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CONJUGACY CLASS SIZES FOR SOME 2-GROUPS OF NILPOTENCY CLASS TWO

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Abstract. In this paper, we investigate the irreducible characters and the conjugacy class sizes for some 2-groups of class 2. The size of conjugacy classes of an element x in a group G is the order of x^G, where x^G is the conjugacy classes containing x. This research is based on the classification of those groups given by Magidin in 2006. We show that the conjugacy class sizes of G are 2^ρ where $0 \le \rho \le \gamma$ and $|G'| = 2^\gamma$.

Keywords: Irreducible characters; conjugacy classes; 2-generator; 2-groups

Abstrak. Dalam kertas ini, kita menyelidik ciri tak terturunkan dan panjang kelas konjugat bagi kumpulan-2 berpenjana-2 dengan kelas nilpoten 2. Panjang kelas konjugat bagi elemen x dalam kumpulan G adalah peringkat x^G di mana x^G ialah kelas konjugat yang mengandungi x. Kajian ini adalah berdasarkan pada klasifikasi kumpulan yang diberikan oleh Magidin pada tahun 2006. Kita akan membuktikan bahawa panjang kelas konjugat bagi G ialah 2^ρ di mana $0 \le \rho \le \gamma$ dan $|G'| = 2^\gamma$.

Kata kunci: Ciri tak terturunkan; kelas konjugat; berpenjana-2; kumpulan-2

1.0 INTRODUCTION

Let G be a finite 2-generator 2-group of class two and let the set $Sz(G)$ denote the size of the conjugacy classes. The size of conjugacy classes of an element x in a group G is the order of x^G; where x^G is the conjugacy classes containing x.

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Studies on the influence of the size of the conjugacy classes on the structure of a finite group have been there over the years. Many researchers produced papers on this topic, for instance [1,2,3,4,5,6,7,8,9]. However, very little is known about how $Sz(G)$ depends on the order of G'. In 1999, How and Chuang characterized groups in which each conjugacy classes has size not greater than 2, called as property $(*)$. The same authors in [9] proved the following statement:

Theorem 1.1 [9]
Let G be a non-abelian 2-group. Then G satisfies property $(*)$ iff $|G'| = 2$.

Theorem 1.1 has suggested that we study the conjugacy class sizes of 2-generator 2-group of class two. We extend the result by How and Chuang for the converse of Theorem 1.1, as the converse is not true specifically.

It may be worth recalling several results concerning the number of the conjugacy classes which have a counterpart in the context of this research. In this research, Magidin's classification [10] is used to formulate the size of each conjugacy classes for 2-generator 2-group of class two.

Theorem 1.2 [10]
Let G be a finite non-abelian 2-generator 2-group of nilpotency class two. Then G is isomorphic to exactly one group of the following types:

i. $G \cong \left\langle a,b \,\middle|\, a^{2^\alpha} = b^{2^\beta} = [a,b]^{2^\gamma} = [a,b,a] = [a,b,b] = e \right\rangle, \alpha,\beta,\gamma \in \mathbb{N}, \alpha \geq \beta \geq \gamma;$

ii. $G \cong \left\langle a,b \,\middle|\, a^{2^\alpha} = b^{2^\beta} = [a,b,a] = [a,b,b] = e, a^{2^{\alpha+\sigma-\gamma}} = [a,b]^{2^\sigma} \right\rangle, \alpha,\beta,\gamma,\sigma \in \mathbb{N},$
$\beta \geq \gamma > \sigma > 0, \alpha + \sigma \geq 2\gamma, \alpha + \beta + \sigma > 3;$

iii. $G \cong \left\langle a,b \,\middle|\, a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a,b]^{2^\gamma} = [a,b,a] = [a,b,b] = e, a^{2^\gamma} = b^{2^\gamma} = [a,b]^{2^{\gamma-1}} \right\rangle,$
$\gamma \in \mathbb{N}.$

The paper is structured as follows. In Section 2, we fix the notations and state a few propositions needed for Theorem 3.1. In the following section, we provide some lemmas to enact the results which are useful for our main result.

2.0 PRELIMINARIES

In this section, we present several propositions that will be useful for the proof of the main result. All groups considered in this paper are finite and we denote by C_n the cyclic group of order n. The commutator of x and y is given as $[x,y]=x^{-1}y^{-1}xy$ and the commutator subgroup of G denoted by G'. The rest of our notation and definition are standard and may be referred from [11] and [12]. Thereafter onwards, we will refer to groups in Theorem 1.2 as Type 1, Type 2 and Type 3 respectively.

The results tabulated in Table 1, 2 and 3 are for the 2-generator 2-groups of class two with the corresponding parameters, where $|G|$ is the order of the group, $|Z(G)|$ is the order of the center of a group, cl_G is the number of the conjugacy classes and $|G'|$ is the order of the commutator subgroup.

Table 1 Illustration of conjugacy classes for 2-generator 2-groups of class two of order at most 4096 for Type 1

α	β	γ	$\|G\|$	cl_G	$\|Z(G)\|$	$\|G'\|$
1	1	1	8	5	2	2
2	1	1	16	10	4	2
3	1	1	32	20	8	2
4	1	1	64	40	16	2
5	1	1	128	80	32	2
6	1	1	256	160	64	2
2	2	1	32	20	8	2
3	2	1	64	40	16	2
2	2	2	64	22	4	4
4	2	1	128	80	32	2
3	3	1	128	80	32	2
3	2	2	128	44	8	4
5	2	1	256	160	64	2
4	2	2	256	88	16	4
3	3	3	512	92	8	8
4	4	4	4096	376	16	16

Table 2 Illustration of conjugacy classes for 2-generator 2-groups of class two of order at most 2048 for Type 2

α	β	γ	σ	$\lvert G\rvert$	cl_G	$\lvert Z(G)\rvert$	$\lvert G'\rvert$
4	2	2	1	128	44	8	4
3	3	2	1	128	44	8	4
5	2	2	1	256	88	16	4
4	3	2	1	256	88	16	4
3	4	2	1	256	88	256	16
4	4	2	1	512	176	32	4
5	4	3	2	2048	368	32	8

Table 3 Illustration of conjugacy classes for 2-generator 2-groups of class two of order at most 4096 for Type 3

γ	$\lvert G\rvert$	cl_G	$\lvert Z(G)\rvert$	$\lvert G'\rvert$
1	8	5	2	2
2	64	22	4	4
3	512	92	8	8
4	4096	376	16	16

We first establish some basic properties that will be used repeatedly. The following propositions essentially simplify the properties of groups of class two.

Proposition 2.1 [13]

Let G be a group. Then $|Irr(G)|$ equals to the number of conjugacy classes of G and $\sum_{\chi\in Irr(G)} \chi(e)^2 = |G|$ where $|Irr(G)|$ denotes the set of all irreducible characters of G.

Proposition 2.2 [13]

$[G,G']$ = the number of linear characters of G.

Proposition 2.3 [14]

Let G be a 2-generator 2-group of class two of Type 1 denoted by $G = G_1(\alpha,\beta,\gamma)$. Then $|G| = 2^{\alpha+\beta+\gamma}$ and $|Z(G)| = 2^{\alpha+\beta-\gamma}$.

Proposition 2.4 [14]
Let G be a 2-generator 2-group of class two of Type 2 denoted by $G = G_2(\alpha, \beta, \gamma, \sigma)$. Then $|G| = 2^{\alpha+\beta+\sigma}$ and $|Z(G)| = 2^{\alpha+\beta-2\gamma+\sigma}$.

Proposition 2.5 [14]
Let G be a 2-generator 2-group of class two of Type 3 denoted by $G = G_3(\gamma)$. Then $|G| = 2^{3\gamma}$ and $|Z(G)| = 2^{\gamma}$.

3.0 CONJUGACY CLASS SIZES

In this section, we break up the proof of Theorem 3.1 into a sequence of lemmas. These lemmas essentially simplify the results on the size of conjugacy classes for 2-generator 2-groups of class two. In [15], Ahmad introduced the notion of a base group. The author defined the base group as a group which is not an extension such that the image is of the same type.

Lemma 3.1 [14]
Let G be a 2-generator 2-group of class two of Type 1 denoted by $G = G_1(\alpha, \beta, \gamma)$. If $G = G_1(\alpha, \beta, \gamma = 1)$ is the base group, then $cl_G = 2^{\alpha+\beta} + |Z(G)| - 2^{\alpha+\beta-2\gamma}$. Otherwise, if $G = G_1(\alpha, \beta, \gamma)$, then G can be reduced to $H = G_1(\alpha, \beta, \gamma - 1)$ down to the base group such that $cl_G = cl_H + \dfrac{|Z(G)|}{2}$.

Lemma 3.2 [14]
Let G be a 2-generator 2-group of class two of Type 2 denoted by $G = G_2(\alpha, \beta, \gamma, \sigma)$. If $G = G_2(\alpha, \beta, \gamma, \sigma = 1)$ is the base group, then $cl_G = 2^{\alpha+\beta+\sigma-\gamma} + 2^{\alpha+\beta-\gamma} - 2^{\alpha+\beta-2\gamma}$. Otherwise, if $G = G_2(\alpha, \beta, \gamma, \sigma)$, then G can be reduced to $H = G_2(\alpha - 1, \beta, \gamma - 1, \sigma - 1)$, down to the base group such $cl_G = 2cl_H + \dfrac{|Z(G)|}{2}$.

Lemma 3.3 [14]
Let G be a 2-generator 2-group of class two of Type 3 denoted by $G = G_3(\gamma)$. If $G = G_3(1)$ is the base group, then $cl_G = 5$. Otherwise, if $G = G_3(\gamma)$, then G can be reduced to $H = G_3(\gamma - 1)$ down to the base group such that $cl_G = 4cl_H + \frac{|Z(G)|}{2}$.

With the aid of these results, we can generalize the main result on the sizes of conjugacy classes for 2-generator 2-groups of class two.

Theorem 3.1
Let G be a 2-generator 2-group of class two. If $|G'| = 2^\gamma$ and $\gamma \in \mathbb{N}$, then $Sz(G) = \left\{2^\rho \mid 0 \leq \rho \leq \gamma\right\}$.

Proof: We may assume that every conjugacy classes of a group G has size at most 2^γ. Note that each element in the center of the group, $Z(G)$ will form its own conjugacy classes. Since the size of the conjugacy classes of $x \in G$ is equal to $[G : C_G(x)]$, the size of conjugacy classes must be 1 or a positive power of 2. Therefore, we claim that every element which is outside $Z(G)$ form its own conjugacy classes. We divide the proof into three cases; for Type 1, Type 2 and Type 3 respectively.

Case 1: Let G be a 2-generator 2-group of class two of Type 1 denoted by $G = G_1(\alpha, \beta, \gamma)$. From the lemmas in the preceding section, $|G| = 2^{\alpha+\beta+\gamma}$, $|Z(G)| = 2^{\alpha+\beta-\gamma}$ and $|G'| = 2^\gamma$. As an immediate consequence of Lemma 3.1, the number of conjugacy classes is given by $cl_G = 2^{\alpha+\beta} + |Z(G)| - 2^{\alpha+\beta-2\gamma}$. Using Proposition 2.1, there are $2^{\alpha+\beta} + 2^{\alpha+\beta-\gamma} - 2^{\alpha+\beta-2\gamma}$ irreducible characters of G. Thus, this yields the following equality:

$$\begin{aligned}|G| &= \sum_{\chi\in Irr(G),\chi(e)=1}\chi(e)^2+\sum_{\chi\in Irr(G),\chi(e)>1}\chi(e)^2\\ &\geq [G:G']+\left(2^{\alpha+\beta}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}\right)\cdot 2^2\end{aligned}$$

Now, applying $|G|=2^{\alpha+\beta+\gamma}$, $|Z(G)|=2^{\alpha+\beta-\gamma}$ and $|G'|=2^{\gamma}$, we obtain the following inequality.

$$\begin{aligned}2^{\alpha+\beta+\gamma} &\geq \frac{2^{\alpha+\beta+\gamma}}{2^{\gamma}}+\left(2^{\alpha+\beta}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}-\frac{2^{\alpha+\beta+\gamma}}{2^{\gamma}}\right)2^{2}\\ 2^{\alpha+\beta+\gamma} &\geq 2^{\alpha+\beta}+\left(2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}\right)2^{2}\\ 2^{\alpha+\beta+\gamma-2} &\geq 2^{\alpha+\beta-2}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}\\ 0 &\geq 2^{\alpha+\beta-2}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}-2^{\alpha+\beta+\gamma-2}\\ 0 &\geq 2^{\alpha+\beta-2}+2^{\alpha+\beta}\left(\frac{1}{2^{\gamma}}-\frac{1}{2^{2\gamma}}-\frac{2^{\gamma}}{2^{2}}\right)\\ 0 &\geq 2^{\alpha+\beta}\left(\frac{1}{2^{2}}+\frac{1}{2^{\gamma}}-\frac{1}{2^{2\gamma}}-\frac{2^{\gamma}}{2^{2}}\right).\end{aligned}$$

This implies that $2^{\alpha+\beta}\left(\frac{2^{\gamma}+1-2^{2\gamma}}{2^{2\gamma}}\right)\leq 0$. In particular, to sustain the inequality, $\frac{2^{\gamma}+1-2^{2\gamma}}{2^{2\gamma}}$ should be less than 0. Thus, this forces $\gamma\geq 1$ since $\gamma\in\mathbb{N}$.

Case 2:: Let $G=G_2(\alpha,\beta,\gamma,\sigma)$. The conjugacy class sizes of $x\in G$ is $cl_G=2^{\alpha+\beta+\sigma-\gamma}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}$. Thus a short computation, making use of Proposition 2.1, shows that

$$\begin{aligned}|G| &\geq [G:G']+\left(2^{\alpha+\beta+\sigma-\gamma}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}-2^{\alpha+\beta+\sigma-\gamma}\right)\cdot 2^2\\ 2^{\alpha+\beta+\sigma} &\geq \frac{2^{\alpha+\beta+\sigma}}{2^{\gamma}}+\left(2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}\right)2^{2}\\ 2^{\alpha+\beta+\sigma-2} &\geq 2^{\alpha+\beta+\sigma-\gamma-2}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}\\ 0 &\geq 2^{\alpha+\beta+\sigma-\gamma-2}+2^{\alpha+\beta-\gamma}-2^{\alpha+\beta-2\gamma}-2^{\alpha+\beta+\sigma-2}\\ 0 &\geq 2^{\alpha+\beta+\sigma}\left(2^{-\gamma-2}-2^{-2}\right)+2^{\alpha+\beta}\left(2^{-\gamma}-2^{-2\gamma}\right).\end{aligned}$$

The inequality holds if and only if both $2^{-\gamma-2}-2^{-2}$ and $2^{-\gamma}-2^{-2\gamma}$ are at most 0. Hence, this yields $\gamma \geq 1$.

Case 3:: Let $G = G_3(\gamma)$. In [14], it is proved that $cl_G = 2^{2\gamma} + |Z(G)| - 1$. By using Proposition 2.5, it is clear that $|G| = 2^{3\gamma}$ and $|Z(G)| = 2^{\gamma}$. Hence, we obtain the following inequality:

$$
\begin{aligned}
|G| &= \sum_{\chi \in Irr(G), \chi(e)=1} \chi(e)^2 + \sum_{\chi \in Irr(G), \chi(e)>1} \chi(e)^2 \\
&\geq [G:G'] + \left(2^{2\gamma} + 2^{\gamma} - 1 - [G:G']\right) \cdot 2^2 \\
2^{3\gamma} &\geq 2^{2\gamma} + \left(2^{2\gamma} + 2^{\gamma} - 1 - 2^{2\gamma}\right) \cdot 2^2 \\
2^{3\gamma-2} &\geq 2^{2\gamma-2} + 2^{\gamma} - 1 \\
0 &\geq \left(2^{2\gamma-2} + 2^{\gamma} - 2^{3\gamma-2}\right) - 1.
\end{aligned}
$$

Note finally that the inequality holds if and only if $2^{2\gamma-2} + 2^{\gamma} - 2^{3\gamma-2} \leq 1$. This yields γ to be at least 1. The computations for all three cases conclude that $\gamma \geq 1$. Hence, the conjugacy class sizes of G is 2^{ρ} where $0 \leq \rho \leq \gamma$ and $|G'| = 2^{\gamma}$. ■

As an immediate consequence, we obtain the following corollary.

Corollary 3.2: If $G = G_1(\alpha, \beta, \gamma = 1)$, then $\left[1, |Z(G)|\right]$ and $\left[2, cl_G - |Z(G)|\right]$ represent the size of conjugacy classes of G with the respective number of conjugacy classes.

4.0 CONCLUSION

This paper has been devoted to the general formula for the conjugacy class sizes for 2-generator 2-groups of class 2.

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