification equation, avoiding the reference frequencies giving unreliable Fourier amplitudes. The underdetermined equation, where the number of equations is smaller than the number of unknowns, is not considered in this study since the number of equations is easily increased solely by increasing the number of reference frequencies even though the number of sensors is limited.

## Verification Using Experimental Data

The feasibility of the proposed damage identification technique and determination method of sensor placement and reference frequency is verified using experimental data done with a cantilever beam.

### Cantilever Beam Model

The cantilever beam is the 90cm long, 3cm wide and 0.32cm thick as shown in Figure 1. It is rigidly fixed to the shaking table using L-shaped metal fittings. Since the length of L-shaped fitting is 5cm, the beam length of the analytical model is set to 85cm. The analytical model is shown in Figure 2. It is divided into 17 elements, and the length of each element is 5cm. The numbering of the nodes and elements is also shown in Figure 2. The Young's modulus and mass density are  $2.15 \times 10^{11} \text{N/m}^2$  and  $7.1 \times 10^3 \text{kg/m}^3$ , respectively.

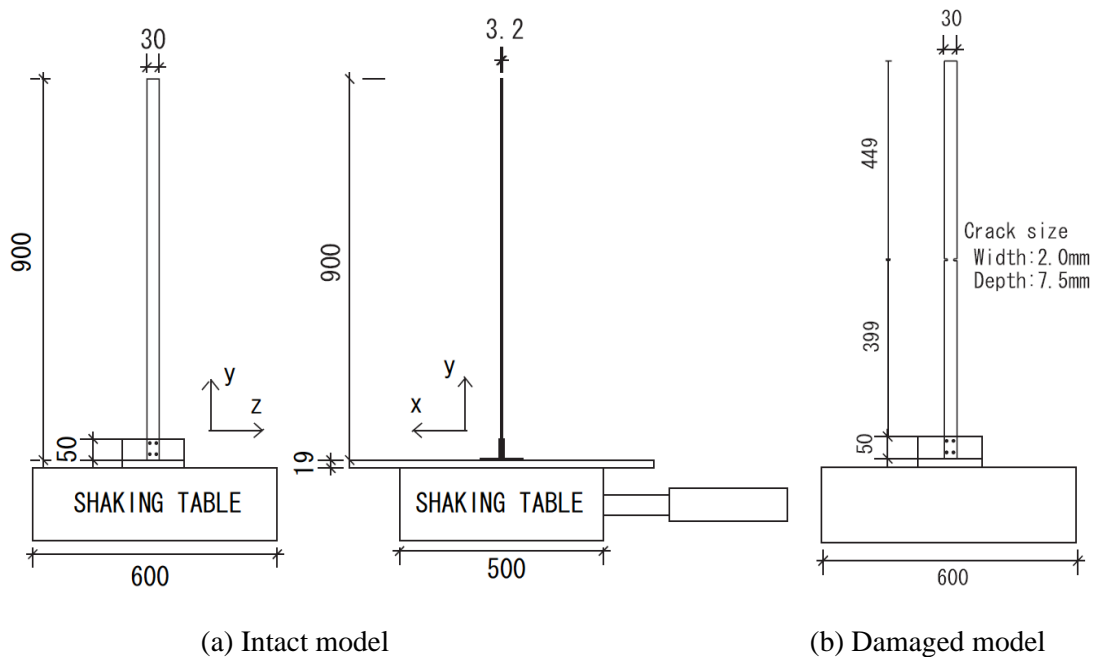


Figure 1. Intact and damaged models and experimental setup



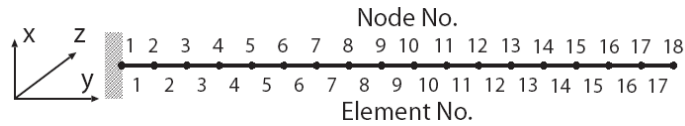


Figure 2. Analytical model of cantilever beam

### Damaged Model

The damage is created by cutting the mid-section of the element No. 9 as shown in Figure1 (b). The damage is made in the both sides of the cantilever beam.

### Input Ground Motion

The cantilever beam is shaken in one direction ( $x$ ) by the shaking table before and after damage. A time history of acceleration input to the cantilever beam is white and shown in Figure 3.

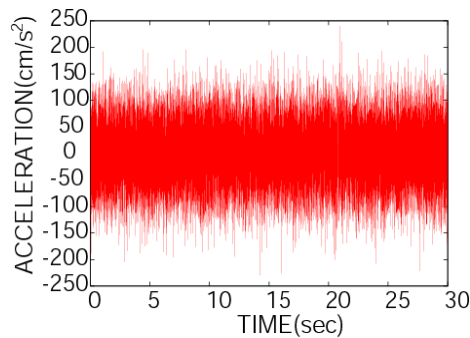
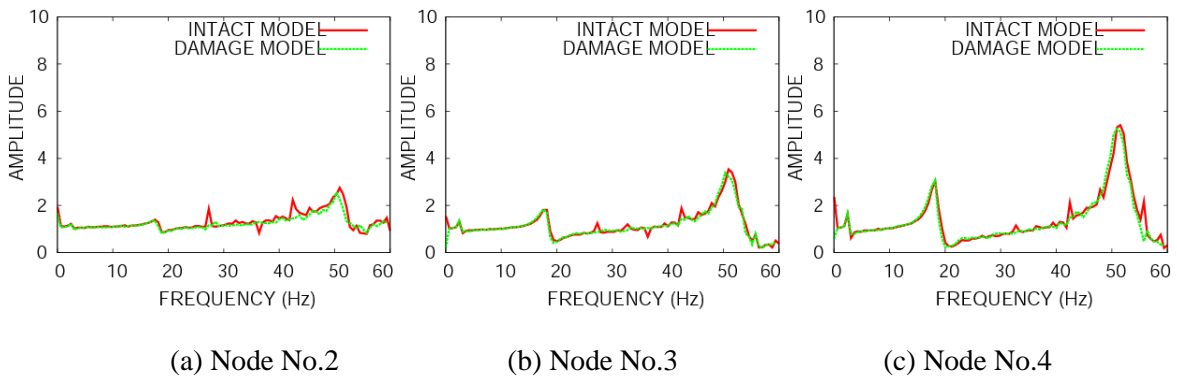


Figure 3. Input ground motion

### Measured FRFs and Natural Frequencies

Acceleration responses are measured at all free nodes Nos.2-18 in  $x$  direction and their FRFs are calculated by dividing their Fourier amplitudes by the Fourier amplitude of the input acceleration. The FRFs of intact and damaged models are shown in Figures 4 and 5. The first three natural frequencies are shown in Table 1.



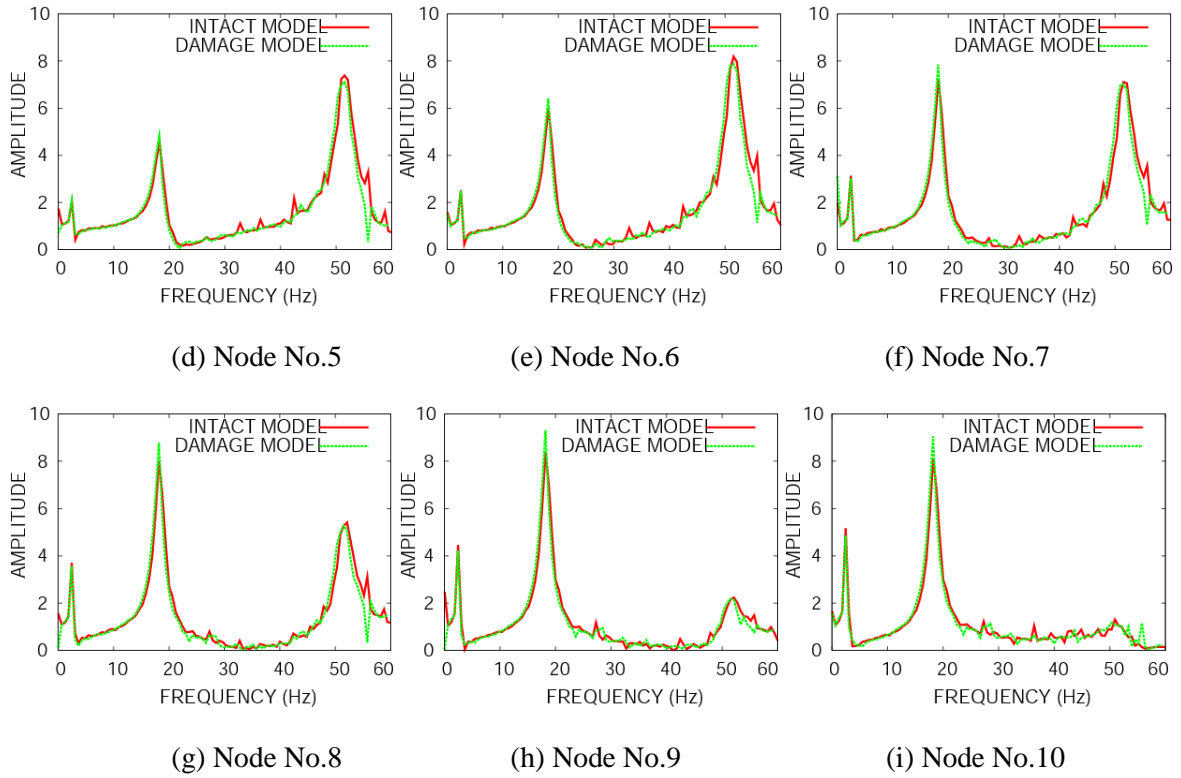
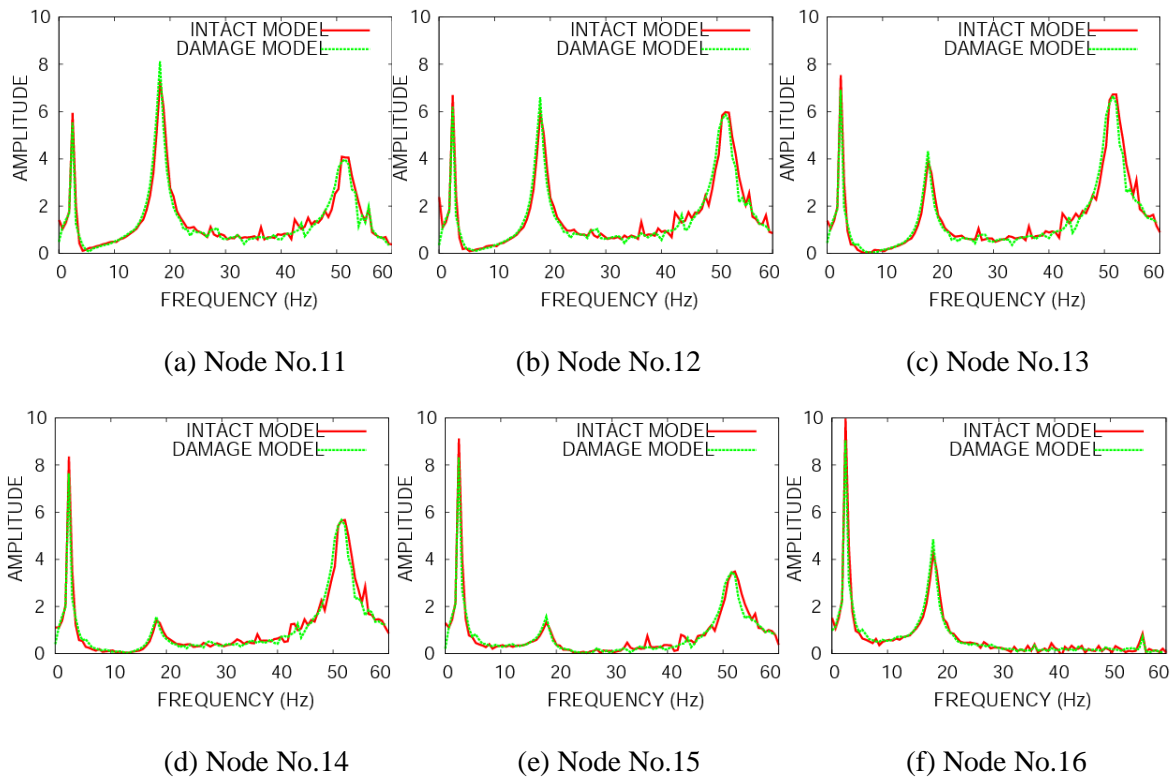


Figure 4. FRF of intact and damaged models (Nodes Nos.2-10)



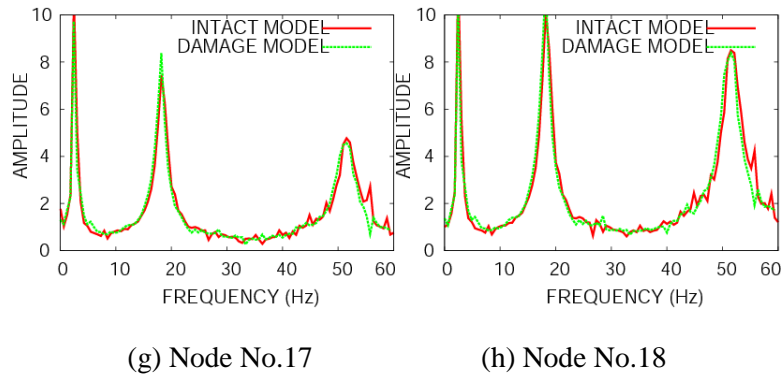


Figure 5. FRF of intact and damaged models (Nodes Nos.11-18)

**Table 1. Natural Frequencies of Cantilever Beam (Hz)**

Model	1 <sup>st</sup> Mode	2 <sup>nd</sup> Mode	3 <sup>rd</sup> Mode
Intact model	2.427	18.28	51.68
Damage model	2.427	18.18	51.51

### Damage Identification

#### *Selection of Reference Frequency*

The finite element model of the intact state must be defined to identify damage. It is first created from a design values, and then their parameters are updated to match the Fourier amplitudes and natural frequencies of the intact model. The structure is modeled as an un-damped system. Using the update finite element model of the intact state, the condition numbers for various reference frequencies from 0Hz -60Hz are calculated assuming that acceleration responses are measured at all free nodes and only one reference frequency is used as shown in Table 2. The calculated condition number with regard to reference frequencies are shown in Figure 6.

**Table 2. Analytical Conditions for Cantilever Beam**

Direction of Ground Shaking	Measurement Node	Direction of Measurement	Number of Reference Frequencies	Number of Data
$x$	2-18	$x$	1	12

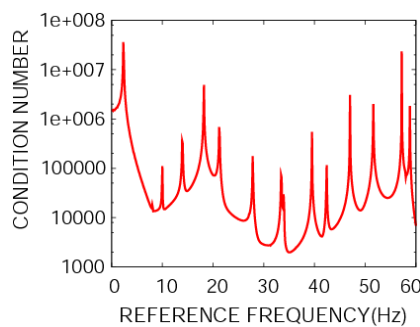


Figure 6. Condition number for cantilever beam

### Combination of Sensor Placement and Reference Frequency

Five combinations of sensor placement and reference frequency are considered as shown in Table 3. From Figure 6, the reference frequencies with the small condition numbers are known. However, some FRFs at the reference frequencies for the small condition numbers, such as 30Hz and 35Hz, seem to be unreliable from Figures 4 and 5. Therefore, the reference frequencies with relatively small condition numbers and the reliable FRFs are chosen. The resonant frequencies have the peak condition numbers as mentioned earlier. But frequencies slightly smaller than the resonant frequencies do not always have peak condition numbers and the change in their Fourier amplitudes before and after damage is large enough not to be overshadowed by the measurement noises because they are sensitive to damage. Such frequencies are chosen, and they are 16.6Hz for case C2 and 50Hz for case C3. For comparison, the reference frequency with the large condition number is also chosen, 1Hz for case C1. Case C4 combines the reference frequencies of cases C2 and C3 to reduce the number of measurement nodes. Case C5 combines the data of cases C2 and C3. The condition numbers for these cases are shown in Table 3. It is expected that the accuracy of cases C5 is the highest, followed by case C3, case C2 and case C4. The accuracy of case C1 is expected to be bad due to its large condition number.

**Table 3. Analytical Conditions for Cases C1-C5**

Case	Direction of Ground Shaking and Measurement	Measurement Node	Reference Frequency (Hz)	Number of Data	Condition Number
C1	x	2-18	1.0	17	1.661E6
C2	x	2-18	16.6	17	6.291E4
C3	x	2-18	50.0	17	1.872E4
C4	x	2,3,6,8,10,12,14,16,18	16.6	18	7.651E4
	x	2,3,6,8,10,12,14,16,18	50.0		
C5	x	2-18	16.6	34	1.384E4
	x	2-18	50.0		

### Damage Identification Results

The identification results are shown in Figure 7. In all cases, the damaged element No. 9 is identified as damaged, but many undamaged elements are identified with positive or negative stiffness reductions. To cope with this matter, the damage identification equations are solved with a precondition that the stiffness is only reduced when being damaged and does not change when not being damaged. Assuming this precondition, the following constrained optimization problem is solved to estimate the stiffness change  $\mathbf{x}$ .

$$\min \|\mathbf{Ax} - \mathbf{b}\|_2 \quad \text{sub to } x \leq 0.0 \quad (39)$$

The identification results are shown in Figure 8. Case C5 with the smallest condition number showed the highest accuracy with only one damaged element detected as damaged. Case C3 with the second smallest condition number identified damage in the same high accuracy with case C5. Case C2 with the third smallest condition number identified three elements as damaged including damaged element No. 9 and two undamaged elements Nos. 2 and 3. However, the identified stiffness reduction of element No. 9 is much larger than those of the other two. Case C4 also identified three elements as damaged including damaged element No. 9. Element No. 10 next to element No. 9 is also identified as damaged and the reason for this is that every two nodes are not measured. The accuracy of case C1

is very bad as expected from its large condition number.

The validity of the damage identification technique and the determination method of the combination of sensor placement and reference frequency is verified through the experiment. The validity of the technique is also verified through the numerical simulations using the same cantilever beam having the same damage with the experiment, where modeling and measurement errors are modeled explicitly as follows. In this simulation, 3% modeling error of uniform random numbers is added to the Young's modulus of each element in the intact state, and 3% measurement error of Gaussian distribution is added to the time history of the acceleration in the damaged state. It is confirmed that the identified results have the same tendency with the experimental verification.

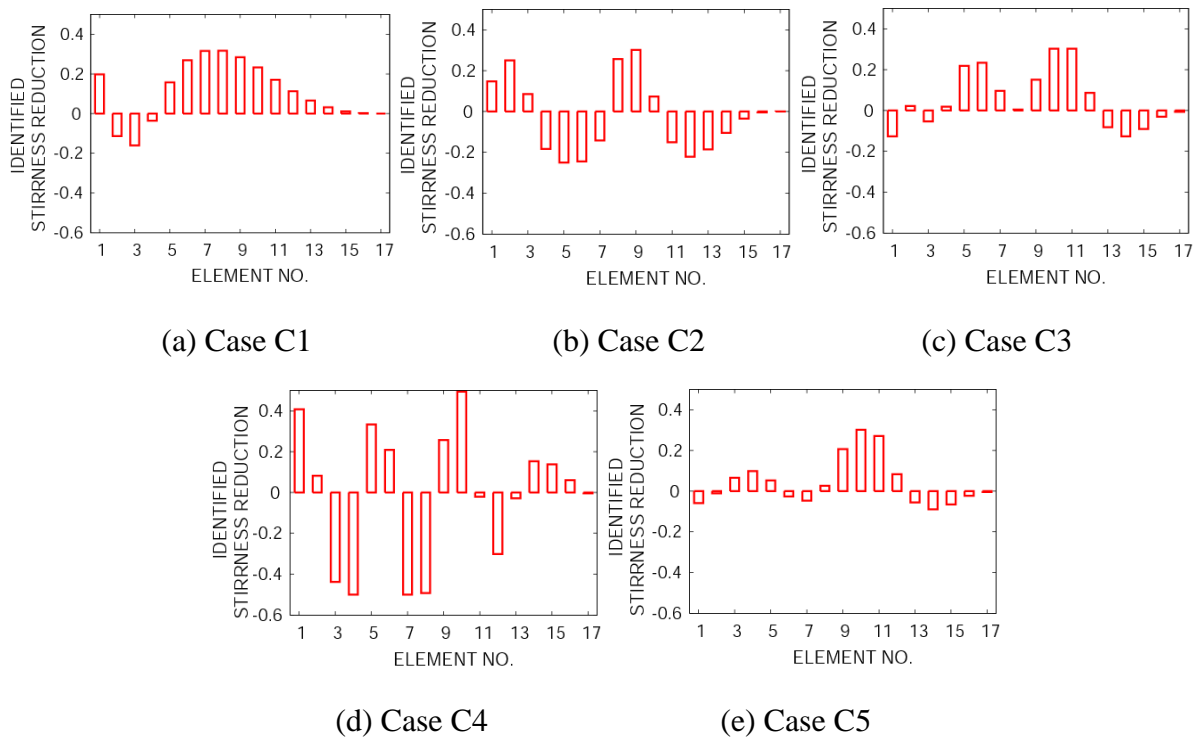
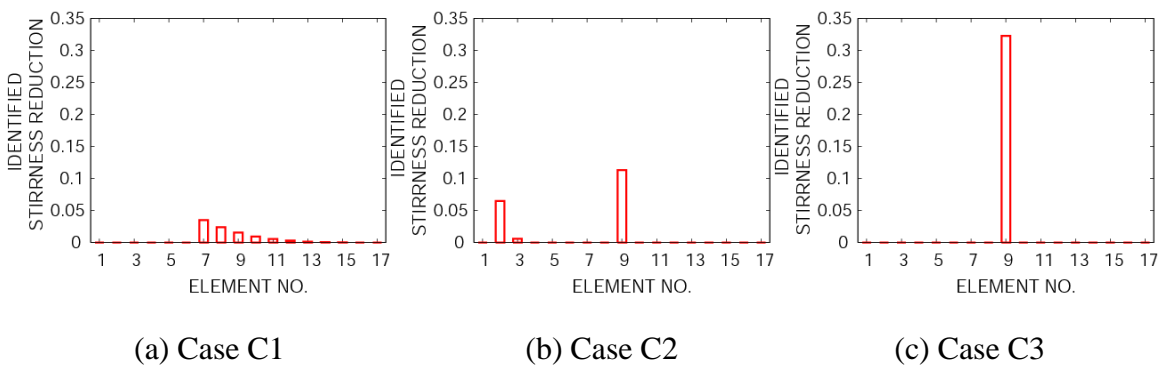


Figure 7. Identification results by solving unconstrained optimization problem



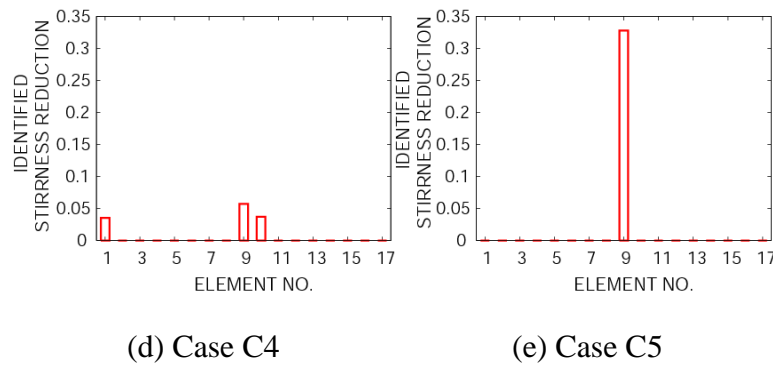


Figure 8. Identification results by solving constrained optimization problem

## Conclusions

A damage identification technique is proposed that is derived from the fact that changes in vibration responses provide information about the location of damage and its severity because damage to structures causes changes in their structural parameters. Damage identification equation is derived from the equations of motion of a structure before and after damage. This identification technique has the advantage that data accumulated simply by changing the measurement point or reference frequency. A determination method of the sensor placement and reference frequency based on the theoretical error of the damage identification technique is also proposed.

The methods are verified by the experimental results done with the cantilever beam. If the Fourier amplitudes at the reference frequency with small condition number is unreliable due to measurement noise, another reference frequency with relatively small condition number and reliable Fourier amplitudes should be used. If the reference frequency is somewhat lower than the natural frequency, the vibration amplitudes are large compared to contaminated noise. Therefore, if such a frequency has relatively small condition number, it could be one of the appropriate reference frequencies.

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