Travelling wave solution to a generalized Kuramoto-Sivashinsky equation

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ABSTRACT

This paper presents a method for the construction of travelling wave solution ot the generalized Kuramoto-Sivashinky equation. The method makes use of a series representation, and is essentially an extension of Hirota's method.

1. INTRODUCTION

The formation of a row of soliton-like pulses in an unstable, dissipative, and dispersive nonlinear system has been observed in the numerical solution of the generalized Kuramoto-Sivashinky (gKS) equation

$$u_t + uu_x + au_{xx} + bu_{xxx} + cu_{xxxx} = 0, \tag{1}$$

where a, b and c are positive constants characterising instability (self-excitation), dispersion, and dissipation, respectively. Equation (1) can be used to describe the long waves on a viscous fluid flowing down an inclined plane [Topper and Kawahara, 1978] and unstable drift waves in plasmas [Cohen et al, 1976].

We remark that, in the limits $b \to 0$ or $c \to 0$, the gks equation can be approximated by the Kuramoto-Sivashinky (KS) equation

$$u_t + uu_x + au_{xx} + cu_{xxxx} = 0, \tag{2}$$

or the Korteweg-de Vries Burgers' (KdVB) equation

$$u_t + uu_x + au_{xx} + bu_{xxx} = 0, \tag{3}$$

respectively.

The KS equation describes chemical reactions which exhibit a turbulent-like behaviour [Wijingaarden, 1972], while the KdVB equation represents the flow of liquids containing gas bubbles [Kuramoto and Tsuzuki, 1976] and the propagation of waves on an elastik tube filled with a viscous fluid [Johnson, 1969]. It is known that the KS equation has travelling wave solution in terms of hyperbolic tangent function, whereas the KdVB equation has travelling wave solutions involving a combination of hyperbolic secant and tangent functions. A detailed discussion of the KdVB equation has been given by Jeffrey and Mohamad [1991].

2. TRAVELLING WAVE SOLUTION OF THE gKS EQUATION

In this section, travelling wave solution of the generalized gKS equation (1) is found by means of the series method. This method uses an approach similar to that due to Hirota when seeking single and multiple soliton solutions of the KdV equation.

We seek a solution of the gKS equation in the form

$$u(x,\ t) = \sum_{j=0}^{\infty} u_j F^{j-3} \tag{4}$$

We express the x and t derivatives of u(x,t) in terms of $u_j(x,t)$and F(x,t). Inserting the expressions obtained from the derivatives into the gKS equation, we find that

$$u_0 = 120cF_x^3, \tag{5a}$$

$$u_1 = -15bF_x^2 - 180cF_xF_{xx} \tag{5b}$$

$$u_2 = \frac{15}{76}\left(16a - \frac{b^2}{c}\right)F_x + 15bF_{xx} = 60cF_{xxx}. \tag{5c}$$

These results are the coefficients of the powers of F^{-7} and F^{-6} and F^{-5} Using equations (5C), the coefficient of F^{-4} Can be written in the form

$$F_t - \frac{b}{76c}\left(7a - \frac{13b^2}{8c}\right)F_x + \frac{15}{152}\left(16a - \frac{b^2}{c}\right)F_{xx} + 5bF_{xxx} + 15cF_{xxxx}$$

$$-\frac{15}{4}bF_{xx}^2\,F_x^{-1} + 15cF_{xx}^3F_x^{-2} - 30cF_{xx}F_{xxx}F_x^{-1} + u_3F_x = 0. \tag{6}$$

In order to find the coefficients $u_j(x,t)$ for $j \geq 4$, we canuse a recursion relation which can be obtained from the condition that the terms multiplying the same powers of F(x,t) are equal. It turns out that the coefficient u_6 is not determined by this formula, and it is necessary to set $u_j = 0$ for $j \geq 4$ in equation(4) A solution u(x,t) of the gKS equation (1) may then be written

$$u(x,t) = \frac{u_0}{F^3} = \frac{u_1}{F^2} = \frac{u^2}{F} + u_3(x,t), \tag{7}$$

where $u_0(x,t)$, $u_1(x,t)$ and $u_2(x,t)$ are expressed in terms of derivatives of F(x,t) according to equations (5). Taking into account equations (5), equation (7) can be re-written as

$$u(x,t) = \frac{15}{76}\left(16a - \frac{b^2}{c}\right)\frac{\partial}{\partial x}\ln F + 15b\frac{\partial}{\partial x^2}\ln F$$

$$+ 60c\frac{\partial^3}{\partial x^3}\ln F + u_3. \tag{8}$$

This equation can be used to find an exact solution of the gKS equation. Now let

$$F(x,t) = 1 + \exp(kx - wt), \tag{9}$$

where k and ω are constants to be determined. After substitution of (9) into (6) and (8), followed by setting $u_3 = 0$, we find that a function F of the form (9) will be a solution of the gKS equation provided

$$\omega - \frac{15}{152}\left(16a - \frac{b^2}{c}\right)k^2 = 0 \text{ and } \frac{b}{76c}\left(7a - \frac{13b^2}{8c}\right)k - \frac{5}{4}bk^3 = 0. \tag{10}$$

From (10), we have

$$\omega = \frac{15}{152}\left(16a - \frac{b^2}{c}\right)k^2 \text{ and } k^2 = \frac{1}{95c}\left(7a - \frac{13b^2}{8c}\right) \tag{11}$$

Thus the solution of the gKS equation may be written

$$u(x,t) = \frac{15}{76}\left(16a - \frac{b^2}{c}\right)\left(\frac{k}{2}\right)\left[1 + \tanh\frac{1}{2}(kx - \omega t)\right]$$
$$+ 15k^2 \operatorname{sech}^2\frac{1}{2}(kx - \omega t)\left[\frac{b}{4} - ck\tanh\frac{1}{2}(kx - \omega t)\right], \tag{12}$$

where k and ω are given by (11).

Equation (12) represents a travelling wave solution to the gKS equation. It is clear that, if $b \to 0$, the travelling wave solution (12) can be reduced to the travelling wave solution of the KS equation

$$u(x,t) = \frac{30}{19}ak\left[1 + \tanh\frac{1}{2}(kx - \omega t)\right]$$
$$- 15ck^3\left[1 - \tanh^2\frac{1}{2}(kx - \omega t)\right]\tanh\frac{1}{2}(kx - \omega t) \tag{13}$$

where from (11), we have

$$\omega = \frac{30}{19}ak^2.$$

This result is in agreement with the result obtained by Kuramoto and Tsuzuki [1976]. (14)

Finally, we remark that this method can be generalized to a larger class of nonlinear evolution equations of the form

$$u_t + auu_x + \sum_{i=2}^{N} a_i u_{ix} = 0 \tag{15}$$

In general, the solution of equation (15) will be of the form

$$u(x, t) = \sum_{j=0}^{N-1} u_j F^{j-N+1}. \tag{16}$$

3. *ACKNOWLEDGEMENTS*

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4. REFERENCES

[1] J. Topper and T. Kawahara, J. Phys. Soc. Jpn. 44,(1978) 663.
[2] B. I. Cohen, J. A. Krommes, W. M Tang, and M. N. Rosenbluth,Nucl. Fision 16, (1976) 971.
[3] L. V Wijingaarden, Ann. Rev. Fluid Mech. 4, (1972) 369.
[4] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. 55, (1976) 356.
[5] R. S. Johnson, Ph.D thesis, University of London, (1969).
[6] A. Jeffrey and M. N. B. Mohamad, Accepted for publication in "Wave Motion" (1991).

APPENDIX I: The SCOPE COMMON statements.

```
SCOPE COMMON
STOCK<-^MATERIAL (150, 25)
LCHM1<-^LEFT_CHAMFER (03, 21, 24, 0, 0, 0, 003, 0, 0)
CYLN1<-^CYLINDER (37, 24, 0, 0, 0, 003, 0, 0)
UCUT1<-^UNDERCUT (10, 13, 0, 0, 0, 040, 0, 0)
CYLN2<-^CYLINDER (20, 18, 0, 0, 0, 050, 0, 0)
RTAP1<-^RIGHT_TAPER (30, 13, 18, 0, 0, 0, 070, 0, 0)
CUT2<-^UNDERCUT (05, 10, 0, 0, 0, 100, 0, 0)
CYN3<-^CYLINDER (25, 13, 0, 0, 0, 105, 0, 0)
CYLN4t<-^CYLINDER (18, 08, 0, 0, 0, 130, 0, 0)
RCHM1<-^RIGHT_CHAMFER (02, 06, 08, 0, 0, 0, 148, 0, 0)
COMPONENT<-STOCK-LCHM1-CYLN1-UCUT1-CYLN2-RTAP1-UCUT2-CYLN3$
-CYLN4-RCHM1
!
! MATERIAL
!
SCOPE MATERIAL
SOLID FAMILY MATERIAL (LENGTH,RADIUS) :MATERIAL
MATERIAL<-CYL (LENGTH, RADIUS) AT (ROTY=90)
RADIUS=25
!
! CYLINDER
!
SCOPE CYLINDER
SOLID FAMILY CYLINDER (LENGTH, RADIUS1, X1, Y1, Z1, $
X2, Y2, Z2) :CYLINDER
OBJ1<-CYL (LENGTH, RADIUS) AT (ROTY=90)
OBJ2<-CYL (LENGTH, RADIUS1) AT (ROTY=90)
CYLINDER<-MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
!
! RIGHT_TAPER
!
SCOPE RIGHT_TAPER
SOLID FAMILY RIGHT_TAPER (LENGTH, RADIUS1, $
RADIUS2, X1, Y1, Z1, Y2, Y2, Z2) :RIGHT_TAPER
OBJ1<-CYL (LENGTH, RADIUS) AT (ROTY=90)
OBJ2<-CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY=90)
RIGHT_TAPER<-MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
!
! UNDERCUT
!
SCOPE UNDERCUT
SOLID FAMILY UNDERCUT (LENGTH, RADIUS1, X1, Y1, Z1, $
X2, Y2, Z2) :UNDERCUT
OBJ1<-CYL (LENGTH, RADIUS) AT (ROTY= 90)
```

```
OBJ2< -CYL (LENGTH, RADIUS1) AT (ROTY=90)
UNDERCUT< -MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
!
! RIGHT_CHAMFER
!
SCOPE RIGHT_CHAMFER
SOLID FAMILY RIGHT_CHAMFER (LENGTH, RADIUS1, RADIUS2, $
X1, Y1, Z1, X2, Y2, Z2) :RIGHT_CHAMFER
OBJ1< -CYL (LENGTH, RADIUS) AT (ROTY=90)
OBJ2< -CONE (LENGTH, RADIUSl, RADIUS2) AT (ROTY=90)
RIGHT_CHAMFER< -MOVE (OBI1-OBJ2) BY (X2, 0, 0)
RADIUS=25
!
! LEFT_CHAMFER
!
SCOPE LEFT_CHAMFER
SOLID FAMILY LEFT_CHAMFER (LENGTH, RADIUS1, RADIUS2, $
X1, Y1, Z1, X2, Y2, Z2) :LEFT_CHAMFER
OBJ1< -CYL (LENGTH, RADIUS) AT (ROTY=270)
OBB2< -CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY =270)
LEFT_CHAMFER< -MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
```

APPENDIX II: The Geometrical Specification File.

```
SCOPE COMMON
STOCK<-^MATERIAL (150, 25)
LCHM1<-^LEFT_CHAMFER (03, 21, 24, 0, 0, 0, 003, 0, 0)
CYLN1<-^CYLINDER (37, 24, 0, 0, 0, 003, 0, 0)
UCUT1<-^UNDERCUT (10, 13, 0, 0, 0, 040, 0, 0)
CYLN2<-^CYLINDER (20, 18, 0, 0, 0, 050, 0, 0)
RTAP1<-^RIGHT_TAPER (30, 13, 18, 0, 0, 0, 070, 0, 0)
UCUT2<-^UNDERCUT (05, 10, 0, 0, 0, 100, 0, 0)
CYLN3<-^CYLINDER (25, 13, 0, 0, 0, 105, 0, 0)
CYLN4<-^CYLINDER (18, 08, 0, 0, 0, 130, 0, 0)
RCHM1<-^RIGHT_CHAMFER (02, 06, 08, 0, 0, 0, 148, 0, 0)
COMPONENT<-STICK-LCHM1-CYLN1-UCUT1-CYLN2-RTAP1-UCUT2-CYLN3$
-CYLN4-RCHM1
!
! MATERIAL
!
SCOPE MATERIAL
SOLID FAMILY MATERIAL (LENGTH, RADIUS) :MATERIAL
MATERIAL-CYL (LENGTH, RADIUS) AT (ROTY=90)
RADIUS=25
!
! CYLINDER
!
SCOPE CYLINDER
SOLID FAMILY CYLINDER (LENGTH, RADIUS1, X1, Y1, Z1, $
X2, Y2, Z2) :CYLINDER
OBJ1<-CYL (LENGTH, RADIUS) AT (ROTY=90)
OBJ2<-CYL (LENGTH, RADIUS1) AT (ROTY=90)
CYLINDER<-MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
!
! RIGHT_TAPER
!
SCOPE RIGHT_TAPER
SOLID FAMILY RIGHT_TAPER (LENGTH, RADIUS1, $
RADIUS2, X1, Y1, Z1, X2, Y2, Z2) :RIGHT_TAPER
OBJ1<-CYL (LENGHT, RADIUS) AT (ROTY=90)
OBJ2<-CONE (LENGTH, RARIUS1, RADIUS2) AT (ROTY=90)
RIGHT_TAPER<-MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
!
! UNDERCUT
!
SCOPE UNDERCUT
SOLID FAMILY UNDERCUT (LENGTH, RADIUS1, X1, Y1, Z1, $
X2, Y2, Z2) :UNDERCUT
OBJ1<-CYL (LENGTH, RADIUS) AT (ROTY=90)
```

```
OBJ2< -CYL (LENGTH, RADIUS1) AT (ROTY=90)
UNDERCUT< -MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=2S
!
!  RIGHT_CHAMFER
!
SCOPE RIGHT_CHAMFER
SOLID FAMILY RIGHT_CHAMFER (LENGTH, RADIUS1, RADIUS2, $
X1, Y2, Z1, X2, Y2, Z2) :RIGHT_CHAMFER
OBJ1< -CYL (LENGTH, RADIUS) AT (ROTY=90)
OBJ2< -CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY=90)
RIGHT_CHAMFER< -MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
!
!  LEFT_CHAMFER
!
SCOPE LEFT_CHAMFER
SOLID FAMILY LEFT_CHAMFER (LENGTH, RADIUS1, RADIUS2, $
X1, Y1, Z1, X2, Y2, Z2) :LEFT_CHAMFER
OBJ1< -CYL (LENGTH, RADIUS) AT (ROTY=270)
OBJ2< -CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY=270)
LEFT_CHAMFER< -MOVE (OBJ1-OBJ2) BY (X2, 0, 0)
RADIUS=25
```

APPENDIX III: An Outline Process Planning File.

Part material details : Medium Carbon Steel Round Bar

Geometry : A round bar of dimensions L150 × R25 (mm)

Op.no	Oper.name	Machine	Tool – type	Lgth	Rad	MinRad	MajRad	Pic	Wid	Dep	Ang	Pattn
00	Stock			150	25							
01	RChamfer	Lathe	Forming	02		06	08				45	
02	Cylinder	Lathe	Turning	18	08							
03	Cylinder	Lathe	Turning	25	13							
04	Undercut	Lathe	End Cut	05	10							
05	RTaper	Lathe	Turning	30		13	18					
06	Cylinder	Lathe	Turning	20	18							
07	Undercut	Lathe	End Cut	10	13							
08	Cylinder	Lathe	Turning	37	24							
09	LChamfer	Lathe	Forming	03		21	24				45	