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A NEW EFFICIENT ODE SOLVER FOR SOLVING INITIAL VALUE PROBLEM*

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Abstract. In this paper we develop a new three-stage, fourth order explicit formula of Runge-Kutta type based on Arithmetic and Harmonic means. The error and stability analyses of this method indicate that the method is stable and efficient for nonstiff problems. Two examples are given which illustrate the fcurth order accuracy of the method..

Keywords: Runge-Kutta method, Harmonic Mean, three-stage, fourth-order, covergence and stability analysis.

1 INTRODUCTION

The requirement of solving initial value problem economically, especially in terms of computing time, while maintaining the high degree of accuracy of the approximate solution has been given a considerable amount of research for the past three decades by many well known authors. Amongst them are [5], [10], [7] and [4]. By having fewer number of stages we could save the evaluation time of derivative functions especially if the function is of a complex form. "It is an intellectual challenge to derive Runge Kutta formulas with as few stages as possible because the number of times the differential equation is evaluated is a significant measure of work." was quoted in [7]. The existence of other form of Runge-Kutta (RK) formulas which are based on other types of means has also contributed methods comparable methods to existing ones, [2], [1], [9] which give us several alternatives for solving certain types of class of problems better. The absence of five-stage fifth order RK methods as proved by [10] is now questionable. This was due to the existence of five-stage fifth order method of Runge-Kutta method based on geometric mean which was developed by Sanugi and Yaacob [1995]. Perhaps by using other types of means, including the arithmethic mean in the increment function one could also obtain other five-stage, fifth order methods.

In this paper we establish a new three-stage fourth order methods [12]. The first of its kind was developed by [11]. Even though it was mentioned in the paper that the stability region of the method RK-N34 is quite small and the numerical results shown are convincing, the class of problems that RK-N34 can solve accurately are limited. With that in mind we try to enhance the reliablity of this method by adjusting the position of an additional parameter a_4 that we introduce in the derivative function. For simplicity, we shall address this new method as RK-NHM34. After comparing the solutions obtained with the classical Runge-Kutta method, we found that our new method RK-NHM34, which based on harmonic mean (with bigger stability region than RK-N34) is more reliable for a larger class of initial value problems.

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2 FORMULATION OF A NEW RK-NHM34 METHOD

We proposed the scheme as

$$(2.1) \qquad y_{n+1} = y_n + h\sum_{j=1}^{3} w_j s_j + 0(h^5), n = 1, 2, 3, \ldots$$

where

$$s_1 = f(x_n, y_n),$$

$$s_2 = f(x_n + c_1 h, y_n + a_1 h s_1),$$

$$s_3 = f(x_n + c_1 h, y_n + h(a_2 s_1 + a_3 s_2 + a_4(\frac{2s_1 s_2}{s_1 + s_2}))).$$

$$c_1 = a_1,\ c_2 = a_2 + a_3 + a_4, \text{ and } h = x_{n+1} - x_n$$

Without lost of generality (2.1) can be written as [8] (2.2)

$$y_{n+1} = y_n + h\sum_{j=1}^{3} w_j s_j + 0(h^5), n = 1, 2, 3, \ldots$$

where

$$s_1 = f(y_n),$$

$$s_2 = f(y_n + a_1 h s_1),$$

$$s_3 = f(y_n + h(a_2 s_1 + a_3 s_2 + a_4(\frac{2s_1 s_2}{s_1 + s_2}))).$$

Note that the term $(\frac{2s_1 s_2}{s_1 + s_2})$ is defined as the harmonic mean for two different real numbers s_1, s_2. In order to achieve the fourth order accuracy we need to solve the seven parameters in (2.2) with the seven equations of order condition. Using MATHEMATICA, a symbolic computation package, we expand (2.2) and $y(x_n + h)$ as Taylor series expansion. On comparison of the coefficients of h^i, i = 1, 2, 3 we obtained the seven equations (mostly nonlinear) shown below.

$$(2.3) \qquad 1 - w_1 - w_2 - w_3 = 0 \qquad : hf$$

$$(2.4) \qquad 1 - 2a_1 w_2 - 2a_2 w_3 - 2a_4 w_3 = 0 \qquad : h^2 ff_y$$

$$(2.5) \qquad 1 - 6a_1 a_3 w_3 - 3a_1 a_4 w_3 = 0 \qquad : h^3 ff_y^2$$

$$(2.6) \qquad 1 - 3a_1^2 w_2 - 3a_2^2 w_3 - 6a_2 a_3 w_3 - 3a_3^2 w_3 - 6a_2 a_4 w_3 - 6a_3 a_4 w_3 - 3a_4^2 w_3 = 0 \qquad : h^3 f^2 f_{yy}$$

$$(2.7) \qquad 1 + 6a_1^2 a_4 w_3 = 0 \qquad : h^4 ff_y^3$$

$$(2.8) \qquad 4 - 12a_1^2 a_3 w_3 - 24a_1 a_2 a_3 w_3 - 24a_1 a_3^2 w_3 - 6a_1^2 a_4 w_3 - 12a_1 a_2 w_3 - 36a_1 a_3 a_4 w_3 - 12a_1 a_4^2 w_3 = 0 \qquad : h^4 f^2 f_y f_{yy}$$

$$(2.9) \qquad 1 - 4a_1^3 w_2 - 4a_2^3 w_3 - 12a_2^2 a_3 w_3 - 12a_2 a_3^2 w_3 - 4a_3^3 w_3 - 12a_2^2 a_3 w_3 - 24a_2^2 a_4 w_3 - 24a_2 a_3 a_4 w_3 - 12a_2^3 a_4 w_3 - 12a_3 a_4^2 w_3 - 4a_4^3 w_3 = 0 \qquad : h^4 f^3 f_{yyy}$$

On solving simultaneously using MATHEMATICA again we obtain two sets of solutions, viz.

Set I: $a_1 = \frac{1}{3},\ a_2 = \frac{35}{24},\ a_3 = \frac{25}{8},\ a_4 = -\frac{15}{4}.$

$w_1 = \frac{1}{10},\ w_2 = \frac{1}{2},\ w_3 = \frac{2}{5}$

Set II: $a_1 = 1,\ a_2 = \frac{3}{8},\ a_3 = \frac{3}{8},\ a_4 = -\frac{1}{4}.$

$w_1 = \frac{1}{6},\ w_2 = \frac{1}{6},\ w_3 = \frac{2}{3}$

Substituting these sets of solution to the scheme (2.1) yield formulas (2.10) and (2.11) respectively.

$$y_{n+1} = y_n + h\psi_n(x_n, y_n, h), \tag{2.10}$$

where

$$\psi_n = \frac{1}{10}(s_1 + 5s_2 + 4s_3)$$
$$s_1 = f(x_n, y_n),$$
$$s_2 = f(x_n + \frac{h}{3}, y_n + h\frac{s_1}{3}),$$
$$s_3 = f(x_n + \frac{5h}{6}, y_n + h(\frac{35s_1}{24} + \frac{25s_2}{8} - (\frac{15}{4})\frac{2s_1s_2}{s_1 + s_2})).$$

and

$$y_{n+1} = y_n + \frac{h}{6}(s_1 + s_2 + 4s_3), \tag{2.11}$$

where

$$s_1 = f(x_n, y_n),$$
$$s_2 = f(x_n + h, y_n + hs_1),$$
$$s_3 = f(x_n + \frac{h}{2}, y_n + h(\frac{3s_1}{8} + \frac{3s_2}{8} - (\frac{1}{4})\frac{2s_1s_2}{s_1 + s_2}))$$

3 ERROR ANALYSIS

The local truncation error (with respect to y) for formula (2.10) is given as LTE(1) where

$$\text{LTE}(1) = y(x_n + h) - y_{n+1} = \frac{1}{25920}(396ff_y^4 + 216f^2 f_y f_{yy} + 264f^3 f_{yy}^2 - 68f^3 f_y f_{yy} + f^4 f_{yyy})h^5 + O(h^6), \tag{3.1}$$

and the local truncation error due to formula (2.11) is given as LTE(2) where

$$\text{LTE}(2) = y(x_n + h) - y_{n+1} = \frac{1}{2880}(84ff_y^4 + 24f^2 f_y f_{yy} - 24f^3 f_{yy}^2 + 28f^3 f_y f_{yy} - f^4 f_{yyy})h^5 + O(h^6), \tag{3.2}$$

If in addition, the function f is linear in y then formula (2.10) is a better choice. This is due to the coefficient of the first term in LTE(1) which is smaller in magnitude compared to that of LTE(2).

At this point we have still not decided which of these two formulas is better for general cases. In order to make a proper decision we need to analyse the stability regions of both formulas and compare to that of the classical RK4. The one that gives a bigger stability region or a stability region at least approximately close to the size of RK4 should be considered in subsequent sections.

4 STABILITY ANALYSIS

We apply the test equation $y' = \lambda y$, where λ is a complex constant to both formula (2.10) and (2.11) respectively. From formula (2.10), the ratio $\frac{y_{n+1}}{y_n}$ gives the stability polynomial $Q_1(z) = \sum_{j=0}^{4}\frac{z^j}{j!} - \frac{z^5}{144} + O(z^6)$, $z = \lambda h$, while formula (2.11) will give the stability polynomial as $Q_2(z) = \sum_{j=0}^{4}\frac{z^j}{j!} - \frac{z^5}{48} + O(z^6)$. We plot the stability regions and compare to that of RK-AM4 classic as shown in Figure (4.1) and Figure (4.2).

It is clear that formula (2.10) gives bigger stability region compared to its counterpart formula (2.11). Thus, we will use this formula as our method to solve some initial value problems which have their own special characteristic.

5 THE CONVERGENCE OF RK-NHM34 METHOD

For the single equation initial value problem

$$y' = f(x, y),\ y(0) = y_0,$$

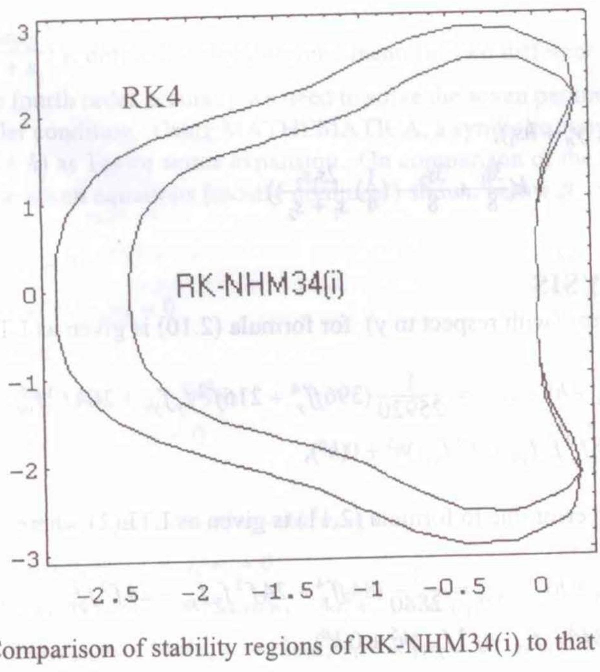

Figure 4.1 Comparison of stability regions of RK-NHM34(i) to that of classical RK4

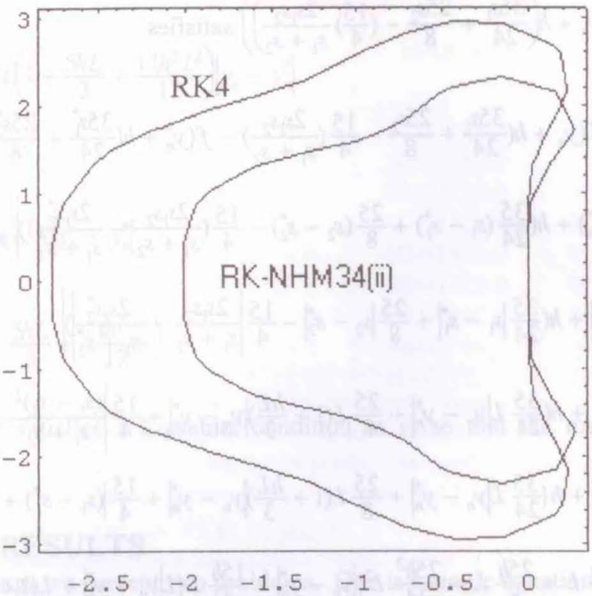

Figure 4.2 Comparison of stability regions of RK-NHM34(ii) to that of classical RK4

we now prove the convergence of the method RK-NHM34 given as

$$(5.1) \qquad y_{n+1} = y_n + \frac{h}{10}(s_1 + 5s_2 + 4s_3),\ n = 1,\ 2,\ 3,\ldots$$

where

$$(5.2) \qquad s_1 = f(y_n),$$

$$(5.3) \qquad s_2 = f\left(y_n + h(\frac{s_1}{3})\right),$$

$$(5.4) \qquad s_3 = f\left(y_n + h(\frac{35s_1}{24} + \frac{25s_2}{8} - (\frac{15}{4})\frac{2s_1s_2}{s_1 + s_2}\right).$$

For f satisfies Lipschitz condition, then from (5.2), $s_1 = f(y_n)$ satisfies

$$(5.5) \qquad |s_1 - s_1^*| = |f(y_n) - f(y_n^*)| \le L|y_n - y_n^*|$$

From (5.3), $s_2 = f(y_n + h\frac{s_1}{3})$ satisfies

$$(5.6) \qquad |s_2 - s_2^*| = \left|f(y_n + \frac{hs_1}{3}) - f(y_n^* + \frac{hs_1^*}{3}\right| \le L\left|(y_n - y_n^*) + \frac{h}{3}(s_1 - s_1^*)\right|$$

$$\le L\left|(y_n - y_n^*) + \frac{h}{3}(f(y_n) - f(y_n^*))\right| \le L\left(1 + \frac{hL}{3}\right)|y_n - y_n^*|$$

From (5.4), $s_3 = f\left(y_n + h\left(\frac{35s_1}{24} + \frac{25s_2}{8} - (\frac{15}{4})\frac{2s_1s_2}{s_1 + s_2}\right)\right)$ satisfies

$$|s_3 - s_3^*| = \left|f(y_n + h(\frac{35s_1}{24} + \frac{25s_2}{8} - \frac{15}{4}(\frac{2s_1s_2}{s_1 + s_2}) - f(y_n^* + h(\frac{35s_1^*}{24} + \frac{25s_2^*}{8} - \frac{15}{4}(\frac{2s_1^*s_2^*}{s_1^* + s_2^*}))\right|$$

$$\leq L\left|(y_n - y_n^*) + h(\frac{35}{24}(s_1 - s_1^*) + \frac{25}{8}(s_2 - s_2^*) - \frac{15}{4}(\frac{2s_1s_2}{s_1 + s_2} - \frac{2s_1s_2^*}{s_1 + s_2^*}))\right|$$

$$\leq L\left|y_n - y_n^*| + h(\frac{35}{24}|s_1 - s_1^*| + \frac{25}{8}|s_2 - s_2^*| - \frac{15}{4}\left|\frac{2s_1s_2}{s_1 + s_2} - \frac{2s_1s_2^*}{s_1 + s_2^*}\right|)\right|$$

$$\leq L\left|y_n - y_n^*| + h(\frac{35}{24}L|y_n - y_n^*| + \frac{25}{8}L(1 + \frac{hL}{3})|y_n - y_n^*| + \frac{15}{4}\left|\frac{(s_1 - s_2)^2}{s_1 + s_2} - \frac{(s_1 - s_2^*)^2}{s_1 + s_2^*}\right|)\right|$$

$$= L\left|y_n - y_n^*| + h\{\frac{35}{24}L|y_n - y_n^*| + \frac{25}{8}L(1 + \frac{hL}{3})|y_n - y_n^*| + \frac{15}{4}|(s_1 - s_1^*) + (s_2 - s_2^*)|\}\right|$$

$$\leq L\{1 + \frac{35h}{24}L + \frac{25h}{8}L + \frac{25h^2}{24}L^2\}|y_n - y_n^*| + \frac{15h}{4}L\{L|y_n - y_n^*| + L(1 + \frac{hL}{3})|y_n - y_n^*|\}$$

$$= \{L(1 + \frac{35h}{24}L + \frac{25h}{8}L + \frac{25h^2}{24}L^2) + \frac{15h}{4}(L^2 + L^2(1 + \frac{hL}{3}))\}|y_n - y_n^*|$$

$$= L(1 + \frac{35h}{24}L + \frac{25h}{8}L + \frac{25h^2}{24}L^2 + \frac{15h}{2}L + \frac{15h^2}{12}L^2)|y_n - y_n^*|$$

$$= L(1 + \frac{145h}{12}L + \frac{55h}{24}L^2)|y_n - y_n^*|$$

Thus,

$$|s_3 - s_3^*| \leq L(1 + \frac{145h}{12}L + \frac{55h^2}{24}L^2)|y_n - y_n^*| \tag{5.7}$$

Now,

$$|\psi_n - \psi_n^*| = \frac{h}{10}|(s_1 - s_1^*) + 5(s_2 - s_2^*) + 4(s_3 - s_3^*)| \tag{5.8}$$

$$\leq \frac{h}{10}\left(|s_1 - s_1^*| + 5|s_2 - s_2^*| + 4|s_3 - s_3^*|\right)$$

$$\leq \frac{h}{10}\left(L|y_n - y_n^*| + 5L(1 + \frac{hL}{3})|y_n - y_n^*| + 4L(1 + \frac{145hL}{12} + \frac{55h^2L^2}{24})|y_n - y_n^*|\right)$$

$$= \frac{hL}{10}\left(1 + 5(1 + \frac{hL}{3}) + 4(1 + \frac{145hL}{12} + \frac{55h^2L^2}{24})\right)|y_n - y_n^*|$$

$$= \frac{hL}{10}\left(10 + \frac{50hL}{3}) + \frac{55h^2L^2}{6}\right)|y_n - y_n^*|$$

Thus,

$$(5.9)\quad |\psi_n - \psi_n^*| \le hL\left(1 + \frac{5hL}{3} + \frac{11h^2L^2}{12}\right)|y_n - y_n^*|$$

or

$$(5.10)\quad |\psi_n - \psi_n^*| \le \phi(L,h)|y_n - y_n^*|$$

where $\phi(L,h) = hL\left(1 + \frac{5hL}{3} + \frac{11h^2L^2}{12}\right)$.

which implies that ψ satisfies a Lipschitz condition in y, so that the method RK-NHM34 is convergent.

6 NUMERICAL RESULTS

To conclude our findings we present two problems. One is a single equation initial value problem and the other is a system consisting of two first order linear equations solved by the method discussed above.

Problem 1

$y' = -\sqrt{1-y^2},\ 0.1 \le x \le 1$

Initial Condition: $y(0.1) = \cos(0.1)$

Exact Solution: $y(x) = \cos(x)$

Stepsize, $h = 0.01$

Problem 2

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.005 & 0.05 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix};\ 0 \le x \le 1$$

Initial Condition: $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$

Exact Solution: $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} e^{0.1x} \\ 0.1e^{0.1x} \end{pmatrix}$

Stepsize, $h = 0.01$

Table (6.1) corresponds to the solution of problem 1 when solved by classical RK4 and RK-NHM34 respectively, while Table (6.2) and (6.3) correspond to the solution of problem 2. All the solutions are printed out for every 10 steps.

Table (6.1)

x	Exact Solution	RK-AM4	RK-NHM34	ABSERR0R RK4	ABSERR0 RK-NHM34
0.1	0.995D+00	0.995D+00	0.995D+00	0.000D+00	0.000D+00
0.2	0.980D+00	0.980D+00	0.980D+00	0.139D–07	0.474D–08
0.3	0.955D+00	0.955D+00	0.955D+00	0.227D–07	0.776D–08
0.4	0.921D+00	0.921D+00	0.921D+00	0.030D–07	0.104D–07
0.5	0.878D+00	0.878D+00	0.878D+00	0.380D–07	0.129D–07
0.6	0.825D+00	0.825D+00	0.825D+00	0.449D–07	0.153D–07
0.7	0.765D+00	0.765D+00	0.765D+00	0.513D–07	0.175D–07
0.8	0.697D+00	0.697D+00	0.697D+00	0.571D–07	0.195D–07
0.9	0.622D+00	0.622D+00	0.622D+00	0.624D–07	0.213D–07
1.0	0.540D+00	0.540D+00	0.540D+00	0.671D–07	0.129D–07

METHOD: RK4 CLASSIC

Table (6.2)

x	Exact1 = u(x)	ERR1	Exact2 = v(x)	ERR2	ERR
0.0	0.100D+01	0.000D+00	0.100D+00	0.000D+00	0.000D+00
0.1	0.101D+01	0.215D–11	0.101D+00	0.264D–10	0.264D–10
0.2	0.102D+01	0.695D–11	0.102D+00	0.531D–10	0.536D–10
0.3	0.103D+01	0.145D–10	0.103D+00	0.803D–10	0.816D–10
0.4	0.104D+01	0.247D–10	0.104D+00	0.108D–09	0.111D–09
0.5	0.105D+01	0.378D–10	0.105D+00	0.136D–09	0.141D–09
0.6	0.106D+01	0.536D–10	0.106D+00	0.164D–09	0.173D–09
0.7	0.107D+01	0.724D–10	0.107D+00	0.193D–09	0.206D–09
0.8	0.108D+01	0.940D–10	0.108D+00	0.222D–09	0.241D–09
0.9	0.109D+01	0.119D––09	0.109D+00	0.252D–09	0.279D–09
1.0	0.111D+01	0.146D–09	0.111D+00	0.282D–09	0.318D–09

METHOD: RK-NHM34

Table (6.3)

x	Exact1 = u(x)	ERR1	Exact2 = v(x)	ERR2	ERR
0.0	0.100D+01	0.000D+00	0.100D+00	0.000D+00	0.000D+00
0.1	0.101D+01	0.747D–12	0.101D+00	0.262D–10	0.262D–10
0.2	0.102D+01	0.413D–11	0.102D+00	0.528D–10	0.530D–10
0.3	0.103D+01	0.102D–10	0.103D+00	0.798D–10	0.805D–10
0.4	0.104D+01	0.190D–10	0.104D+00	0.107D–09	0.109D–09
0.5	0.105D+01	0.305D–10	0.105D+00	0.135D–09	0.139D–09
0.6	0.106D+01	0.448D–10	0.106D+00	0.163D–09	0.169D–09
0.7	0.107D+01	0.620D–10	0.107D+00	0.192D–09	0.202D–09
0.8	0.108D+01	0.821D–10	0.108D+00	0.221D–09	0.236D–09
0.9	0.109D+01	0.105D–09	0.109D+00	0.250D–09	0.272D–09
1.0	0.111D+01	0.131D–09	0.111D+00	0.281D–09	0.310D–09

Note: For both Tables (6.2)and (6.3) we define the errors as

$$\text{ERR1} = \text{ABS}(u(x+nh) - u_n),$$
$$\text{ERR2} = \text{ABS}(v(x+nh) - v_n),$$
$$ERR = \sqrt{(\text{ERR1})^2 + (\text{ERR2})^2}$$

where n = 0, 1, 2,..., 10.

7 CONCLUSION

In this paper we have established a new three-stage fourth order Runge-Kutta formula based on Arithmetic and Harmonic means. By including the Harmonic mean in S_3, we are able to reduce the number of stages and at the same time we have increased the order of formula. Judging from the results as shown in Table 5.1 - 5.3, we could say that this new method is comparable to the classical RK4. The smaller stability region as shown in Fig. 4.1 indicate that this method requires a 'smaller' step size compared to that of classical RK4.

To summarize our discussion, we present two tables, Table (7.1) and (7.2). Table (7.1) shows the two formulas used, and Table (7.2) shows the amount of computational work involved per step for each method. From Table (7.2) it is clear that the RK-NHM34 is more favourable since it requires fewer function evaluations while the arithmetic operatio is similar with the classical RK4 method.

Table 7.1 The two formulas used in the numerical examples.

Method	Formula y_{n+1} =
RK-AM4	$y_n + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4)$
RK-NHM34	$y_n + \frac{h}{6}(s_1 + 5s_2 + 4s_3)$

Table 7.2 Comparison of the amount of computational work in each step.

Method	+ / −	× / ÷	FCN
Classical RK4	4	3	4
RK-NHM34	3	4	3

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