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An Example on Computing the Irreducible Representation of Finite Metacyclic Groups by Using Great Orthogonality Theorem Method

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Graphical abstract

$$\left\langle a,b \middle| a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, a^b = a^r \right\rangle$$

Abstract

Representation theory is a study of real realizations of the axiomatic systems of abstract algebra. For any group, the number of possible representative sets of matrices is infinite, but they can all be reduced to a single fundamental set, called the irreducible representations of the group. This paper focuses on an example of finite metacyclic groups of class two of order 16. The irreducible representation of that group is found by using Great Orthogonality Theorem Method.

Keywords: Irreducible representation; metacyclic groups; Great Orthogonality Theorem Method

Abstrak

Teori perwakilan merupakan satu kajian mengenai kesedaran nyata bagi sistem aksiom dalam aljabar niskala. Untuk sebarang kumpulan, bilangan yang mungkin bagi set perwakilan matriks adalah tak terhingga, tetapi ia boleh terturun kepada satu set asas yang dipanggil perwakilan tak terturunkan bagi kumpulan. Artikel ini memfokus kepada satu contoh kumpulan metakitaran terhingga bagi kelas dua berperingkat 16. Perwakilan tak terturunkan bagi kumpulan tersebut telah didapati dengan menggunakan Kaedah Teori Agung Ortogon.

Kata kunci: Perwakilan tak terturunkan; kumpulan metakitaran; Kaedah Teori Agung Ortogon.

■1.0 INTRODUCTION

A group G is metacyclic if it contains a cyclic normal subgroup H and the quotient G/H is also cyclic. Many works on metacyclic groups have been done [1, 2, 3]. In this paper, the irreducible representation of finite metacyclic groups is given. There are also many studies on irreducible representations for some groups including symmetric group by Murnaghan [4], finite classical groups by Lusztig [5] and finite metacyclic groups with faithful irreducible representations by Sim [6]. In this paper we use Great Orthogonality Theorem method to obtain the irreducible representations of finite metacyclic groups. In 2006 the same method has been used to obtain irreducible representations of groups of order 8 by Sarmin and Fong [7].

■2.0 PRELIMINARIES

According to Beuerle [8], there are fourteen types of finite metacyclic groups. We first state some results on the number of conjugacy class of some finite metacyclic groups.

Proposition 2.1[9]
Let G be a finite metacyclic p-group, $G \approx \left\langle a,b \middle| a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, a^b = a^r \right\rangle$, where $\alpha, \beta, \epsilon, \gamma \in \mathbb{N}$ with $\alpha, \beta > 0$ and ϵ, γ non-negative, $r = p^{\alpha-\beta} \pm 1$ and p prime. Then

(i) $|G| = p^{\alpha+\beta}$;
(ii) $|Z(G)| = \langle a^{P^{\gamma}}, b^{P^{\gamma}} \rangle$ *and* $|Z(G)| = P^{\alpha+\beta-2\gamma}$.

Theorem 2.1 [9]
Let G be a non-abelian metacyclic p-group, where p is any prime number. Let G be one of the groups in the following list:

1) $G \approx \left\langle a,b \middle| a^{p^{\alpha}} = b^{p^{\beta}} = 1, [a,b] = a^{p^{\alpha-\gamma}} \right\rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, and $\beta \geq \gamma \geq 1$;
2) $G \approx \left\langle a,b \middle| a^{p^{\alpha}} = b^{p^{\beta}} = 1, [a,b] = a^{p^{\alpha-\gamma}} \right\rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}$, $\gamma - 1 < \alpha < 2\gamma$, and $\beta \geq \gamma$;
3) $G \approx \left\langle a,b \middle| a^{2^{\alpha}} = b^{2^{\beta}} = 1, [a,b] = a^{2^{\alpha-\gamma}} \right\rangle$ where $\alpha, \beta, \gamma, \epsilon \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, and $\gamma \leq \beta$.

4) $G \approx \left\langle a,b \middle| a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, [a,b] = a^{p^{\alpha-\gamma}} \right\rangle$, where $\alpha,\beta,\gamma,\epsilon \in \mathbb{N}, \gamma - 1 < \alpha < 2\gamma,\ \gamma \leq \beta$ and $\alpha < \beta + \epsilon$.

5) $G \approx \left\langle a,b \middle| a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-\epsilon}}, [a,b] = a^{2^{\alpha-\gamma}} \right\rangle$, where $\alpha,\beta,\gamma,\epsilon \in \mathbb{N}, 1 + \gamma < \alpha < 2\gamma$, and $\beta \geq \gamma, \alpha < \beta + \epsilon$,

Then

$$K(G) = P^{\alpha+\beta}\left(\frac{1}{P^{\gamma}} + \frac{1}{P^{\gamma+1}} - \frac{1}{P^{2\gamma+1}}\right),$$

where $K(G)$ is the conjugacy class of G.

3.0 GREAT ORTHOGONALITY THEOREM METHOD

According to Cotton [10] and Sathyanarayana [11], there are five important rules concerning irreducible representations and their characters. Here we rewrite these important rules for some types of finite metacyclic groups, given in Theorem 2.1,

1. The sum of the squares of the dimensions of the irreducible representations of a group is equal to the order of the group, that is,
$$\sum l_i^2 = l_1^2 + l_2^2 + l_3^2 + \cdots = p^{\alpha+\beta} \qquad (1)$$
where l_i is the dimension of the ith representation and $p^{\alpha+\beta}$ is the order of a group[Proposition 2.1].

2. The sum of the squares of the characters in any irreducible representation equals the order of the group, that is,
$$\sum_R [\chi_i(R)]^2 = p^{\alpha+\beta}, \qquad (2)$$
where R is the various operations in the group, $\chi_i(R)$ is the character of the representation of R in the ith irreducible representation and $p^{\alpha+\beta}$ is the order of the group[Proposition 2.1].

3. The vectors whose components are the characters of two different irreducible representations are orthogonal, that is,
$$\sum_R \chi_i(R)\chi_j(R) = 0 \quad \text{when} \quad i \neq j, \qquad (3)$$
where $\chi_i(R)$ is the character of the representation of R in the ith irreducible representation.

4. In a given representation (reducible or irreducible), the characters of all matrices belonging to operations in the same class are identical.

5. The number of irreducible representations of this group is.
$$Irr(G) = P^{\alpha+\beta}\left(\frac{1}{P^{\gamma}} + \frac{1}{P^{\gamma+1}} - \frac{1}{P^{2\gamma+1}}\right), \qquad (4)$$

where $\alpha,\beta,\gamma,\epsilon \in \mathbb{N}$ and $Irr(G)$ is the number of irreducible representations of group [Theorem 2.1].

3.1 Irreducible Representations of Finite Metacyclic Groups of Class Two of Order 16

Let G be a non-abelian metacyclic p-group of nilpotency class two. Then G is isomorphic to this group:

$$G \approx \left\langle a,b \middle| a^{p^{\alpha}} = b^{p^{\beta}} = 1, [a,b] = a^{p^{\alpha-\gamma}} \right\rangle,$$ where $\alpha,\beta,\gamma \in \mathbb{N},\ \alpha \geq 2\gamma, and\ \beta \geq \gamma \geq 1;$

We will find the irreducible representations of this type where $p = 2, \alpha = \beta = 2\ and\ \gamma = 1$. Then $|G| = P^{\alpha+\beta} = 2^{2+2} = 16$ and the group presentation is

$$\langle a,b | a^4 = b^4 = 1, [a,b] = a^2 \rangle.$$

Thus the elements of the group are

$\{1, a, a^2, a^3, b, ab, a^2b, a^3b, b^2, ab^2, a^2b^2, a^3b^2, b^3, ab^3, a^2b^3, a^3b^3\}$.

According to (Theorem 2.1), the number of classes as follows:

$$K(G) = 2^{2+2}\left(\frac{1}{2^1} + \frac{1}{2^{1+1}} - \frac{1}{2^{2\times1+1}}\right) = 10.$$

Then there are ten classes in this group as listed in Table 1:

Table 1 Classes in finite metacyclic group of class two of order 16

C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
1	a, a^3	a^2	b, a^2b	ab, a^3b	b^2	ab^2, a^3b^2	a^2b^2	b^3, a^2b^3	ab^3, a^3b

Then there are ten irreducible representations in this group.
Using Equation (1) in rule 1,

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 + l_6^2 + l_7^2 + l_8^2 + l_9^2 + l_{10}^2 = 2^{2+2} = 16.$$

Therefore, possible values for l_i are $1, 1, 1, 1, 2, 1, 1, 1, 1$ and 2, then this means that there are eight 1-dimensional and two 2-dimensional irreducible representations. However, in any group, there will be a 1-dimensional representation whose character are all equal to 1, then

$$1^2 + 2(1)^2 + 1^2 + 2(1)^2 + 2(1)^2 + 1^2 + 2(1)^2 + 2(1)^2 + 1^2 + 2(1)^2 = 16$$

For remaining 1-dimentional representations, we will use rule 3 equation (3). These 1-dimentional representations to be orthogonal to the first representation, there will have to be eight $+1$'s and eight -1's.

Now, our fifth and tenth representation will be of dimension 2. Hence,

$$\chi_4(C_0) = 2 \text{ and } \chi_9(C_0) = 2$$

In order to find out the values of $\chi_4(C_1)$, $\chi_4(C_2)$, $\chi_4(C_3)$, $\chi_4(C_4)$, $\chi_4(C_5)$, $\chi_4(C_6)$, $\chi_4(C_7)$, $\chi_4(C_8)$ and $\chi_4(C_9)$ we make use of the orthogonality relationships that is in rule 3 Equation (3):

$$\sum_R \chi_0(R)\chi_4(R) = [1][2] + [1]\chi_4(C_1) + [1]\chi_4(C_2) + [1]\chi_4(C_3) + [1]\chi_4(C_4) + [1]\chi_4(C_5) + [1]\chi_4(C_6) + [1]\chi_4(C_7) + [1]\chi_4(C_8) + [1]\chi_4(C_9)$$

$$\sum_R \chi_1(R)\chi_4(R) = [1][2] + [1]\chi_4(C_1) + [1]\chi_4(C_2) + [-1]\chi_4(C_3) + [-1]\chi_4(C_4) + [1]\chi_4(C_5) + [1]\chi_4(C_6) + [1]\chi_4(C_7) + [-1]\chi_4(C_8) + [-1]\chi_4(C_9)$$

$$\sum_R \chi_2(R)\chi_4(R) = [1][2] + [-1]\chi_4(C_1) + [1]\chi_4(C_2) + [1]\chi_4(C_3) + [-1]\chi_4(C_4) + [1]\chi_4(C_5) + [-1]\chi_4(C_6) + [1]\chi_4(C_7) + [1]\chi_4(C_8) + [-1]\chi_4(C_9)$$

$$\sum_R \chi_3(R)\chi_4(R) = [1][2] + [-1]\chi_4(C_1) + [1]\chi_4(C_2) + [-1]\chi_4(C_3) + [1]\chi_4(C_4) + [1]\chi_4(C_5) + [-1]\chi_4(C_6) + [1]\chi_4(C_7) + [-1]\chi_4(C_8) + [1]\chi_4(C_9)$$

$$\sum_R \chi_5(R)\chi_4(R) = [1][2] + [1]\chi_4(C_1) + [1]\chi_4(C_2) + [1]\chi_4(C_3) + [1]\chi_4(C_4) + [-1]\chi_4(C_5) + [-1]\chi_4(C_6) + [-1]\chi_4(C_7) + [-1]\chi_4(C_8) + [-1]\chi_4(C_9)$$

$$\sum_R \chi_6(R)\chi_4(R) = [1][2] + [1]\chi_4(C_1) + [1]\chi_4(C_2) + [-1]\chi_4(C_3) + [-1]\chi_4(C_4) + [-1]\chi_4(C_5) + [-1]\chi_4(C_6) + [-1]\chi_4(C_7) + [1]\chi_4(C_8) + [1]\chi_4(C_9)$$

$$\sum_R \chi_7(R)\chi_4(R) = [1][2] + [-1]\chi_4(C_1) + [1]\chi_4(C_2) + [1]\chi_4(C_3) + [-1]\chi_4(C_4) + [-1]\chi_4(C_5) + [1]\chi_4(C_6) + [-1]\chi_4(C_7) + [-1]\chi_4(C_8) + [1]\chi_4(C_9)$$

$$\sum_R \chi_8(R)\chi_4(R) = [1][2] + [-1]\chi_4(C_1) + [1]\chi_4(C_2) + [-1]\chi_4(C_3) + [1]\chi_4(C_4) + [-1]\chi_4(C_5) + [1]\chi_4(C_6) + [-1]\chi_4(C_7) + [1]\chi_4(C_8) + [-1]\chi_4(C_9)$$

From rule 2 Equation (2) in section (3.0) we have:

$$4 + 2[\chi_4(C_1)]^2 + [\chi_4(C_2)]^2 + 2[\chi_4(C_3)]^2 + 2[\chi_4(C_4)]^2 + [\chi_4(C_5)]^2 + 2[\chi_4(C_6)]^2 + [\chi_4(C_7)]^2 + 2[\chi_4(C_8)]^2 + 2[\chi_4(C_9)]^2 = 16$$

This gives $\chi_4(C_1) = \chi_4(C_3) = \chi_4(C_6) = \chi_4(C_8) = \chi_4(C_9) = 0$, $\chi_4(C_2) = \chi_4(C_7) = -2$ and $\chi_4(C_5) = 2$.
The tenth irreducible can be shown similarly.

Then the complete set of characters of the five irreducible representations is given in Table 2.

Table 2 Irreducible representations of $G \cong \langle a, b \mid a^4 = b^4 = 1, [a, b] = a^2 \rangle$

	C_0	$2C_1$	C_2	$2C_3$	$2C_4$	C_5	$2C_6$	C_7	$2C_8$	$2C_9$
Γ_0	1	1	1	1	1	1	1	1	1	1
Γ_1	1	1	1	−1	−1	1	1	1	−1	−1
Γ_2	1	−1	1	1	−1	1	−1	1	1	−1
Γ_3	1	−1	1	−1	1	1	−1	1	−1	1
Γ_4	2	0	−2	0	0	2	0	−2	0	0
Γ_5	1	1	1	1	1	−1	−1	−1	−1	−1
Γ_6	1	1	1	−1	−1	−1	−1	−1	1	1
Γ_7	1	−1	1	1	−1	−1	1	−1	−1	1
Γ_8	1	−1	1	−1	1	−1	1	−1	1	−1
Γ_9	2	0	−2	0	0	−2	0	2	0	0

4.0 CONCLUSION

Great Orthogonality Theorem Method has been used to find the irreducible representations of some types of finite metacyclic groups. Five important rules to find the irreducible representations of finite metacyclic groups of class two of order 16 with the character table have been applied.

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