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# Some Applications of Elliptic Integrals of First Kind

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**Graphical abstract**

$$\int R(x, \sqrt{P_n(x)})dx$$

**Abstract**

One of the most important applications of elliptic integrals in engineering mathematics is their usage to solve integrals of the form $\int R(x, \sqrt{P_n(x)})dx$ (Eq. 1), where $R$ is a rational algebraic function and $P_n(x)$ is a polynomial of degree $n = 3,4$ with no repeated roots. Nowadays, incomplete and complete elliptic integrals of first kind are estimated with high accuracy using advanced calculators. In this paper, several techniques are discussed to show how definite integrals of the form (Eq. 1) can be converted to elliptic integrals of the first kind, and hence be estimated for optimal values. Indeed, related examples are provided in each step to help clarification.

*Keywords*: Elliptic integral; first kind; Legendre's form

**Abstrak**

Salah satu aplikasi paling penting bagi kamiran eliptik dalam matematik kejuruteraan adalah kegunaannya dalam menyelesaikan kamiran dalam bentuk $\int R\left(x, \sqrt{P_n(x)}\right) dx$ (Eq. 1), yang mana $R$ adalah fungsi aljabar nisbah dan $P_n(x)$ adalah polinomial berdarjah $n = 3,4$ dengan tiada punca yang berulang. Kini, kamiran eliptik jenis pertama yang tidak lengkap dan lengkap telah dianggarkan dengan kejituan yang tinggi mengunakan kalkulator canggih. Dalam kertas kerja ini, beberapa teknik dibincangkan bagi menunjukkan bagaimana kamiran tak tentu dalam bentuk (Eq. 1) boleh ditukarkan kepada kamiran eliptik dalam jenis pertama, dan oleh itu, dianggarkan kepada nilai optimum. Malah, contoh-contoh berkaitan diberikan dalam setiap langkah bagi membantu penjelasan.

*Kata kunci* : Kamiran eliptik; jenis pertama; bentuk Lagendre



## ■1.0 INTRODUCTION

The solution of many problems in physics, chemistry and engineering leads to integrals. Indeed, integrals have tremendous applications. They have not only been used in simple calculations such as area and volume, but also, in complex differential equations and complicated astrophysics calculations. Not all indefinite integrals are solvable. However, in the mentioned fields usually definite integrals are used, which more often can be evaluated.

In this paper, some techniques are presented to show how some difficult definite integrals can be altered to elliptic integrals of first kind. Since these integrals can be accurately estimated by using advanced calculators or online calculating websites such as [1, 2], the methods that are given in this paper can be found very helpful.

Elliptic integrals are of three kinds that each kind can be stated in two forms, namely Jacobi and Legendre [3]. Hereby, we only consider elliptic integrals of first kind in their Legendre's forms.

Let $k$ be a real number such that $0 < |k| < 1$. Legendre's forms of incomplete and complete elliptic integrals of first kind are as follows, respectively.

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}}$$

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}}.$$

If $k = 0$, then obviously $F(\phi, k) = \phi$. The complete form is mostly denoted as $K(k)$. By substituting $v = sin\theta$ and $dv = cos\theta\, d\theta = \sqrt{1-v^2}\, d\theta$ in Legendre's form, we can obtain Jacobi's form of these integrals. Moreover, letting $\phi = \frac{\pi}{2}$ implies $x = \sin\frac{\pi}{2} = 1$. Therefore, complete integral in Jacobi's form is denoted as $F_1(1, k)$. Jacobi's form of the incomplete one is given in the following:

$F_1(x,k) = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}.$

In case of interest to see more details concerning other kind of elliptic integrals refer to [3].

There are some methods to evaluate elliptic integrals of first kind. For instance, by successively applying the following formula we may estimate $F(\phi,k)$. This method is called Landen's transformation [4]:

$$F(\phi,k) = \sqrt{\frac{k_1k_2\dots}{k}}\int_0^{\Phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \sqrt{\frac{k_1k_2\dots}{k}}\ln\tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right).$$

Where $\Phi = \lim_{n\to\infty}\phi_n$, $k_n\sin\phi_n = \sin(2\phi_{n+1}-\phi_n)$, $\phi_0=\phi$, $k_{n+1} = \frac{2\sqrt{k_n}}{(1+k_n)}$, and $k_0 = k$.

In [5] by giving an example we have shown that only after two times applying Landen's transformation, an answer which is accurate to 3 decimals would be found.

Moreover, we presented there a proof of the following equivalency to provide a method for evaluating complete integrals of the first kind:

$$F\left(\frac{\pi}{2},k\right) = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1.3}{2.4}\right)^2k^4+\left(\frac{1.3.5}{2.4.6}\right)^2k^6+\cdots\right\}.$$

For more details and examples refer to [5]. In the following tables we present the amount of some elliptic integrals of first kind for some real values of $k$ and $\phi$. These amounts will be used to evaluate the given integrals in the next examples. Table 1 refers to incomplete elliptic integrals of first kind and Table 2 shows some amounts of the complete ones.

**Table 1** Some amounts of incomplete elliptic integrals of first kind

| Entry | $k$ | $\phi$ | $F(\phi,k)$ |
|---|---|---|---|
| 1 | $\frac{2}{3}$ | $\frac{\pi}{6}$ | 0.5342136 |
| 2 | $\sqrt{\frac{5}{6}}$ | $\frac{\pi}{4}$ | 0.8606394 |
| 3 | $\sqrt{\frac{1}{2}}$ | $\frac{\pi}{4}$ | 0.8260038 |
| 4 | $\sqrt{\frac{5}{6}}$ | $\frac{\pi}{3}$ | 1.2421478 |
| 5 | $\frac{2}{3}$ | $\frac{\pi}{3}$ | 1.1295143 |

All values given in Table 1 are calculated using [1].

**Table 2** Some amounts of complete elliptic integrals of first kind

| Entry | $k$ | $F\left(\frac{\pi}{2},k\right)=K(k)$ |
|---|---|---|
| 1 | $\sqrt{\frac{1}{2}}$ | 1.8540666 |
| 2 | $\frac{2}{3}$ | 1.809701 |
| 3 | $\sqrt{\frac{5}{6}}$ | 2.3406433 |

All values given in Table 2 are calculated using [2].

## 2.0 RESULTS AND DISCUSSION

Let $P_n(x)$ be a polynomial of degree 3 or 4 with no repeated real roots. Then, infinite integrals of the form $\int R(x,\sqrt{P_n(x)})dx$, cannot be solved in ease. However, definite form of these integrals can be converted to elliptic integrals of first kind. In this section we present transformations that can be applied in this matter. In all given transformations, $n$ and $m$ are supposed positive real numbers.

**Theorem 3.1** Let $P_4(x) = (x^2+m^2)(x^2+n^2)$ be a polynomial with no real roots. If $m \le n$, then

$$I = \int_{x_0}^{x_1}\frac{dx}{\sqrt{P_4(x)}} = \frac{1}{n}\left[F\left(\theta_1,\frac{\sqrt{n^2-m^2}}{n}\right)-F\left(\theta_0,\frac{\sqrt{n^2-m^2}}{n}\right)\right].$$

Where, $\theta_i = \tan^{-1}\left(\frac{x_i}{m}\right)$ for $i = 0,1$.

**Proof.** The transformation $x = m\tan\theta$ helps us to convert $I$ to an algebraic summation of elliptic integrals of first kind. Thus, we have $dx = m\sec^2\theta\, d\theta$, and $\theta = \tan^{-1}\left(\frac{x}{m}\right)$. By substituting the new variable, $\theta$, we find:

$$I = \int_{x_0}^{x_1}\frac{dx}{\sqrt{(x^2+m^2)(x^2+n^2)}} = \int_{\theta_0=\tan^{-1}\left(\frac{x_0}{m}\right)}^{\theta_1=\tan^{-1}\left(\frac{x_1}{m}\right)}\frac{m\sec^2\theta\, d\theta}{\sqrt{m^2(1+\tan^2\theta)^2[n^2-(n^2-m^2)\sin^2\theta]}}$$

$$= \int_{\theta_0}^{\theta_1}\frac{m\sec^2\theta\, d\theta}{\sqrt{m^2n^2\sec^4\theta\left(1-\frac{n^2-m^2}{n^2}\sin^2\theta\right)}} = \int_{\theta_0}^{\theta_1}\frac{d\theta}{\sqrt{n^2\left(1-\frac{n^2-m^2}{n^2}\sin^2\theta\right)}}$$

$$= \frac{1}{n}\left[F\left(\theta_1,\frac{\sqrt{n^2-m^2}}{n}\right)-F\left(\theta_0,\frac{\sqrt{n^2-m^2}}{n}\right)\right].$$

Note that $x^2+n^2 = m^2\tan^2\theta+n^2 = m^2\sin^2\theta+n^2\cos^2\theta = m^2\sin^2\theta+n^2(1-\sin^2\theta)$. The following solved examples illustrate the application of previous theorem.

**Example 3.2** Evaluate the following definite integral:

$$I = \int_{\sqrt{\frac{5}{3}}}^{\sqrt{15}}\frac{dx}{\sqrt{(x^2+5)(x^2+9)}}.$$

**Solution.** According to Theorem 3.1, we have $m=\sqrt{5}$, $n=3$, $\theta_0 = \tan^{-1}\left(\frac{\sqrt{5/3}}{\sqrt{5}}\right) = \frac{\pi}{6}$ and $\theta_1 = \tan^{-1}\left(\frac{\sqrt{15}}{\sqrt{5}}\right) = \frac{\pi}{3}$. Applying entries 1 and 5 in Table 1 implies:
$I = \frac{1}{3}\left[F\left(\frac{\pi}{3},\frac{2}{3}\right)-F\left(\frac{\pi}{6},\frac{2}{3}\right)\right] = 0.1984336.$

**Example 3.3** Evaluate the following definite integral:

$$I = \int_{\sqrt{\frac{1}{2}}}^{\infty}\frac{dx}{\sqrt{(2x^2+1)(x^2+1)}}.$$

**Solution.** Since $2x^2+1 = 2\left(x^2+\frac{1}{2}\right)$ we have $m = \sqrt{\frac{1}{2}}, n = 1$, $x = \sqrt{\frac{1}{2}}\tan\theta$, $\theta_0 = \tan^{-1}(1) = \frac{\pi}{4}$ and $\theta_1 = \lim_{k\to\infty}\tan^{-1}(k) = \frac{\pi}{2}$. Thus, according to Theorem 3.1 and Tables 1 and 2, we find:
$I = \frac{1}{\sqrt{2}}\left[F\left(\frac{\pi}{2},\sqrt{\frac{1}{2}}\right)-F\left(\frac{\pi}{4},\sqrt{\frac{1}{2}}\right)\right] = 0.7269501.$

**Theorem 3.4** Let $P_3(x)$ be a cubic polynomial that decompose to linear factors. If $P_3(x) = (x-a)(x-b)(x-c)$ where $a < b < c$ are real numbers. Then

$$I = \int_{x_0}^{x_1} \frac{dx}{\sqrt{P_3(x)}} = \frac{2}{\sqrt{c-a}}\left[F\left(\theta_1, \sqrt{\frac{b-a}{c-a}}\right) - F\left(\theta_0, \sqrt{\frac{b-a}{c-a}}\right)\right].$$

Where, $\theta_i = \tan^{-1}\left(\sqrt{\frac{x_i-c}{c-b}}\right)$ for $i = 0,1$.

**Proof.** In this case, we use two transformations. First, letting $x = u^2 + c$ implies $dx = 2udu$ which leads to

$$P_3(x) = u^2(u^2+c-a)(u^2+c-b).$$

Next, using $u = \sqrt{c-b}\,\tan\theta$ and $du = \sqrt{c-b}\,\sec^2\theta\, d\theta$ according to Theorem 3.1, leads to the following procedure:

$$\int \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \int \frac{2udu}{\sqrt{u^2(u^2+c-a)(u^2+c-b)}}$$

$$= \int \frac{2\sqrt{c-b}\sec^2\theta\, d\theta}{\sqrt{[(c-b)\tan^2\theta+(c-a)][(c-b)\tan^2\theta+(c-b)]}}$$

$$= \int \frac{2d\theta}{\sqrt{(c-b)\sin^2\theta+(c-a)\cos^2\theta}} = \int \frac{2d\theta}{\sqrt{(c-a)-(b-a)\sin^2\theta}}$$

$$= \frac{2}{\sqrt{c-a}}\int \frac{d\theta}{\sqrt{1-\frac{b-a}{c-a}\sin^2\theta}}.$$

Hence, $k = \sqrt{\frac{b-a}{c-a}}$ and furthermore

$$I = \frac{2}{\sqrt{c-a}}\left[F\left(\theta_1, \sqrt{\frac{b-a}{c-a}}\right) - F\left(\theta_0, \sqrt{\frac{b-a}{c-a}}\right)\right].$$

Note that $\theta_0 = \tan^{-1}\left(\frac{u_0}{\sqrt{c-b}}\right) = \tan^{-1}\left(\sqrt{\frac{x_0-c}{c-b}}\right)$ and $\theta_1 = \tan^{-1}\left(\frac{u_1}{\sqrt{c-b}}\right) = \tan^{-1}\left(\sqrt{\frac{x_1-c}{c-b}}\right)$.

**Example 3.5** Evaluate the following definite integral:

$$I = \int_9^{\infty} \frac{dx}{\sqrt{(x-2)(x-7)(x-8)}}.$$

**Solution.** In this problem we have $a = 2, b = 7$ and $c = 8$. Thus, according to previous theorem letting $x = u^2 + 8$ and $u = \tan\theta$ leads to $\theta_0 = \tan^{-1}(1) = \frac{\pi}{4}$ and $\theta_1 = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$. Moreover, $k = \sqrt{\frac{b-a}{c-a}} = \sqrt{\frac{5}{6}}$. Hence, by using Table 1 we find:

$$I = \frac{2}{\sqrt{6}}\left[F\left(\frac{\pi}{3}, \sqrt{\frac{5}{6}}\right) - F\left(\frac{\pi}{4}, \sqrt{\frac{5}{6}}\right)\right] = 0.3115003.$$

**Corollary 3.6** Let $P_4(x)$ be a polynomial that decompose to linear factors, such that $P_4(x) = (x-a)(x-b)(x-c)(x-d)$ where $a < b < c < d$ are real numbers. Then, $I = \int_{x_0}^{x_1} \frac{dx}{\sqrt{P_4(x)}}$ can be converted to an integral of the form given in Theorem 3.4.

**Proof.** $x = \frac{At+B}{Ct+D}$, we can reduce the degree of the polynomial that is under the radical to three; and so then alter the integral to the case given in Theorem 3.4. To determine coefficients $A, B, C$ and $D$, we replace $(x,t)$ by $(a,0), (b,1)$ and $(c,\infty)$, respectively. The following equations will be found: $= \frac{B}{D}, b = \frac{A+B}{C+D}, c = \frac{A}{C}$. Thus,

$$A = \frac{cb-ca}{c-b}D, B = aD, C = \frac{b-a}{c-b}D.$$

We may easily consider $D = 1$ and find the other coefficients. Note that, $dx = \frac{(AD-BC)dt}{(Ct+D)^2}$.

The following example shows the application of Corollary 3.6 in details.

**Example 3.7** Reduce the degree of the under radical polynomial using Corollary 3.6.

$$I = \int \frac{dx}{\sqrt{(x+2)(x-2)(x-3)(x-7)}}.$$

**Solution.** Since $a = -2, b = 2$ and $c = 3$ we obtain $A = 12, B = -2, C = 4$ and $D = 1$. Therefore, $x = \frac{12t-2}{4t+1}$ and $= \frac{20dt}{(4t+1)^2}$. Consequently, we find:

$$I = \int \frac{\frac{20dt}{(4t+1)^2}}{\sqrt{\left(\frac{12t-2}{4t+1}+2\right)\left(\frac{12t-2}{4t+1}-2\right)\left(\frac{12t-2}{4t+1}-3\right)\left(\frac{12t-2}{4t+1}-7\right)}}$$

$$= \int \frac{\frac{20dt}{(4t+1)^2}}{\sqrt{\frac{(20t)(4t-4)(-5)(-16t-9)}{(4t+1)^4}}} = \int \frac{dt}{\sqrt{t(t-1)(16t+9)}}.$$

The last integral is of the form given in Theorem 3.4. Note that in the new integral we have $a = -\frac{9}{16}, b = 0$ and $c = 1$. Thus, it can be solved using transformations $t = u^2 + 1$ and $u = \tan\theta$.

If $P_4(x)$ is of the form $(x^2-m^2)(x^2-n^2)$, $(x^2-m^2)(n^2-x^2)$ or $(m^2-x^2)(n^2-x^2)$, then Corollary 3.6 can be applied. However, in part (a) of the following theorem another transformation has presented for the last one in case $m < n$.

**Theorem 3.8** Let $P_4(x)$ be a polynomial of degree 4 with at least a pair of real roots, and $= \int_{x_0}^{x_1} \frac{dx}{\sqrt{P_4(x)}}$.

a. If $P_4(x) = (m^2-x^2)(n^2-x^2)$ where $m < n$, then $I = \frac{1}{n}\left[F\left(\theta_1, \frac{m}{n}\right) - F\left(\theta_0, \frac{m}{n}\right)\right]$.

b. If $P_4(x) = (x^2-m^2)(n^2+x^2)$, then $I = \frac{1}{\sqrt{n^2+m^2}}\left[F\left(\theta_1, \frac{n}{\sqrt{n^2+m^2}}\right) - F\left(\theta_0, \frac{n}{\sqrt{n^2+m^2}}\right)\right]$.

**Proof.** The proof is presented for each case separately.

a) Let $P_4(x) = (m^2-x^2)(n^2-x^2)$ and set $x = m\sin\theta$. Then $dx = m\cos\theta\, d\theta$ and $\theta = \sin^{-1}\left(\frac{x}{m}\right)$. Hence,

$$I = \int_{x_0}^{x_1} \frac{dx}{\sqrt{P_4(x)}} = \int_{x_0}^{x_1} \frac{dx}{\sqrt{(m^2-x^2)(n^2-x^2)}}$$

$$= \int_{\theta_0=\sin^{-1}\left(\frac{x_0}{m}\right)}^{\theta_1=\sin^{-1}\left(\frac{x_1}{m}\right)} \frac{m\cos\theta\, d\theta}{\sqrt{m^2n^2\cos^2\theta\left(1-\frac{m^2}{n^2}\sin^2\theta\right)}}$$

$$= \int_{\theta_0}^{\theta_1} \frac{d\theta}{n\sqrt{(1-\frac{m^2}{n^2}\sin^2\theta)}} = \frac{1}{n}\left[F\left(\theta_1, \frac{m}{n}\right) - F\left(\theta_0, \frac{m}{n}\right)\right].$$

b) In this case, transformation $x = m\sec\theta$ is used. Therefore, $dx = m\sec\theta\tan\theta\, d\theta$, and $\theta = \sec^{-1}\left(\frac{x}{m}\right)$. Thus, we have:

$$I=\int_{x_0}^{x_1}\frac{dx}{\sqrt{P_4(x)}}=\int_{x_0}^{x_1}\frac{dx}{\sqrt{(x^2-m^2)(n^2+x^2)}}=$$

$$\int_{\theta_0=\sec^{-1}\left(\frac{x_0}{m}\right)}^{\theta_1=\sec^{-1}\left(\frac{x_1}{m}\right)}\frac{m\sec\theta\tan\theta\, d\theta}{\sqrt{m^2\tan^2\theta\sec^2\theta\,(n^2\cos^2\theta+m^2)}}=$$

$$\int_{\theta_0}^{\theta_1}\frac{d\theta}{\sqrt{(n^2+m^2-n^2\sin^2\theta)}}=\int_{\theta_0}^{\theta_1}\frac{d\theta}{\sqrt{(n^2+m^2)\left(1-\frac{n^2}{n^2+m^2}\sin^2\theta\right)}}=$$

$$\frac{1}{\sqrt{n^2+m^2}}\left[F\left(\theta_1,\frac{n}{\sqrt{n^2+m^2}}\right)-F\left(\theta_0,\frac{n}{\sqrt{n^2+m^2}}\right)\right].$$

**Example 3.9** Evaluate the following definite integral:
$I=\int_{\sqrt{5/2}}^{\sqrt{5}}\frac{dx}{\sqrt{(5-x^2)(6-x^2)}}$.

**Solution.** According to Theorem 3.8-(a), we have $m=\sqrt{5}$, $n=\sqrt{6}$, $\theta_0=\sin^{-1}\left(\frac{\sqrt{5/2}}{\sqrt{5}}\right)=\pi/4$ and $\theta_1=\sin^{-1}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)=\pi/2$. Thus
$I=\frac{1}{\sqrt{6}}\left[F\left(\frac{\pi}{2},\sqrt{\frac{5}{6}}\right)-F\left(\frac{\pi}{4},\sqrt{\frac{5}{6}}\right)\right]=0.6042091.$

**Example 3.10** Evaluate the following definite integral:
$I=\int_{2\sqrt{5}}^{\infty}\frac{dx}{\sqrt{(x^2-5)(x^2+4)}}$.

**Solution.** In this example, we have $m=\sqrt{5}$, $n=2$, $\theta_0=\sec^{-1}\left(\frac{2\sqrt{5}}{\sqrt{5}}\right)=\pi/3$ and $\theta_1=\lim_{k\to\infty}\sec^{-1}\left(\frac{k}{1}\right)=\pi/2$. Once more we use Theorem 3.8. Considering part (b) of this theorem, we find
$I=\frac{1}{3}\left[F\left(\frac{\pi}{2},\frac{2}{3}\right)-F\left(\frac{\pi}{3},\frac{2}{3}\right)\right]=0.2267289.$

## ■4.0 CONCLUSION

Sometimes elliptic integrals of the first kind can be used to evaluate definite integrals of the form $\int_{x_0}^{x_1}\frac{dx}{\sqrt{P_4(x)}}$. If $P_4(x)$ is a polynomial with no real roots, then $x=m\tan\theta$ is applied to convert the given integral to an algebraic sum of elliptic integrals. In case, two roots of $P_4(x)$ are real, then transformation $x=m\sec\theta$ is used. Finally, let all roots of $P_4(x)$ are real numbers, i.e. it can be decomposed to linear factors. Then, two steps should be taken. In the first step by substituting $x=\frac{At+B}{Ct+D}$ the degree of the problem is decreased to 3. In the second step transformation $x=m\tan\theta$ is applied.


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