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ONE STEP COSINE-TAYLORLIKE METHOD FOR SOLVING STIFF EQUATIONS

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Abstract. This paper discusses the derivation of an explicit Cosine-Taylorlike method for solving stiff ordinary differential equations. The formulation has resulted in the introduction of a new formula for the numerical solution of stiff ordinary differential equations. This new method needs an extra work in order to solve a number of differentiations of the function involved. However, the result produced is better than the results from the explicit classical fourth-order Runge-Kutta (RK4) and the implicit Adam-Bashforth-Moulton (ABM) methods. When compared with the previously derived Sine-Taylorlike method, the accuracy for both methods is almost equivalent.

Keywords: Explicit method; stiff ordinary differential equations; Runge-Kutta; implicit method; Adam-Bashforth-Moulton; Sine-Taylorlike

Abstrak. Makalah ini membincangkan penghasilan kaedah tak tersirat bak Cosine-Taylor untuk menyelesaikan persamaan pembezaan biasa kaku. Perumusannya menghasilkan pengenalan kepada satu rumus baru bagi penyelesaian berangka bagi persamaan pembezaan biasa kaku. Kaedah baru ini memerlukan penghitungan tambahan yakni melakukan beberapa terbitan bagi fungsi yang terlibat. Walau bagaimanapun, keputusan yang diperoleh adalah lebih baik berbanding hasil yang didapati apabila menggunakan kaedah tak tersirat Runge-Kutta peringkat-4 dan kaedah tersirat Adam-Bashfiorth-Moulton (ABM). Perbandingan yang dibuat dengan kaedah bak Sine-Taylor menunjukkan kejituan bagi kedua-dua kaedah adalah hampir setara.

Kata kunci: Kaedah tak tersirat; persamaan pembezaan biasa kaku; Runge-Kutta; kaedah tersirat; Adam-Bashforth-Moulton; bak Sine-Taylor

1.0 INTRODUCTION

A stiff system contains one or more fast decay processes along with relatively slow processes, such that the shortest decay "time constant" is much smaller than the total span of interest in the independent variable, which is usually "time" [1]. Systems of ordinary differential equations arise frequently in almost every discipline of science and engineering that usually exhibited stiffness characteristic, as a result of modeling and simulation activities. It has been suggested that the problem of stiffness is very

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difficult to be solved by explicit methods but recently many explicit methods were introduced and developed to solve the stiff problems like those in [4 - 6].

In this paper, we introduced a Taylorlike explicit method which can be used to solve stiff problems and give a good accuracy. It was shown previously in [1] that the explicit one-step method for stiff problems could be represented by the composition of a polynomial and an exponential function of the form

$$PE(t) = a_0 + t\left(a_1 + t\left(a_2 + t\left(a_3 + t\left(a_4 + a_5 t\right)\right)\right)\right) + Ab_1 e^{b_2 t}. \tag{1.1}$$

In [2], taking $A = 1$, we had calculated the values of a_i, $i = 0,1,2,3,4,5$ and b_i, $i= 1,2$. In [3], we had substituted A with a trigonometric function, $A = \sin z_n h$. Based on the same theory for the solution of a differential equation with complex eigenvalues, we replaced A by $\cos z_n h$ to produce a Cosine-Taylorlike method. Provided that $f^{(5)}.f^{(6)} \neq 0$, we obtain the equation

$$[PE(t) = y_n + (t-t_n)\left(f_n + (t-t_n)\left(\frac{f_n'}{2} + (t-t_n)\left(\frac{f''}{6} + (t-t_n)\left(\frac{f'''}{24} + (t-t_n)\frac{f_n^{(4)}}{120}\right)\right)\right)\right) +]$$
$$\frac{f_n^{(5)}\cos(z_n h)}{z_n^6}\left(e^{z_n(t-t_n)} - 1 - z_n(t-t_n) - \frac{1}{2}\left(z_n(t-t_n)\right)^2 - \frac{1}{6}\left(z_n(t-t_n)\right)^3 - \frac{1}{24}\left(z_n(t-t_n)\right)^4\right.$$
$$\left. - \frac{1}{120}\left(z_n(t-t_n)\right)^5\right), \tag{1.2}$$

where $z_n = \dfrac{f_n^{(6)}}{f_n^{(5)}}$.

Letting $t = t_{n+1}$, we obtain the following formula:

$$[y_{n+1} = y_n + h\left(f_n + h\left(\frac{f_n'}{2} + h\left(\frac{f_n''}{6} + h\left(\frac{f_n'''}{24} + h\frac{f_n^{(4)}}{120}\right)\right)\right)\right) +]$$
$$\frac{f_n^{(5)}\cos(z_n h)}{z_n^6}\left(\exp(z_n h) - 1 - hz_n\left(1 + hz_n\left(\frac{1}{2} + hz_n\left(\frac{1}{6} + hz_n\left(\frac{1}{24} + \frac{z_n h}{120}\right)\right)\right)\right)\right). \tag{1.3}$$

2.0 STABILITY

Theorem 1.1

The explicit Cosine-Taylorlike method is A-stable.

Proof:

On applying equation (1.3) to the test equation $y' = \lambda y$ with Re $(\lambda) < 0$, we obtain

$$[y_{n+1} = y_n + h\left(\lambda y_n + h\left(\frac{\lambda^2 y_n}{2} + h\left(\frac{\lambda^3 y_n}{6} + h\left(\frac{\lambda^4 y_n}{24} + h\frac{\lambda^5 y_n}{120}\right)\right)\right)\right) +]$$

$$\frac{\cancel{\lambda^6}\, y_n \cos(\lambda h)}{\cancel{\lambda^6}}\left(\mathrm{e}^{\lambda h} - 1 - \lambda h\left(1 + \lambda h\left(\frac{1}{2} + \lambda h\left(\frac{1}{6} + \lambda h\left(\frac{1}{24} + \frac{\lambda h}{120}\right)\right)\right)\right)\right)$$

$$[= y_n + \lambda h y_n\left\{1 + \frac{\lambda h}{2} + \frac{(\lambda h)^2}{6} + \frac{(\lambda h)^3}{24} + \frac{(\lambda h)^4}{120}\right\} +]$$

$$y_n \cos(\lambda h)\left(e^{\lambda h} - 1 - \lambda h\left(1 + \lambda h\left(\frac{1}{2} + \lambda h\left(\frac{1}{6} + \lambda h\left(\frac{1}{24} + \frac{\lambda h}{120}\right)\right)\right)\right)\right)$$

$$[= y_n\left(1 + h\lambda + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{120}\right) +]$$

$$y_n \cos(\lambda h)\left(e^{\lambda h} - \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{120}\right)\right)$$

$$[= y_n\left(e^{\lambda h}\cos(\lambda h) + (1 - \cos(\lambda h))\left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!}\right)\right),]$$

i.e.,

$$y_{n+1} = y_n\left(e^{\lambda h}\cos(\lambda h) + (1 - \cos(\lambda h))\left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!}\right)\right)$$

which gives us

$$y_{n+1} = Q(\lambda h)\, y_{n},$$

where

$$
\begin{aligned}
Q(\lambda h) &= e^{\lambda h}\cos(\lambda h) + (1-\cos(\lambda h))\left(1+\lambda h+\frac{(\lambda h)^2}{2}+\frac{(\lambda h)^3}{3!}+\frac{(\lambda h)^4}{4!}+\frac{(\lambda h)^5}{5!}\right)\\
&\approx e^{\lambda h}\cos(\lambda h) + (1-\cos(\lambda h))e^{\lambda h}\\
&= e^{\lambda h}.
\end{aligned}
\tag{2.1}
$$

Since $y_{n+1} = e^{\lambda h} y_n$ for $n = 0,1,2,\dots$,

$$y_1 = e^{\lambda h} y_0\,;\, y_2 = e^{\lambda h} y_1 = e^{2\lambda h} y_0\,;\,\dots\,;\, y_k = e^{\lambda h} y_{k-1} = e^{k\lambda h} y_0.$$

For any fixed point $t = t_n = nh$, we have

$$y_n = e^{n\lambda h} y_0.$$

Since $|e^{n\lambda h}| \rightarrow 0$ as $n \rightarrow \infty$ for all λh with Re $(\lambda) < 0$, hence $y_n \rightarrow 0$ as $n \rightarrow \infty$ and consequently the method is *A*-stable. ■

Theorem 1.2

The explicit Cosine-Taylorlike method is also *L*-stable.

Proof:

We have shown that on applying equation (1.3) to the test equation $y' = \lambda y$, with Re $(\lambda) < 0$, we obtain

$$y_{n+1} = e^{\lambda h} y_n.$$

From Theorem 1.1, the method is *A*-stable. Since $|e^{\lambda h}| \rightarrow 0$ as Re$(\lambda h) \rightarrow -\infty$, hence it is *L*-stable. ■

Figure 1 and Figure 2 illustrate the stability polynomial and stability region of the method, respectively.

3.0 NUMERICAL RESULTS

The formula (1.3) was tested on the stiff ordinary differential equation

$$y'(t) = -100y(t) + 99e^{-t};\quad y(0) = 1, \tag{3.1}$$

and the result is compared to the exact solution which is represented by

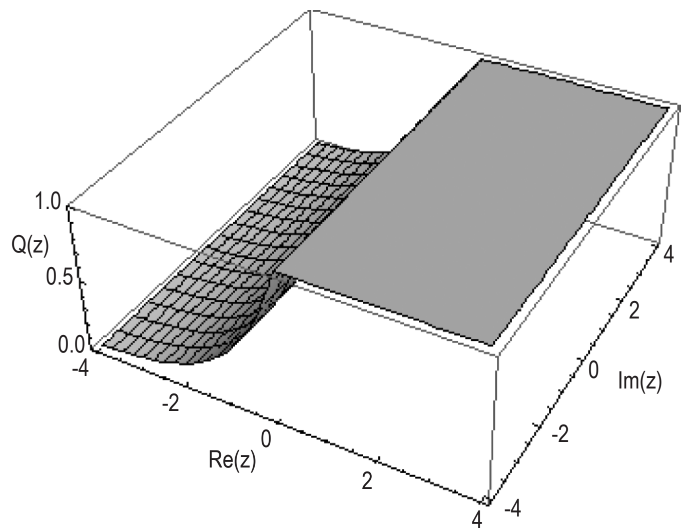

Figure 1 The stability polynomial given by the method above

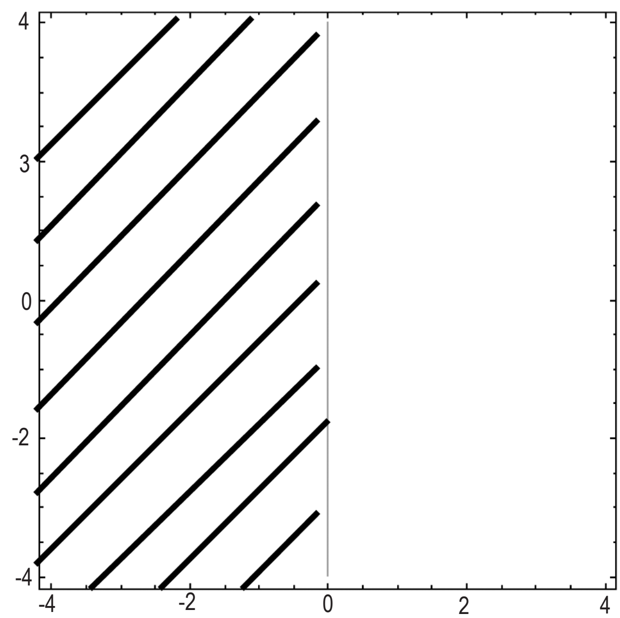

Figure 2 The stability region given by the method above

$$y(t) = e^{100t}\left(-1 + e^{99t}\right).$$

We also solved equation (3.1) using three other methods; namely the classical fourth-order Runge-Kutta (RK4), the implicit Adam-Bashforth-Moulton (ABM) and the Sine-Taylorlike (STL6)[3] methods. By taking $h = 0.02$, the relative errors for the four methods applied were compared and presented in Table 1.

The relative errors for the four methods used are plotted and illustrated in Figure 3.

4.0 DISCUSSION AND CONCLUSION

This research has generally discussed a one-step explicit method in solving stiff ordinary differential equations, namely, the Cosine-Taylorlike method. The results showed excellent relative errors of the Cosine-Taylorlike method compared to the classical Runge-Kutta and the Adam-Bashforth-Moulton methods. The one-step explicit Sine Taylorlike method, which we had derived earlier, also showed comparable relative errors to the Cosine-Taylorlike method. We have proved that the Cosine-Taylorlike method is both *A*-stable and *L*-stable. We realize that the function evaluations of the methods is about the same for every method but we believe that the cost of computation is much 'cheaper' in the explicit formula compared with the implicit ones, and is reflected in the efficiency of the method in dealing with stiffness. We conclude that the proposed method is comparable to our previous Sine-Taylorlike method.

Table 1 Relative errors of RK4, ABM, STL6 and CTL6 methods on the Stiff ODE

t	**Exact value**	**Adam-Bashforth-Moulton (ABM)**	**Relative error Classical Runge-Kutta (RK4)**	**A=Sin(λh)[3]**	**A=Cos(λh)**
0.1	0.90479201811	4.447120572E-03	3.114502171E-01	5.017718352E-05	5.008859210E-05
0.2	0.81873075102	3.050605404E-05	9.914111543E-01	2.517399007E-09	2.517263404E-09
0.3	0.74081822068	5.109507959E-05	2.556514973E+00	2.847424224E-14	1.269351746E-13
0.4	0.67032004604	5.118873038E-05	3.273900726E+00	9.987236512E-14	1.490632315E-15
0.5	0.60653065971	5.118915632E-05	1.006547891E+02	9.884420906E-14	0.000000000E+00
0.6	0.54881163609	9.327128023E-02	8.502726223E+02	9.649510832E-14	1.820662421E-15
07	0.49658530379	5.118915823E-05	5.181725175E+03	1.107796595E-13	1.207285896E-14
0.8	0.44932896412	5.118915826E-05	2.508607182E+04	1.001928370E-13	2.223762103E-15
0.9	0.40656965974	5.118915825E-05	9.224755720E+04	1.058149352E-13	7.372911615E-15
1.0	0.36787944117	5.118915826E-05	1.738914072E+05	9.793082493E-14	0.000000000E+00

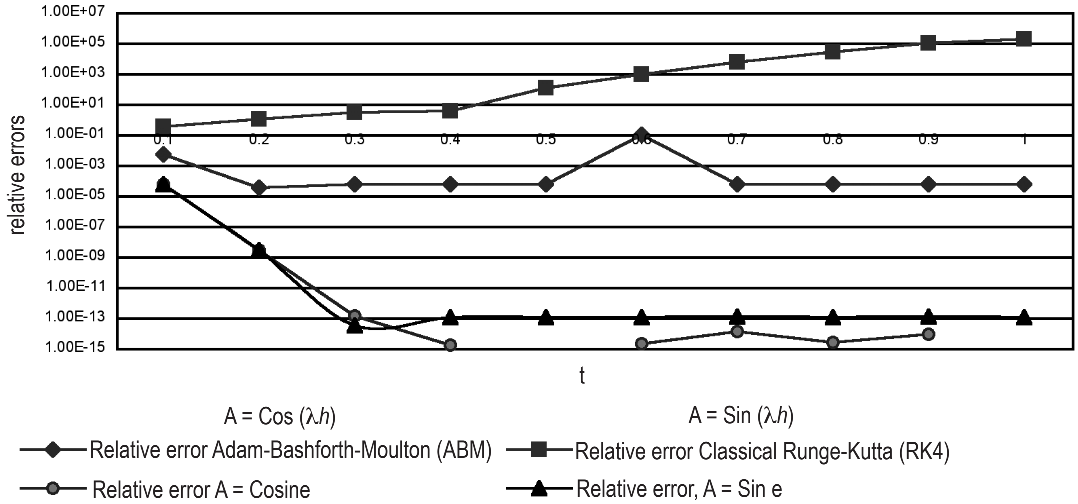

Figure 3 Relative errors of the four methods used for the stiff problem

Table 2 Number of evaluation of the functions in methods used

Methods	**Function Evaluation in One Iteration**
Classical Runge-Kutta	5
Adams-Bashforth-Moulton	6
Sine-Taylorlike	6
Cosine-Taylorlike	6

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