A Note On The Calculation And Application Of The Third Absolute Central Moment Of Chi-square Distribution

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ABSTRAK

Kertas ini mengemukakan satu cara mencari momen memusat mutlak ketiga bagi pembolehubah rawak khi-kuasadua. Aturcara SAS (1988) iaitu PROBCHI digunakan bagi menyelesaikan pengamiran berangka. Kegunaannya dalam membina batas yang tepat ke atas ralat penghampiran dalam Teorem Had Memusat diberikan.

ABSTRACT

This paper presents a way of calculating the third absolute central moment of a chi-square random variable. The SAS (1988) function PROBCHI is used to evaluate the numerical integrations. An application of this result in the construction of an exact bound on the error of approximation in the Central Limit Theorem is presented.

INTRODUCTION

In applied statistics, the first two moments of a random variable are of great importance. The first moment locates the centre of the probability density function. The second moment about the mean is the variance which measures the spread of the distribution. The third central moment about the mean is called the skewness of a distribution, and it is a measure of asymmetry of the distribution. The third absolute central moment always exist and is nonnegative. Berry (1941), Esseen (1946) et al., have used the third absolute central moment for the construction of error bounds.

The purpose of this paper is to shown a way of calculating the third absolute central moment of a chi-square random variable. The SAS function PROBCHI is used to evaluate the numerical integration. An example on the application of the third absolute central moment in the Berry-Esseen Theorem is given.

METHODOLOGY

Let Y be a random variable from the chi-square distribution with one degree of freedom. Then, the third absolute central moment of Y is evaluated in the following way.

$$E|Y-1|^3 = \int_0^\infty |Y-1|^3 (f) dy. \qquad \text{Eqn. 1}$$

where f(y) corresponds to the density of Y. Thus,

$$E|Y - 1|^3 = \int_0^\infty \frac{(1/2)^{1/2}}{\Gamma(1/2)} |Y - 1|^3 y^{1/2-1} e^{-y/2} dy$$

<div align="right">Eqn. 2</div>

$$= (1/\sqrt{2\Pi}) \int_0^\infty |Y - 1|^3 Y^{-1/2} e^{-y/2} dy. \text{ since } \Gamma(1/2) = \sqrt{\Pi}.$$

Note that the absolute function, $|y - 1|^3$ can be split into two non-negative function as follows:

$$|Y - 1|^3 = \begin{cases} (1 - Y)^3 \text{ for } 0 < Y < 1 \\ (Y - 1)^3 \text{ for } 1 < Y < \infty \end{cases}$$

Hence,

$$E|Y - 1|^3 = (1/\sqrt{2\Pi}) \left\{ \int_0^1 (1 - y)^3 y^{-1/2} e^{-y/2} dy + \int_1^\infty (y - 1)^3 y^{-1/2 - y/2} dy \right\}$$

<div align="right">Eqn. 3</div>

$$\doteq (1/\sqrt{2\Pi}) \{ C_1 T_1 - 3C_2 T_2 + 3C_3 T_3 - C_4 T_4 \}$$

Where,

$$C_i = 2^{i-(1/2)} \Gamma\{i(1/2)\}$$
$$= \{2^{i-1/2}(2i) i \sqrt{\Pi} / 4^i (i!)$$

and

$$T_i = \int_0^1 (1/C_i) y^{(i-1/2)-1} e^{-y/2} dy - \int_1^\infty (1/C_i) y^{(i-1/2)-1} e^{-y/2} dy$$
$$i = 1, 2, 3, 4.$$

Observe that the integrands in the expression representing T_i are the probability density function of chi-square random variables. To help evaluate (3), the SAS (Statistical Analysis System) function PROBCHI is used. From the SAS output we obtain

$$E|Y - 1|^3 = 8.69152$$

<div align="right">Eqn. 4</div>

APPLICATION

An application of the third absolute central moment can be found in the Berry-Esseen Theorem (Serfling, 1988). Let the distribution function of the normalized sum be

$$G_n(t) = P(S^*_n < t).$$

where

$$S^*_n = \left\{ \sum_i^n X_i - E\left\{ \sum_{i=1}^n X_i \right\} \right\} / \left\{ \text{var} \sum_{i=1}^n X_i \right\}^{1/2}$$

For independent and identically distributed random variable $\{X_i\}$ with mean μ and variance $\sigma^2 > 0$, an exact bound on the error of approximation is provided by the following theorem due to the Berry and Esseen.

Theorem

Let $\{X_i\}$ be independent and identically distributed with mean μ and variance $\sigma^2 > 0$. Then

$$\text{Sup}_t|G_n(t) - \Phi(t)| \leq \{33/4\}(E|X_i - \mu|^3)/(\sigma^3\sqrt{n}), \text{ for all n.} \qquad \text{Eqn. 5}$$

The main idea of this theorem is that, the third absolute central moment can be used to construct a bound on the error of approximation in the Central Limit Theorem.

EXAMPLE

We will present an application of the third absolute central moment in the Berry-Esseen Theorem for characterising the error when using the standard normal distribution to approximate the distribution of the normalized sample variance. Let

$$S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1) \qquad \text{Eqn. 6}$$

be the sample variance. Note that if the random variable $\{X_i\}$ $i = 1, 2, .., n$, are independent and identically distributed $N(0, \sigma^2)$, using the Helmerts matrix (Graybill, 1969), the summation term on the right-hand side of (6) can be written as

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n-1}U_i^2, \qquad \text{Eqn. 7}$$

where the random variable $\{U_i\}$ is independent and identically distributed $N(0, \sigma^2)$. Hence, $\sum_{i=1}^{n-1}U_i^2$ is distributed as $\sigma^2\chi^2_{(n-1)}$ (Graybill, 1976).

Let $Y_i = U_i^2$. Therefore, (6) and (7) show that the sample variance can be written as the average of independent and identically distributed random variable $\{Y_i\}$, $i = 1, 2,..., n-1$, where $\{Y_i\}$ is independent and identically distributed with mean σ^2 and variance $2\sigma^4 > 0$.

In this example, $G_n(t) = P(S_n^* \leq t)$ is the distribution function of S_n^*, where $S_n^* = (s^2 - \sigma^2)/(\sqrt{2\sigma^4/(n-1)})$ is the standardized sample variance. Hence,

$$G_n(t) = P(\sqrt{(n-1)}(s^2 - \sigma^2)/\sqrt{2\sigma^4} \leq t).$$

Without loss of generality, let $\sigma^2 = 1$, therefore

$$G_n(t) = P(\sqrt{(n-1)}(s^2 - 1)/\sqrt{2} \leq t)$$
$$= P((n-1)s^2 \leq \sqrt{2(n-1)}t + (n-1)).$$

Hence,

$$G_n(t) = F(\sqrt{2(n-1)}t + (n-1)) \qquad \text{Eqn. 8}$$

since $(n-1)s^2$ is distributed as $\chi^2(n-1)$, and F denotes the distribution function of a chi-square random variable. Therefore, the left-hand side of (5) can be written as

$$\sup_t|G_n(t) - \Phi(t)| \qquad \text{Eqn. 9}$$
$$= \sup_t|\int_0^u 1/\Gamma[(n-1)/2]2^{[(n-1)/2]-1}x^{[(n-1)/2]-1}e^{-x/2}dx - \int_{-\infty}^t 1/\sqrt{2\Pi}e^{-x^2/2}dx|$$

where $u = \sqrt{2(n-1)}t + (n-1)$. Recall that the chi-square random variable takes on only positive values, thus (8) implies that, $t \geq -\sqrt{(n-1)/2}$.

Using the result from (4), the right-hand side of (5) becomes,

$$(33/4)(E|Y-1|^3)/\sqrt{8n} = (33/4)8.69152/\sqrt{8n}. \qquad \text{Eqn. 10}$$

Now, the Berry-Esseen Theorem can be used to characterize the error of approximation for the distribution of S^*_n. The folowing table gives the preliminary results from the simulation study for comparing (9) and (10).

Table 1: Preliminary results form simulation study for comparing (9) and (10).

$n-1$	t	$\sup_t |G_n(t) - \Phi(t)|$	$\dfrac{(33/4)E|Y-1|^3}{(\sigma^3 \sqrt{\eta})}$
9	0.0287	0.0628394	8.017
18	0.00	0.0443647	5.816
36	0.00736	0.0313453	4.168

DISCUSSION

In this paper a way of evaluating the third absolute central moment of a chi-square random variable is presented. The SAS function PROBCHI helps evaluate the numerical integrations. Preliminary results from the simulation study show that as the degree of freedom $(n-1)$ increases, the supremum error of approximation for $G_n(t)$ decreases. Hence, from the large, the sample theory, the distribution of S_n^* converges to the standard normal distribution as n increases.

Since the purpose of this paper is only to illustrate the method and application of the third absolute central moment, we will not be concerned for the large values of the upperbound. However, interested readers are referred to Serfling (1988).

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