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UNDERGRADUATE MATHEMATICS EDUCATION: TEACHING MATHEMATICAL THINGKING OR PRODUCT OF MATHEMATICAL THOUGHT ?

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Abstract. This paper reports on an investigation into students' thinking about mathematics and their mathematical behaviour when faced with a problem. It is found that students perceived mathematics as a fixed body of knowledge to be learned. When solving a problem, students demonstrate little intellectual independence and lack the ability to think for themselves. This is a matter of some concern. The findings indicate that the mathematical environment may not be providing students with the experiences to encourage them to be creative and reflective. It is suggested that mathematicians need to move away from teaching students the product of mathematical thought to teaching them mathematical thinking.

1 INTRODUCTION

Over the years little has changed in the learning and teaching of advanced mathematics at UTM. The traditional sequence of presenting mathematics from definition, theorem, proof and illustrations continues to be the dominant method. Teaching students the product of mathematical thought can no longer meet the demands of the changing modern world. Studies have shown that the traditional approach has not only failed with the majority of the students, but more disturbingly also successful students. Students find great difficulties in constructing their own understanding of the mathematical concepts ([4], [13], [31], [8], [1], [38], [28]) and have a narrow view of the mathematics that consequently shape their mathematical behaviour ([26], [37]). It is likely that such difficulties would also be observed among Malaysian students, in particular at UTM.

At the rate things are changing in the society, we are uncertain that the mathematics we are teaching now will be valid in 10 or 15 years' time. We are also uncertain on how the mathematics learned will be used by the students in their future. The advancement in technology — the creation of calculating devices such as calculators and computers — requires us to know more than just how to use procedures or to obey rules. The more pressing need in mathematics education is to instill among students the ability to adapt the mathematical performance to varied circumstances and to ensure that students can think for themselves. With the current development in the country's policy, there are already calls from certain quarters to completely "redesign" the education system in Malaysia as the present one is unable to support the country's vision of becoming a developed nation by the year 2020. Institutions of higher learning are being urged to upgrade their education pedagogy to meet the changing needs of society ([6]). A distinguished figure in Malaysia's education, Royal Professor Ungku Abdul Aziz suggested that what Malaysia needs is a large

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number of educated people who can, amongst other things, "think" ([11]). Certainly, for members of the mathematics department such calls should not be left unheeded; the right time seems to have presented itself.

Mathematics courses at UTM have become so routine that they have become inclined more towards teaching students procedures that can be easily tested rather than developing flexible new skills. It is believed that we learn mathematics so that we can appreciate fully the nature of the subject matter, to apply it in everyday life and to help in understanding the world that we live in. The mathematics education goals at present only seem to provide students with a sense of the subject matter. The emphasis is more on the content rather than on the processes. For Malaysia to attain her high goals, in addition to teaching students to get a sense of the subject matter, mathematics teaching should also include preparing students to become effective thinkers and independent problem solvers. Mathematics learning must then go beyond learning accumulated mathematical thought. Certainly we need to know the mathematical facts and standard procedures in order to be able to do mathematics. But knowing them without its functional meaning is totally limiting. Hiebert & Lefevre ([10]) succinctly state that:

> Students are not fully competent in mathematics if either kind of knowledge [conceptual and procedural knowledge] is deficient or if they both have been acquired but remain separate entities. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing.
>
> p. 9

Research findings indicate that thinking mathematically or problem-solving can be taught with some success. For instance, Mason & Davis [14] explored how people can develop their mathematical thinking, learning, and teaching by reflecting on their own experience. They argued that the technique of using meaningful vocabulary can help students to become more reflective and effective in mathematical learning. It was observed that students not only notice the use of the vocabulary and advice from tutors, but also remember it when the same language pattern (e.g. specialising, generalising, a slang "What do I want?" etc.) was repeatedly used and their attention was explicitly drawn to it. Whilst a study by Rogers ([23], [24]) reports on an American institutions success in creating a learning environment that develops students to their potential in the learning of advanced and abstract concepts in mathematics. Their initial focus was on changing the students' perception of mathematics as a difficult and an almost impossible subject to one that they are capable of doing. She observed that encouraging students behaviours such as high self-esteem, confidence in their mathematical abilities, and the ability to work independently are closely linked to the faculty's approach to teaching mathematics. The emphasis is on the negotiation of mathematical meaning and students growth and development rather than the transmission of knowledge and skills. Their teaching practices show that students who learn to think mathematically are able to reconstruct ideas and learn independently. Some of the techniques used include active student participation, group work in class and outside of class, and constructivist approaches to developing the subject matter. Further, Rogers suggests that the attitude towards teaching the students to think mathematically requires a "caring teacher" in the sense of helping the students grow and actualise themself.

This gives us an indication that students can alter their methods of doing mathematics when they are aware and conscious of the meta-processes in thinking mathematically. Whilst the importance of an answer declined, working on the process was emphasised. Consequently students might be motivated to persevere, which could result in more positive attitude and perhaps reduce the fragility of students mathematical knowledge ([25], [1], [21]).

In this study I looked at students' thinking about the mathematics they are doing and their problem-solving performance in an attempt to bring to light some factors that were responsible for their mathematical behaviour. The data presented were collected during an experiment designed to encourage mathematical thinking (based around the book Thinking Mathematically by Mason, Burton & Stacey, [16]). Further data after a delayed period of six months were also shown. The findings indicate that majority of the students had reached a position where they perceived mathematics as a fixed body of knowledge to be remembered. In the words of Skemp ([32]), university mathematics is seen as "the product of mathematical thought rather than the process of mathematical thinking". Futhermore, they are not developing problem-solving skills in formal undergraduate mathematics courses. This should be a matter of some concern.

2 THE CONTEXT

The students taking part in the research were a mixture of third, fourth and fifth year undergraduates aged 18 to 21 in SSI (Industrial Science, majoring in Mathematics) and SPK (Computer Education) at UTM. They were chosen to follow an experimental course in mathematical problem-solving given by the author of this paper. They were taking mathematics as a core subject; students need to pass to obtain their respective degree. These students had followed a relatively advanced level of mathematics courses at the time of this study. Thus, one may assume that they have achieved considerable experience in university mathematics. More importantly, they have the mathematical knowledge to solve all the problems given. The data on the students' performance and perceptions were collected by direct observation in the classroom. Students' written comments and interviews with selected students supplemented the data from classroom observation.

3 THE RESULTS

3.1 Classroom Observation

Initially, the students were very confused with the rationale set for the problem-solving course. A course which emphasised students' involvement in mathematics contrasted starkly with their established perceptions which generally involved passive acceptance of course material during lectures. During standard mathematics lecturing the students are likely to have very limited opportunities to make their own mathematical decisions and this, coupled with the underlying formality of the culture, contrived to make them feel lost and uncomfortable. When faced with a problem, students began attacking it quickly and mechanically. They did not pause to think whether whatever they are doing were appropriate. They want to escape from the task as quickly as possible. They kept asking questions such as "What shall I do now?", "Is this the right way of doing it?", "What is the answer?", when they became stuck after their frantic attack on the given problem. Such questioning served to reinforce the view that personal decision making on what to do next and the development of strategies for solving problems were not part of their usual mathematical behaviour. It was clear that their mathematical thinking is influenced greatly by their beliefs about mathematics and problem solving. As one student explained when asked about his attempt to solve the Warehouse problem (see [16], p. 1):

> I tried several times on my own using trial and error. I got stuck. I asked my friend how he did it and gathered some information. I got stuck again. I copied his solution and tried to understand it. Then I tried to solve the problem again on my own.
>
> SPK, year 5

It is suggested that such behaviour depicts this student's usual way of solving mathematical problems in regular mathematics courses – imitating what he is taught rather than figuring out solutions for themselves. And he is not alone. One may suggest (from experience) that for many of the students, it is a common practice to find solutions by looking them up at the end of mathematics texts or by getting them from their friends or lecturers who have solved the problems. Having to solve an unfamiliar mathematical problem appears too threatening to the majority of the students.

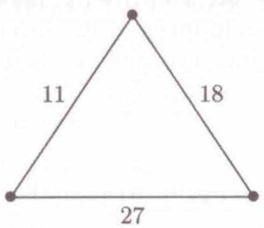
It is observed that a great proportion of the students frequently had ideas that may have been useful in getting a possible solution but they simply did not know how to use them. Additionally, it was noticed that they felt reluctant to put forward any ideas that they were not certain about. They did not like thinking aloud or sharing their ideas with others because they feared that their suggestion were wrong or may get rejected. The fear of failure removed them from the habit of answering questions in front of others or putting forward any suggestions. Furthermore, they did not know what a conjecture is and had no notion how to make one:

> Before the idea of conjecturing was introduced I have no idea what it really is. It's difficult to use conjecturing without being taught about how to make conjectures. ...I was not inclined to making conjectures due to the ingrained attitude of mathematics being 'right' or 'wrong'.
>
> SSI, year 3

The students were given a couple of problems to solve as a group. Although students were requested to work co-operatively in their groups, there was hardly any discussion between the individuals. Each of them appeared to be more concerned with getting the work done on their own. Group contributions were only made after they had first tried to solve the problem alone. The students showed enormous resistance to co-operative activity.

The following problem were given to the students as part of a written assessment. They have taken a linear algebra course the semester before and all have passed the exam. Thus they have the required mathematical knowledge to solve the problem:

> A secret number is assigned to each vertex of a triangle. On each side of the triangle is written the sum of the secret numbers at its ends. Find a simple rule for revealing the secret numbers. For example, secret numbers 1, 10, 17 produce



> Generalise to other polygons.
>
> [16], p. 160.

Students have the necessary algebra to write down the three equations involving secret numbers x, y, z at the vertices and to show how these lead to a unique solution. However, when generalising to four secret numbers at the corners of a square, it happens that the equations are either inconsistent (usually) or have an infinite number of solutions. This did not evoke the knowledge of solving linear equations to the majority of the students. Thus before they move to find that the pentagon once more has a unique solution they were already confused. It led them to make a conjecture that odd polygons have a unique solution. But when it come to even polygons, it constitutes a problem.

Although half of the students managed to formulate a correct solution for the triangle, only 16% of the students related the problem to the content of their linear algebra course. The majority simply solve the problem mechanically and could not construct meaningful explanations. It may be suggested that the majority of the students have not understood enough of the subject matter to see its relevance; it is conjectured that their understanding is instrumental (in the sense of Skemp, [33]). Consequently, when given a problem formulated in a context which is slightly different, they failed to make the link.

After ten weeks of problem-solving, students gradually displayed some positive reactions; they began to think for themselves, to share their thoughts with others in the group and to write a rubric commentary outlining their problem-solving activity. Students learnt to cope with their emotions and their obsession with getting correct answers. Their written entries in the weekly assignment became more explicit and self-analytical. They presented many desirable qualities such as a willingness to struggle with the task and the desire to reflect on their own problem-solving:

> Not knowing how to proceed, we decided to try and get a better grip on the question. ...we had isolated our weakness and acted to overcome it.
>
> We felt very negative when we couldn't find a solution as easily as before. We mulled over the problem for a while at this point, until we suddenly realised it was our own preconceptions holding us back!
>
> We needed to find 'why'. At this point we are stuck! We try to gather the information so far in order to find a path forward.

When the students came to realise that they had to figure out the answer themselves and were responsible for their own progress, they stopped asking for the right answer. They began to explore their own mathematical knowledge, to select and use it to formulate a method of solution of their own:

> ...I got stuck when I thought there had to be a unique solution. Then I started to think about using a parameter and obtaining a family of solutions which is the same idea for solving simultaneous equations. The problem solving techniques encourage me to apply ideas to areas which I may not normally consider them appropriate.
>
> SPK, year 4

> The emphasis has always been to get the correct answer. This puts a great deal of pressure on the students. The [problem solving] course focuses more on how you get the answer. This allows me to re-assess my capacity. I had confidence in my ability to try out things. ...I was able to feel in control of the problem, get involved in it, enjoy extending it and come to a resolution that I was satisfied with.
>
> SPK, year 5

Although the changes were slow to come, the majority of the students gradually learned to generate mathematical ideas, to talk about them and be critical of suggestions given by their friends. Their discussion became livelier as they moved towards doing things that they could explain to their friends, rather than simply satisfying the course requirements or pleasing the tutor. Their problem-solving became "a more creative activity, which includes the formulation of a likely conjecture, a sequence of activities testing, modifying and refining..." ([36], p. 18). In the final assessment, more than half of the students managed to come up with their own original, though mathematically acceptable, way of solving a problem. This indicates that it is possible to get the students to create their own solution methods, even though this took longer than the previous experience using procedural methods in routine problems.

The following selected phrases indicate a major shift in the students' thinking about problem-solving as a result of the course. In particular, the change was from an attitude which focused on getting the right answer from the teacher or friends, or wanting to escape from the task as quickly as possible, to one which reflected an ability to figure out the solution and defend the results. Students came to realise that there were different approaches to a problem. More importantly, they began to see that a solution depended on the decision to use a method which was more appropriate to the circumstances rather than on doing the right calculation. Consequently, they appeared to gain confidence in solving problems independently and were capable of thinking up their own strategies; they came to appreciate the problem-solving experience that they had had:

> Problem solving gives both meaning and value to the study of maths. It encourages us to apply what we know and to plan an approach to solving a problem. ...The discussion helps to sharpen our understanding of mathematical concepts and gives us the chance to negotiate its meaning in our own terms. It is also gives us the opportunity for the presentation of alternatives and makes us realise the fact that there is more than one way to solve most problems. ...We each learn to take some responsibility for what occurs. I feel this is not always encouraged by the maths course at the university.
>
> SSI, year 3

> Solving the problem requires a great deal of time and thought. I have never actively thought about the processes that my mind undergoes while attempting to solve a problem. Usually I am eager to start a question. I attacked it having not really gained all the information. Frequently I will come to a point where I can continue no further. By this time I no longer have enough motivation to continue. So I just abandon it. ...After a long history of failure I surprised myself that I managed to solve this problem.
>
> SPK, year 4

> Before I took this course I could probably have solved the problem, but it would have taken me longer, and I would not have had such a coherent solution at the end.
>
> SSI, year 3

Students' comments illustrate what they can do when they are given the opportunity to think in a mathematical manner. Opinions expressed suggest that the majority of the students are capable of benefiting from the course in many ways. This was particularly noticeable in the way they restructured their views about mathematics and gained the confidence to make mathematical decisions independently. There are also indications of an improvement in their mathematical thinking. For instance, in seeing there is more than one way to solve a problem and in having a willingness to try out new ideas without giving up too easily. Students became more positive about themselves in learning the mathematics and in their mathematical ability.

During the course the students had been encouraged to view their activities in three phases – entry, attack and review, with appropriate activities for each. Did students use this structure as a framework for meaningful problem-solving? To investigate the manner in which students attack a problem, six groups (comprising of 3 or 4 students who had worked together, see Table 1) were selected and given a problem which was relatively easy to state but did not have a straightforward algorithmic solution. The groups of students picked were representative of the subject areas and of the academic achievement at the end of previous semester.

Table 1. The 6 groups of students selected for interview

	Students	Course	Degree Classification	Gender
group 1	Sam	5 SPK	II–1	M
	Abel	4 SPK	II–2	M
	Henry	4 SPK	II–1	M
group 2	Sue	4 SPK	I	F
	Teresa	4 SPK	II–1	F
	Sasha	5 SPK	II–1	F
group 3	Rob	3 SSI	II–1	M
	Kline	3 SSI	II–1	M
	Ian	3 SSI	I	M
group 4	Hanna	5 SPK	II–1	F
	Katy	5 SPK	I	F
	Terry	5 SPK	I	M
group 5	Bob	5 SPK	II–2	M
	Yvonne	5 SPK	II–1	F
	Alma	4 SPK	II–1	F
	Pauline	5 SPK	II–2	F
group 6	Matt	5 SPK	II–1	M
	Al	4 SPK	II–2	M
	Holmes	5 SPK	III	M
	Ricky	5 SPK	II–2	M

The interview data provided some evidence of qualitatively different thinking between the various groups (see [20]). The students were given a problem to work on, as follows:

> A man lost on the Nullarbor Plain in Australia hears a train whistle due west of him. He cannot see the train but he knows that it runs on a very long, very straight track. His only chance to avoid perishing from thirst is to reach the track before the train has passed. Assuming that he and the train both travel at constant speeds, in which direction should he walk?
>
> [16], p. 183.

During the problem-solving session, it could be seen that three of the six groups (the lower attaining groups 5 and 6 and the younger group 3) followed the techniques taught in the problem-solving course very rigidly. Of these three, the two groups 5 and 6 seemed to be doing it more religiously than group 3. They were more concerned to cover each phase in a sequence and so they could be seen to be working procedurally throughout. They interpreted the problem-solving technique as a procedure that they have to follow step by step, it was as if they believed that precision in following each phase would guarantee them a solution. Most of their time was spent looking for formulas that could be used. In none of these beginnings of solutions have the students thought in a broader conceptual fashion, for instance to consider the direction of the train, or to draw a diagram. In contrast, the other groups (1, 2, and 4) were more involved in considering plausible ways to solve the problem by creating their own solution method.

It can be seen that four of the six groups (groups 1, 2, 3 and 4) gave some evidence that they are able to carry out the mathematical processes to some extent. They show that they are capable of making judgements on the content and in making mathematical decisions for themselves. They also question the meaning of the task.

> The problem is very challenging. It does not require a specific formula or procedure that you have to apply to solve it. It is quite difficult. We got an idea what the answer is but to proof it is the hardest part.
>
> group 1

> The problems in the problem-solving course are interesting. Like this one. We have to think, work out what we want, what we do know before we actually work out what we don't know. ...The course is beneficial. It makes us sit down and see where to start.
>
> group 3

> We only managed to understand the question better towards the end of the discussion time. But I think we can solve the problem if we have more time. It is not difficult, but to generalise and to prove is very difficult. ...We will keep on thinking about it until we get the answer.
>
> group 2

> At the moment we are still not satisfied with the solution. Not until we can show that it is correct. The problem looks difficult, but once involved we think we can do it. ... We found the course very helpful. Usually when faced with mathematical problems we just ploughed straight into it. But now we tried, for instance when solving this problem to structure our attempts properly.
>
> group 4

However, the other two groups (5 and 6) have the notion that mathematical problems consist of direct application of facts and procedures. Their lower attainments in their examinations suggests they have less secure knowledge to bring to the solution process. Thus they are in an interesting position where they have built up their confidence to tackle problems and yet they find the problems very difficult.

> We found it [the problem] very difficult. We are unsure of which formulas or methods to use. Even if we got a solution, we don't know whether our solution is right. ...Unlike problems in the problem-solving course, most of the problems in the maths course are simply applications of a ready rule. There is always a definite answer at the end.
>
> group 6

> We tried to generate a few possible ideas. But we felt a bit put off because we couldn't recall the formulas. ...The problems are totally different from those in the maths course. In maths we always know what method to use. Here we have to find it out ourselves. ...I think we have more confidence now. Before the [problem-solving] course we probably would have given up very easily.
>
> group 5

Although none of the groups could provide a complete solution to the problem within the time limit, they were at least able to tackle the problem to start with. All the student groups were very willing to tackle the problem without any overt sign of anxiety. It appears that the meta-support students received during the problem-solving course was sufficient to give them a sense of well-being. Even though the problem remained unfinished, groups 1, 2, 3, and 4 considered that they could solve the problem given more time, although based on their responses this may involve a lot more effort than they may have thought. Meanwhile, the other two groups were seeking formulae appropriate for a solution and using the overall strategy of problem-solving as a procedure to attack the problem. Their response to problem-solving shows the same procedural format as their approach to traditional mathematics problems.

In [19], we reported that problem-solving activities has the effect of changing students' attitudes. Before the course the students generally regarded mathematics as abstract facts and procedures to be committed to memory, and had a range of negative attitudes such as fear of new problems, being unwilling to try new approaches, and giving up all too easily. After the course, students' attitudes changed in what was considered a positive manner. However, six months after returning to regular mathematics courses students revert to their previous positions. This indicates that the positive changes developed during problem-solving is certainly one which the standard mathematics courses do not achieve. Therefore, one may conjecture that the mathematics teaching does not give support to the students' growth of mathematical thinking, in the way it is encouraged in the problem-solving course.

4 STUDENTS COMMENTS

Students were requested to give some comments on the mathematics that they are doing as they currently see it. Opinions expressed bring to light some factors that were responsible for their attitudes towards mathematics.

4.1 Pre-test comments

It is apparent that their feelings about the subject matter runs high for many of them and they display various conceptions about mathematics. Students perceived mathematics as a static and abstract discipline. To them mathematics is a very difficult subject and they are suffering cognitive strain in trying to cope with it:

> Mathematics is too abstract. It is very difficult to understand especially the mathematics that we learn at the university. The practice is OK but the theory is more difficult than practice.
>
> SPK, year 4

> Mathematics is full of definitions and proofs that are very abstract and complex. I find it very boring and thus did not feel like working hard enough to understand my maths courses.
>
> SPK, year 4

> I quite enjoy maths, especially when I can understand it. However, at the pace it is taught, it is often very confusing and difficult to follow. ...I am feeling stressed!!
>
> SSI, year 3

> I find mathematics at the university extremely difficult partly because it is too abstract. Furthermore, the fact that it is delivered in a dull atmosphere makes it very boring. ..I could never encompass the whole lot.
>
> SPK, year 5

> I did not find any of the maths lectures exciting. The atmosphere is not conducive to questioning. ...Maths is becoming harder and harder.
>
> SPK, year 4

To some students, mathematics means nothing to them. They are not interested with what the human mind has achieved over the years. It seems that they are finding the subject too difficult to comprehend. Perhaps, because they fail to grasp the rationale of the mathematics taught, they do not sense any particular loss from not knowing it. Therefore, they do it mainly to pass the exams:

> I am basically studying maths to get a degree. The way maths is taught here, it seems as though it is difficult and boring. There is no opportunity to display one's creativity. This makes it real dull and frustrating.
>
> SPK, year 5

> The mathematics that I learn at the university is so alien because it is too abstract and everything seems pointless to me. Often the maths course I have taken has been both unintelligible and torturous.
>
> SPK, year 5

Whilst to some students university mathematics is so formal that they simply resort to rote-learning the materials to reproduce in examinations. Many of these students appear successful.

> ...I spent a lot of time remembering the formulas and algorithms. The abstract nature of mathematics is just above me!
>
> SPK, year 4
>
> The emphasis in the exam is to get the correct answer. And it is possible to gain a good grade by rote learning the syllabus and solutions to the examples in tutorial sheets.
>
> SPK, year 5

However, there is a minority who are attracted by the intellectual challenge and do feel a sense of satisfaction generated by their mathematics.

> I enjoy the challenge that maths gives and have great pleasure when I get correct answers. ...I feel a great sense of satisfaction when I am able to understand new concepts and solve problems.
>
> SSI, year 3
>
> I enjoy maths most of the time. After many attempts, the satisfaction of a correct answer is very rewarding. I feel all the effort put in is worthwhile.
>
> SPK, year 5

4.2 Post-test comments

Responses following the course indicate that the students' view changes dramatically. Problem-solving helps them to say that mathematics is not simply a body of procedures to be learned by memorising, it is also a process of thinking.

> I find many aspects of mathematics challenging. I think it trains the brain to think in a logical and structured way. This is the first time that I have actually used maths to think. Before I just learnt maths to pass the exam.
>
> SPK, year 4
>
> I am beginning to think instead of just doing the tutorial questions. Mathematics is not merely computation as I had believed. A lot of effort is required before a solution method becomes apparent in solving a problem. I think I am learning more because I understand what is going on.
>
> SSI, year 3
>
> I spent a lot of time remembering the formulas and algorithms. ...Now that I know about mathematical thinking, my interest and desire to learn maths have increased.
>
> SPK, year 4

4.3 Comments after six months of standard mathematics

For longer term effect, students' newly built confidence must be further encouraged and reinforced. It is believed that what students had experienced in the problem-solving course is not what they normally encounter at the university mathematics courses. For many it was an isolated experience.

Since following the course I know mathematics is about solving problems. But whatever mathematics I am doing now doesn't allow me to do all those things. There are just more things to be remembered.

SPK, year 5

I find the mathematics I am doing now confusing and sometimes pointless. I wonder what the point of some courses is and why you are doing it.

SPK, year 4

At the moment I am finding difficulty with maths because I am just not enjoying it. Too much emphasis is put on getting the right answer and not on method and understanding.

SPK, year 4

...The content is emphasised over everything else. We are crammed with lots of bland mathematical abstract theory.

SSI, year 3

I realise that knowledge is the cornerstone to learning, but it's by thinking and reflecting upon the experience that we build on the knowledge and learn. I think this aspect is not always encouraged by the maths course.

SPK, year 4

The opinions expressed suggest that the difficulty faced by mathematics students gives them little room for creative thinking. The emphasis is on the procedural aspects which can be successfully tested. This give an indication that the environment provides very little encouragement for students to think mathematically nor is it supportive of reinforcing their positive attitudes built during the problem-solving course. Thus, six months after returning to standard mathematics teaching the students were back to their former position; being almost submissive, to obey the rules rather than to challenge them.

I do maths according to the lecturers' style. If he says this particular proof or theorem is not important, I'll skip them. I don,t want to burden myself learning things that won't be tested.

SPK, year 5

We have a great deal of work already, without spending extra time on solving problems that is of our own invention and won't be asked in the exams. ...It is perfectly possible to gain a good grade by rote learning the materials.

SPK, year 4

5 DISCUSSION

The general observation is not only true in UTM but all over the world – that the current method of teaching through lecturing is failing. One might hope that the acquisition of a rich body of mathematical knowledge would naturally lead to the ability to apply the knowledge to solve problems. But regrettably that may not be so. Selden, Mason & Selden, ([29] and [30]) report similar observation in their studies amongst undergraduates taking Calculus courses. They found that not only average students could not solve nontrivial problems in Calculus, even good students stumble over them. Their conclusion also highlights shortcomings in traditional methods of teaching mathematics; although students can pass their mathematics, many do not have the ability to apply their mathematical knowledge creatively. Dreyfus ([7]) argued that the inability to use mathematical knowledge in a flexible manner to solve problems is due to lack of insight into the processes that had led mathematicians to their creations. Indeed how can we expect students to have these qualities when they never experience mathematics beyond learning established facts and carrying out standard mathematical procedures? As Freudenthal ([9]) succinctly said:

> ...the only thing the pupil can do with the ready-made mathematics which he is offered is to reproduce it.
>
> p.117

Doing mathematics certainly must involve thinking. But doing mathematics in university courses tends to be a reproductive process. Students are presented theorems and proofs which they then learn and reproduce. Then they are given some measure of problem-solving at the end to encourage them to think of related problems themselves. That is, problem-solving is seen as no more than just a skill to be acquired. The opportunity for them to think, to articulate their own questions, to challenge conjectures and to reflect on their problem-solving is very limited. The kind of mathematics students are doing do not encourage them to think in a mathematical manner. Burton ([3]) asserts that mathematical thinking does not emerge automatically from learning and mastering of mathematics. It requires some degree of groundwork and training. Accordingly undergraduate mathematics courses were found to have an almost insignificant effect in changing the quality of mathematical thinking of the students ([22]).

Schoenfeld ([26]) in his study among 17-year old college students suggests that students have come to separate school mathematics — mathematics that they know in their mathematics classes — from the "abstract mathematics, the discipline of creativity, problem solving, and discovery, about which they are being told but which they have not experienced" (p. 349). Consequently, students' behaviour seems to be driven much more by their classroom experiences during problem-solving situations. Our classroom observation showed similar tendency amongst the students, and therefore confirm Schoenfeld's finding.

Mathematicians want their students to think mathematically and have positive attitudes. It is what we desire but do we get it? Probably because we fear the students will not understand we teach in a way that promotes precisely Freudenthal's conjecture. One possible reason is the question of efficiency. We need to cover so much material in a limited time and the only way to do it is to push through. As one of the staff commented:

> ...We gave them little room to do their own thinking. But we cannot change it because the system does not allow us to do so. ...So we end up teaching them what they need to know.

Hence mathematics teaching is based on mastering separate discrete facts and procedures and pays little attention to students' mathematical thinking. This phenomena reinforced Davis's ([5]) assertion that:

> ... in mathematics courses, students are often given some examples that they can imitate. The teacher does not know how the students are thinking about the work, but on the contrary, tests only for successful imitation.
>
> p. 13

He points out that this may give short term success for students who are capable of imitating very complicated things but as he clearly puts it "in the long run how the students think about things becomes the decisive factor in performance and future learning" (ibid., p. 13).

Moreover, students who are being "crammed with lots of bland mathematical abstract theory" eventually became deflated. It makes them wonder what the point of some courses is and why they are doing them. They are being trained to think in a static and rather restricted manner. The mathematics teaching has led students to view mathematics as a collection of facts and procedures that they need to memorise to remember ([18]), students pick up the rhetoric but not the substance in their mathematics learning ([26]). Tall ([34]) argued that as long as students are only taught to comprehend the accumulated wisdom of mathematical thought, mathematics teaching will fail the majority of our students.

What is mathematical thinking? Schoenfeld ([27]) describes what it means to think mathematically as follows:

> Learning to think mathematically means (a) developing a mathematical point of view — valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using these tools in the service of the goal of understanding structure — mathematical sense making.
>
> p.335

He argues that the fundamental aspects of thinking mathematically include core knowledge, problem-solving strategies, effective use of ones resources, having a mathematical perspective, and engagement in mathematical practices. Whilst Tall ([36]) observed that the nature of advanced mathematics involves processes that call for thought and creativity. Mathematical thinking is therefore considered a part of the living process of human thought and not just proof and deduction. According to Tall, there is a full cycle of mathematical thinking: intuition is followed by the making of conjectures from abstractions, leading to definition and to the final stage of proof. Although the cognitive processes involved in advanced mathematical thinking are also found in elementary mathematics, it is the possibility of formal definition and deduction that distinguishes advanced mathematical thinking. The mathematical thinking processes teachers hope to provoke in the students do not happen by themselves. Even if they do happen the students might not be conscious of them. Many features of these processes need to be made very explicit to the students to the point that the students are conscious of them ([7]).

Indeed it is the creative processes of mathematical thinking together with the possibility of human error that bring mathematics into existence. Accordingly these may need to be considered when structuring mathematical instruction if it is to generate students creativity. An over-conscientious concentration on mathematical content may obscure the mathematical thinking that is responsible for the derivation or application of particular aspects of mathematics. Thus an environment should be created to direct the student's attention entirely to the processes that are essential to successful mathematical thinking. Problem-solving as truly practised by mathematicians is seen as the art of thinking mathematically. When one is solving a mathematical problem, one is thinking mathematically. Therefore through active participation in problem-solving together with the opportunity to reflect on their mathematical activities, we can teach students to think in a mathematical manner.

It was observed that the students doing the experimental problem-solving course are willing to work hard as well as to struggle. They are willing to have periods when they feel under stress because they do not understand. Therefore, lecturers could play a vital role in helping the students to make personal meaningful constructions by formulating the mathematical knowledge in a such a way that it is easier for the students to make it on their own. Lecturers could in fact reduce the lecture content enormously by focusing only on the important ideas together with methods of constructing the less important ideas and leave other essential ideas to be worked out by the students. Encouraging students to think for themselves would prompt them to start filling in the details. It is suggested that for a more effective teaching, lecturers need to be aware of problem-solving techniques, to have an understanding of them and to have a problem-solving attitude themselves. When lecturers and students both share an understanding of the problem-solving processes, it would make the lecturers effort in getting students to think mathematically more explicit and meaningful to the students. Mason ([15]) suggested that mathematics teachers should try to be with their students, entering their experience, and exposing one's own experience to them. To do this, he points out that it requires one to work on one's mathematical being, re-awakening the awareness that one possesses the powers that are essential to think mathematically. As he puts it:

> Through that self-discovery arises the opportunity to enter the experience of other people, because I can only help somebody else work on mathematics if I can enter their experience, but I can only enter their experience if I am fully cognisant with my own experience.
>
> p. 57

It is clear that it is necessary to reflect on our own mathematical thinking to pass these thinking processes on to students. Being covert about the power of our own working methods may have served students badly. Tall ([36]) succinctly said:

> We cheated our students because we did not tell the truth about the way mathematics works, possibly because we sought the Holy Grail of mathematical precision, possibly because we rarely reflected on, and therefore never realized, the true ways in which mathematicians operate.
>
> p. 255

It is evident that during the initial stage of the problem-solving course the students showed no intellectual autonomy to solve the problems on their own. Their limited view of mathematics and problem-solving prevented them using their mathematical knowledge. In addition, due to rote learning, some may not understand much of the mathematics they use during problem-solving. Certainly it is not our aim to turn every student of mathematics into a fully fledged mathematician. But, most of us would want our graduates to have confidence to tackle anything new and able to think for themselves. Research in undergraduate mathematics education is a relatively new and developing area ([12]). The current trend in mathematics education is towards conceptualising mathematics as a living subject with the development of mathematical thinking becoming a priority ([27], [17], [36]). In this recent development, problem-solving has been emphasised as a process to construct mathematical knowledge as well as a process in the application of mathematical knowledge. We cannot continue to ignore research in this field. The question of how we might shape mathematical instruction if we wish students to think mathematically for example, requires further investigations. Mathematicians themselves should do it, for others surely lack the mathematical knowledge to research it in depth ([35]).

6 SUMMARY

When considering students thinking about mathematics, it was observed that their comments were very emotional in the sense that they expressed a high degree of frustration about their university mathematics. They view mathematics courses as a fixed body of knowledge to be learnt rather than a living subject in which they think for themselves. The findings show that students have little intellectual independence that we desire them to possess nor have they the ability to think for themselves. The students involved in this research have long since learned that what matters most is to be able to carry out the procedures to do mathematics. They do not feel any loss by their lack of understanding; a system which merely assesses the products of learning allows them to be successful. Responses following the experimental course indicate that the students' views change dramatically. Problem-solving helps them to say mathematics is not simply a body of procedures to be learned by memorising, it is also a process of thinking. However, students' attitudinal changes were mainly on a short term basis. Opinions expressed suggest that the quantity and difficulty of the mathematics give students little room for creative thinking.

When given the opportunity to think mathematically, the majority of students showed that they are capable of carrying out various processes of mathematical thinking and engage actively in problem-solving. The interviews emphasise that there are differences in the quality of the students' thinking. For instance, some lower attaining students, when faced with a problem appear to be more concerned about recalling and applying learned techniques to solve the problem rather than looking for insights, methods and reasons. Perhaps their contextual understanding of mathematical concepts is limited. Thus they lack confidence in carrying out the mathematical performance. Their reaction to the given mathematical problem indicates that they see the problem-solving knowledge as just another procedure. While solving problems, their emphasis is on applying learned techniques or ready rules to the task. They were using a procedural method and were not truly doing problem-solving. This gave an indication of the way they do mathematics; in a procedural and a non conceptual way. Students' tendency to lay emphasis on procedural aspects remains. It may be suggested that the change following the problem-solving course was away from being very procedural to weakly procedural, not to non-procedural.

It is a matter of some concern that the system may not be providing students with the experiences to encourage them to be creative and reflective. If Malaysia is to achieve her high aims — to have a large number of educated people who can think — then some changes should take place in mathematics education. In particular, in the learning and teaching of mathematics amongst the undergraduates at UTM. It is suggested that mathematicians need to move away from teaching students mathematical thought to teaching them mathematical thinking; if they wish students to think mathematically.

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