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Laboratory Measurement of Displacement on Shallow Foundation in Uniform Sand using Particle Image Velocimetry Technique

Siti Noor Linda Taib[a*], Lau See Hung[a], Prabir Kumar Kolay[b]

[a]*Universiti Malaysia Sarawak, Faculty of Engineering, 94300 Kota Samarahan, Sarawak, Malaysia*
[b]*Southern Illinois University, Carbondale, U.S.A*

*Corresponding author: tlinda@feng.unimas.my

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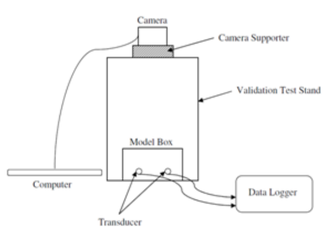
Graphical abstract



Abstract

Particle Image Velocimetry (PIV) is one of the non-intrusive techniques recently being utilized for measuring soil displacement in geotechnical engineering. The work discussed in this paper focuses on the application of an image processing tool known as MatPIV, a freeware readily available to the general users. The test programmes involved measurement and visualization of the behavior of displaced soil under various type of loadings and sizes of the foundation plate. Prior to testing, validation test was performed to study on the validity of measurements produced by MatPIV by comparing its measurements with the actual. Boundary analysis was also performed to justify on the configuration chosen for the model box used in both validation and experimental set-ups. In addition, a set of 70 displacement data was also studied in the validation test and a further statistical analysis on the data showed an acceptable accuracy of 98.8% for the MatPIV analysis when compared to the actual measured displacement. Hence, further test programmes which consisted of various soil-structure interaction conditions were performed. This comprehensive analysis showed that the diameter of loading rods and the sizes of foundation plate do affect the behavior of soil displacement under centric structural loading and was shown successfully via MatPIV.

Keywords: Foundation; MatPIV; particle image velocimetry (PIV); soil displacement; soil-structure interaction

Abstrak

Kaedah *Particle Image Velocimetry (PIV)* adalah salah satu teknik pengukuran tanpa terobosan yang baru-baru ini diaplikasi bagi mengukur pergerakan tanah di dalam bidang kejuruteraan geoteknikal. Kertas penyelidikan ini mengfokus kepada penggunaan teknik memproses imej dan gerakan iaitu MatPIV, aplikasi percuma di laman sesawang yang boleh diperolehi umum. Penyelidikan ini melibatkan pengukuran dan observasi ke atas sifat-sifat pergerakan tanah disebabkan oleh bebanan dan saiz plat tapak yang berbeza. Sebelum ujian dilaksanakan, ujian validasi dijalani dahulu bagi mengukuhkan bacaan yang diperolehi melalui MatPIV dengan membandingkannya kepada bacaan sebenar. Analisa ke atas impak batas kotak model juga dijalankan bagi mengesahkan dimensi yang sesuai untuk kotak model yang akan digunakan. Selain dari itu, sebanyak 70 data bacaan pergerakan tanah juga diukur melalui MatPIV dan bacaan sebenar dan analisa statistik menunjukkan yang data-data tersebut mencapai tahap ketepatan sebanyak 98.8%. Oleh itu, dengan tahap ketepatan yang tinggi ini, ujian selanjutnya yang melibatkan pelbagai kondisi interaksi antara tanah dan struktur dapat dilaksanakan dengan yakin. Analisa yang komprehensif ini memberi rumusan berikut di mana diameter rod beban dan saiz plat tapak memberi kesan ke atas sifat-sifat pergerakan tanah yang dibebankan secara memusat dan ini dapat ditunjukkan dengan jayanya melalui MatPIV.

Kata kunci: Tapak; MatPIV; particle image velocimetry (PIV); pergerakan tanah; interaksi tanah dan struktur

1.0 INTRODUCTION

Particle image velocimetry (PIV) has been widely used in fluids analysis for measuring the instantaneous flow of velocity field within a short time interval [1]. The vast development of photography technology triggers the growing interest on PIV. Thus, significant amount of studies in developing the theories and applications of PIV in fluid and particle analysis had been carried out. The PIV measuring technique has benefited in the characterization of soil fabric in the particle analysis identifying the pattern of deformed soil [2]. In addition, this method is also used to illustrate the evolution of soil microstructure that is by analyzing the flow pattern of soil deformation during shear ([3]; [4]; [5] and [6]).

In geotechnical engineering, PIV had been used in monitoring soil displacement during shear without introducing any intrusive tracing particles in the observed sample for the centrifuge modeling ([7]; [8]; [9]; [10]; [11]; [12]; [13]; [14]; [15]). The traditional method for measuring the planar deformation in geotechnical centrifuge test is to mark an exposed plane of soil with targets or to colour the layers of soil with dye or ink ([16]; [17]). The deformation of the targets or layers was then captured using video or film photography by monitoring only the mid plane of the sample [18].

White *et al*., 2003 [14] first described the use of PIV was beneficial as no intrusive tracing particles were utilized within the deforming soil. Instead, the method relied solely on the soil texture to track the displacement of given element of soil between two subsequent images. In view of this, studies were carried out to design a new set-up in the UNIMAS geotechnical laboratory to enable the study for measuring and visualizing the profile of soil displacement for a simple soil-structure under structural loading. The set-up would then be introduced in undergraduate laboratory and graduate practices in UNIMAS.

This paper presented the adaptability of MatPIV [19], a freeware readily available to general users, with a simple soil-structure interaction in geotechnical engineering. This was carried out by first, modeling the test set-up and devising a standard method to validate the image-space and object space measurements prior to the study of the soil profile and the effects on load-displacement behavior. Observation on the key phenomena such as displacement profile of the soil is essential in understanding the mechanism of a shallow foundation system.

2.0 MATERIALS AND METHODS

2.1 The Validation Test Set-Up

The validation test was performed before the actual test and was performed to validate the MatPIV measurement by comparing it with the actual measurement of displacement. Figure 1 shows the set-up of this validation test which consists of a digital camera, a device for supporting the camera, a desktop for downloading the images, Unconfined Compression Strength (UCS) machine with attached loading fixture, loading plate, displacement transducer and a data logger. The size of the model box is chosen based on a numerical analysis performed to investigate the effect of boundary condition to the displacement of shallow foundation and is discussed in Section 2.2.

Validation test was carried out by capturing images of displaced soil located directly beneath a loading plate which was displaced by a known distance measured using the Linear Variable Displacement Transducer (LVDT). The images of initial and final movements of the soil were captured and analysed using MatPIV to obtain the magnitude of displacement. The analysed displacement was then compared to the measured value to obtain the distribution and percentage of errors produced in the MatPIV analysis using basic statistical analysis. A second validation test was also performed on a slightly different set-up in which a horizontal displacement was introduced to a board with target markers on it. The detail of the set-up is discussed in Section 3.2.

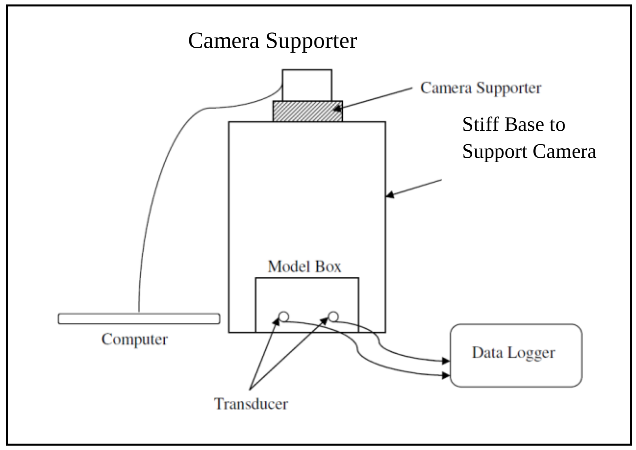


Figure 1 Top view of the validation test set-up (not to scale)

2.2 The Model Box Design

The decision on the size for the model box is based on the boundary analysis performed using PLAXIS software. In the modeling, a number of assumptions were made regardless of the type of soil sample applied. Firstly, the size of the foundation plate was assumed to be a square with dimension of 55 × 55 mm and the depth of the soil sample was 100 mm. Secondly, the foundation plate was considered to be very stiff and rough. Thirdly, it was assumed that the loading was applied directly at the center of the foundation plate and fourthly the stress was distributed evenly to the soil particles to ensure symmetrical outcomes in both sides of the sample from the central axis. Therefore, it was sufficient to analyze only half of the model box as shown in Figure 2.

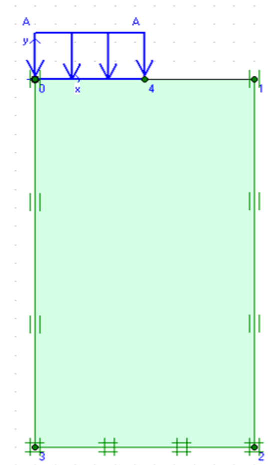


Figure 2 PLAXIS modeling for symmetrical model box

Analyses were carried out by increasing the applied centric loading and the width of sample box (w) relative to the size of the foundation plate (b). The flow and the mode of soil displacement were then studied to comprehend the behavior of the soil displacement in relative to the width of the model box and the effects of the model box's size in restraining the movement of soil particles in an analysis. Figure 3 shows the flow and the mode of soil displacement for a constant 5 N/mm^2 structural loading with an increased of width, w of the model box

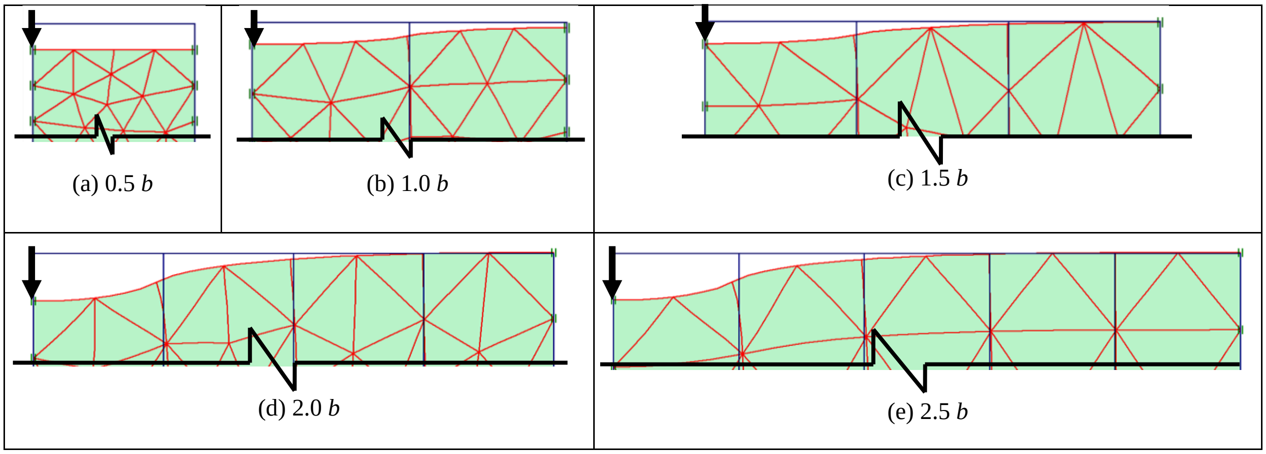


Figure 3 Effects of soil displacement in relative to the size of the model box (*w*) under constant loading

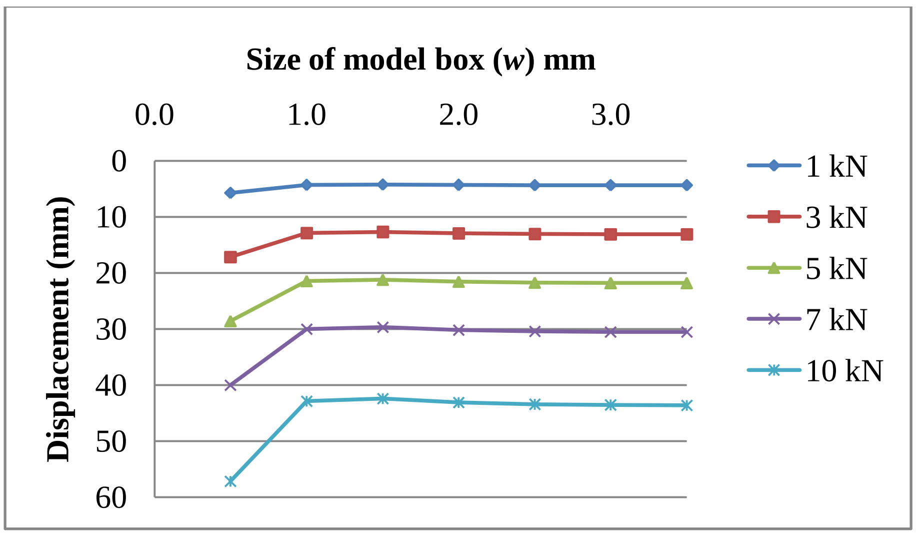


Figure 4 Graph of displacement (mm) versus the ratio of the width of the model box (w)

The studies showed that when the width of model box, w, equals to 0.5b, the soil displacement was fully restrained by the width of the model box. Therefore, extent of the soil displacement could not be examined as displayed in Figure 3 (a) in which displacement was found to be uniform throughout the width of the box. However, as the width of the model box increased, the displacement of the soil was observed to be distributed to the right and gradually diminished to a slight effect of displacement at the right end as observed in Figure 3 (b). In order to design a model box to cater for a soil displacement analysis, the size of the model box was studied to ensure that there would be minimal restraining effects onto the soil sample when loading is applied as shown in Figure 3 (d) and 3 (e).

Figure 4 shows the effects of the width of model box, w, to the amount of displacement under increasing centric loading. The graph shows that when the size of model box increased from 0.5b to 1.5b, the displacement of the soil sample increased at a significant value. The limited width of the model box restricted the further movement of the soil particles and led to the apparent displacement occurring uniformly within the boundary. Further increment of the width for model box from 1.5b to 2.5b shows that the displacement of the soil sample maintained constant at the point of measurement and reduced indicating distribution of displacement in the soil. This verifies that the size of the model box ranging from 1.5b to 2.5b was sufficient to cater for the displacement of the soil sample without introducing much restriction to the movement of the soil particles.

Vesic, 1973 [20] tabulated 15 theoretical solutions since 1940 to obtain the ultimate bearing capacity of a foundation other than estimations. There had been little experimental verification on any of the methods except by using model footings. However, Vesic suggested that using models of b = 25 – 75 mm and w = 25 – 200 mm was popular and acceptable because the ultimate load can be developed in a small prepared box of soil in the laboratory using commonly available compression machines on the order of 400 kN capacity. In view of this, a factor of 1.5b was chosen for the modeling of the model box. Thus, a model box of width and depth of 150 mm was chosen to avoid boundary effect problem and to produce ultimate load in this simple soil-structure interaction test.

2.3 The Experimental Set-Up

In this research, experiments were carried out using different size of loading rods (5 and 10 mm in diameter). Various sizes (i.e. width, length) and shapes (i.e. square, rectangular) of foundation plate were also tested to study on their effects on the profile and magnitude of the soil displacement. The general experimental set-up is shown in Figure 5. A UCS machine was used to load the foundation plate via a loading rod. The model box was filled with sand and a digital camera which was connected to a desktop was focusing the target area beneath the foundation. An available software from the camera's manufacturer was used to capture the images automatically at designated time interval.

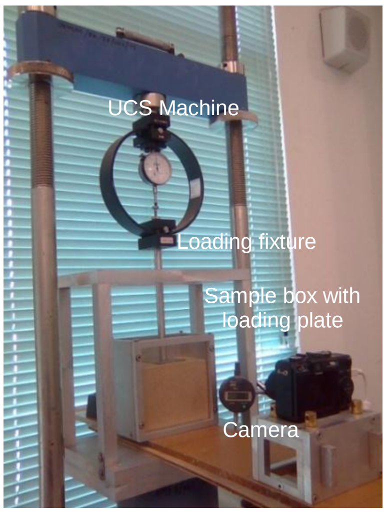


Figure 5 Experimental set-up

2.4 Soil Properties

The load tests were carried out on a bed of uniform sand in which the physical properties were predetermined. Several tests were carried out which include the sieve analysis, specific gravity, Proctor test and Triaxial Unconsolidated Undrained (UU) test. The soil's physical properties are shown in Table 1.

Table 1 Physical properties of the uniform sand

Analysis	Result
Sieve analysis	Poorly graded sand passing 600µm retain 20µm
Specific gravity	2.64
Angle of friction, ϕ (°)	34.4
Cohesion, c, kN/m^2	0
Dry density, γ_{d}, kN/m^3	14.2
Moisture content, %	19
Average void ratio, e	0.86

2.4 The Experimental Programme

The experimental works were carried out on various widths and lengths of the foundation plate (i.e., 40×40 mm, 40×45 mm, 40×60 mm and 40×80 mm) with two diameters of loading rods (5 and 10 mm) as specified earlier. All the plates were centrically loaded via the different size loading rods.

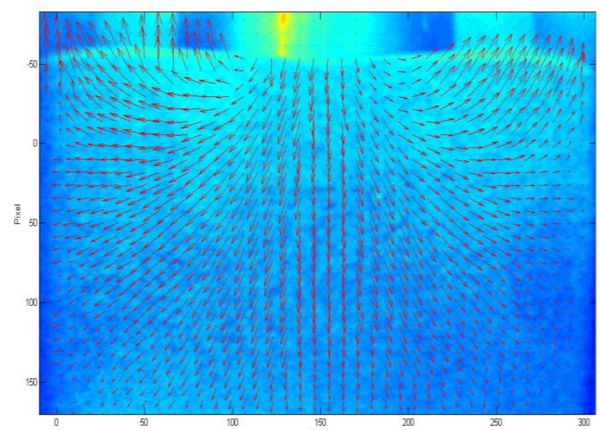
■3.0 RESULTS AND DISCUSSION

3.1 Preliminary Validation Test

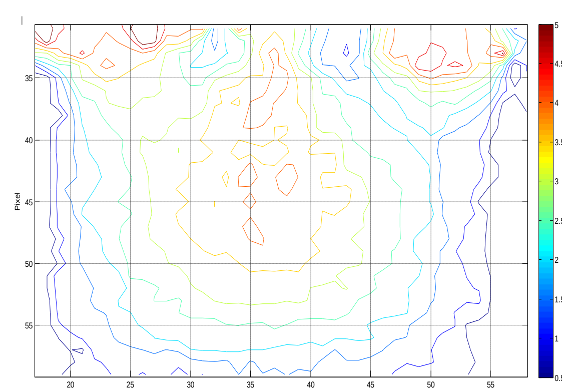
The validation test was carried out to validate the relationship of a point in the object-coordinate system to the corresponding point in the image-place coordinate system. In order to emulate the actual condition of the simple soil-structure condition in the test programme, the first validation test was carried out by placing uniform sand in a model box with a 5 mm thick Perspex glass.

A known displacement measured using the LVDT was introduced to the soil. Images of the displaced soil were captured using a camera that was positioned directly perpendicular to the plane of the target area and stored in a dedicated desktop for the MatPIV analysis.

The analysis for a validation test was shown in Figure 6. Figure 6(a) shows the direction of the soil movement, while, the magnitude of the displacement is shown in Figure 6(b). In Figure 6(b), the different color of the contour line represents the amount of displacement in the sample. The maximum displacement of the soil particles measured using the MatPIV is 5.042 mm compared to the measured value of 5.069 mm using the LVDT. The measurement from MatPIV is found to be marginally close to the LVDT's measurement.



(a)Enlarged vectors on the direction of displacement



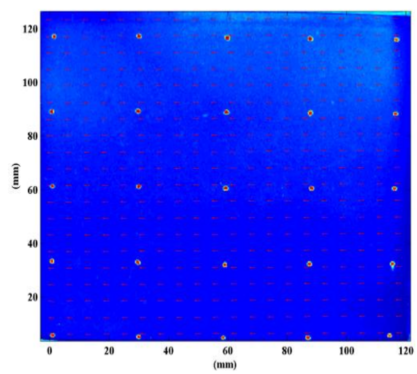
(b) Magnitude of the displacement (mm)

Figure 6 MatPIV's results of validation test

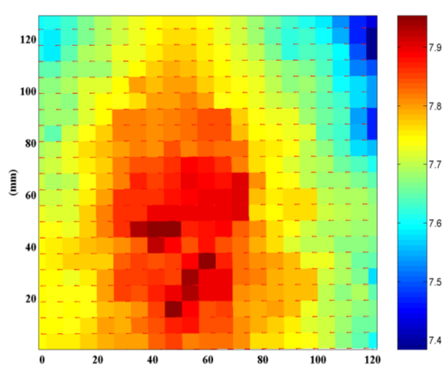
3.2 Further Validation with Statistical Analysis

A set of 70 data for another condition of validation test was carried out to study on the distribution of the data. In this study, the second validation test involves capturing a grid of 30 mm apart white target dots in a dark background as shown in Figure 7. This much simpler set-up was opted for as only horizontal displacement was measured as opposed to measuring the much scattered movement of soil particles. The calibration board was located directly beneath a 5 mm perspex glass and displaced horizontally by a known displacement of 7.773 mm. The result of the analysis in Figure 8 shows that the values of the MatPIV measurement were dispersed between 7.678 – 7.953 mm range.

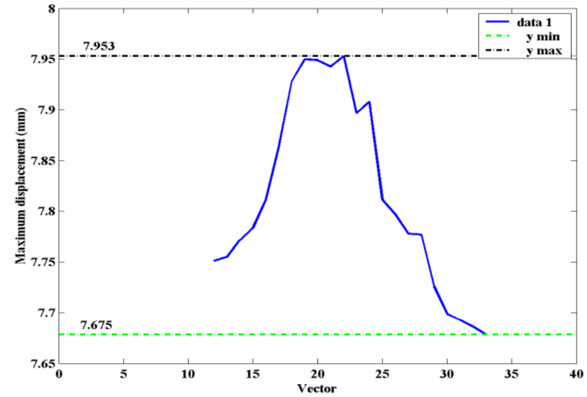
Figure 7 Validation board



(a) Direction of displacement



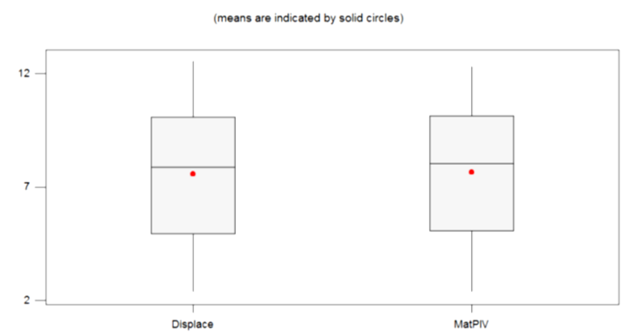
(b)Magnitude of displacement (mm)



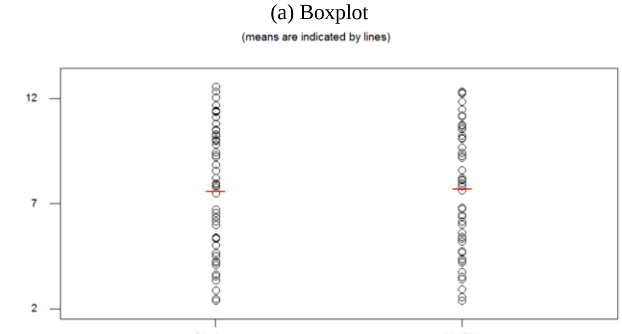
(c) Range of displacement (mm)

Figure 8 Validation test for 7.773 mm horizontal displacement measured using LVDT

Figure 9 shows the boxplot and dotplots for the 70 data set of displacement measured using LVDT and MatPIV. The mean error for the displacement was -0.088 for a 95% confidence interval with accuracy up to 98.8%. With excellent accuracy value, the reading from the MatPIV is considered acceptable.



(a) Boxplot



(b) Dotplots

Figure 9 Boxplot and dotplots for measured displacement versus MatPIV

3.3 Load Displacement Analysis

Eight analyses were carried out to study on the effects of various diameters of loading fixtures and increments of the size of the foundation plates on the profile and magnitude of displacement in a sand sample. The images were captured at every 10 seconds during continuous load being applied to the sand sample via the UCS machine. Images were then stored and analysed using

MatPIV [19] to study the profile and magnitude of soil displacement as well as the strain of the soil.

Foundation plates with different length (*l*) over the width (*b*) ratio of 1.0, 1.125, 1.5 and 2.0 were studied for the effects of the increase in size of the foundation plate to the soil profile and the magnitude of soil displacement under continuous centric loading. The images of the deformed soil sample were incessantly recorded with the camera and then transferred to the desktop for further analysis.

Figure 10 shows the images of both the 5 mm and 10 mm loading rods centrically placed on different sizes of the foundation plate prior to the PIV analysis. The stress acting on the surface of the soil sample and the displacement in the sand sample were then studied to measure the load-displacement analysis of the foundation plate.

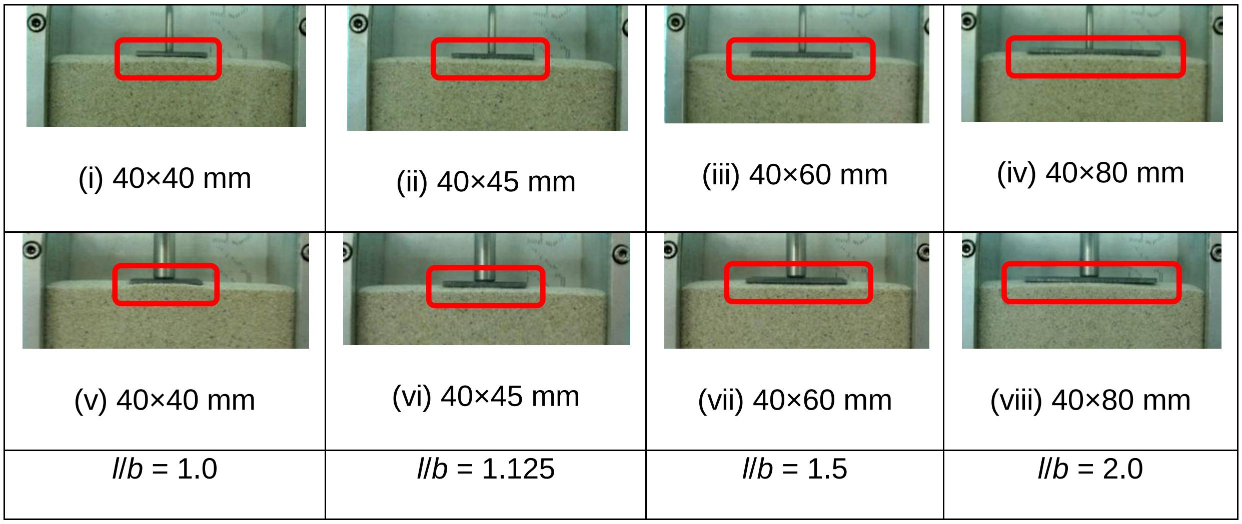
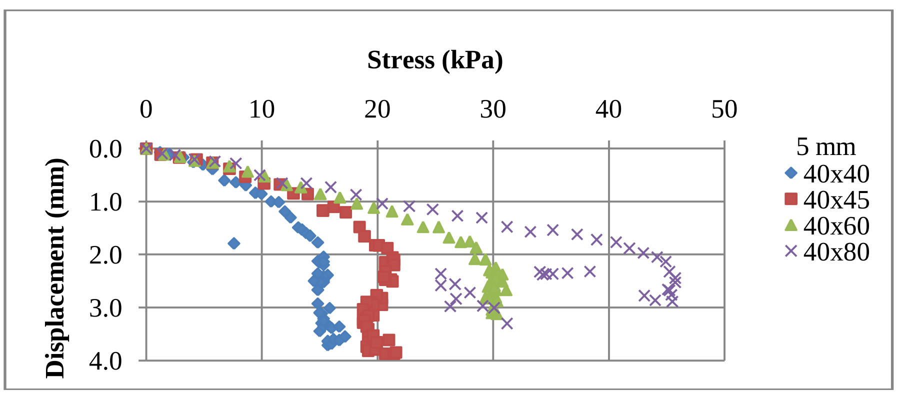
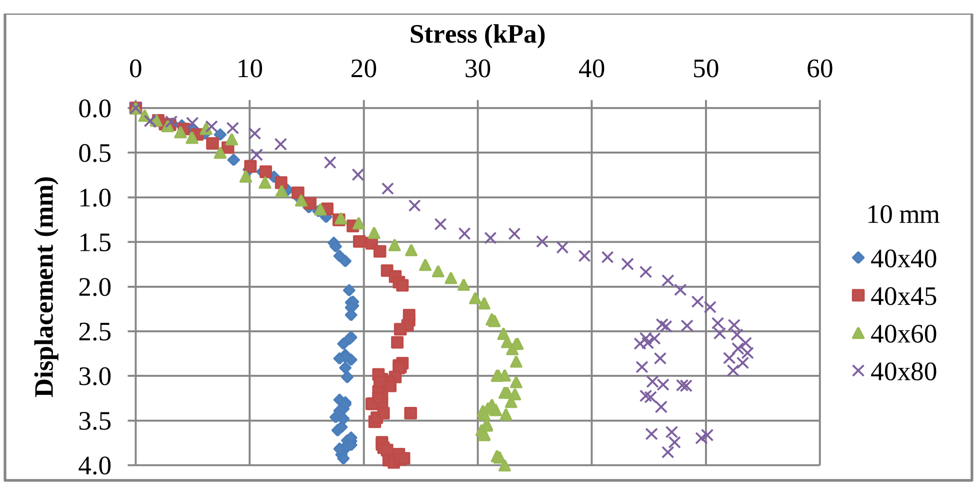


Figure 10 Centrically loaded foundation plate of *l*/*b* Ratio of 1.0, 1.125, 1.5 and 2.0 for 5 mm and 10 mm loading rods



(a) 5mm loading rod



(b) 10mm loading rod

Figure 11: Effect of different sizes of foundation loading plate and different loading rod diameters on bearing capacity and displacement

The load-displacement graph of the soil sample for the foundation plates under various structural loadings was plotted to identify the ultimate load that could be sustained by the soil under each foundation plate and is shown in Figure 11.

For a 5 mm loading rod, the soil's bearing capacity under 40 × 40 mm (*l/b* = 1.0) plate was recorded to be approximately 15 kPa. When the size of the foundation plate increased to 40 × 45 mm (*l/b* = 1.125), 40 × 60 mm (*l/b* = 1.5) and 40 × 80 mm (*l/b* = 2.0) the ultimate bearing capacity was read at 21 kPa, 31 kPa and 45 kPa respectively. However, in the study of a 10 mm diameter loading rod, the soil's bearing capacity for 40 × 40 mm (*l/b* = 1.0) plate was measured to be approximately 18.9 kPa. When the size of the foundation plate increased to 40 × 45 mm (*l/b* = 1.125), 40 × 60 mm (*l/b* = 1.5) and 40 × 80 mm (*l/b* = 2.0), the ultimate bearing capacity was read as 23 kPa, 33 kPa and 53 kPa respectively.

In general, it is found that when the (*l/b*) ratio of the foundation plate increased, the bearing capacity of the foundation increased accordingly regardless of the diameter of the loading rod being used in the analysis. On the other hand, when the diameter of the loading rod increased from 5 mm to 10 mm, the stress acted upon the foundation plate increased as well. It is concluded that the increased in the length over width ratio (*l/b*) of foundation plate increased the bearing capacity of the foundation. This was due to the increase of the surface area of the foundation plate in contact with the soil sample.

3.4 Profile, Magnitude and Strain of Soil Displacement from MatPIV Analysis

Four different types of the foundation plates (40 × 40 mm, 40 × 45 mm, 40 × 60 mm and 40 × 80 mm) were selected for studying the effects of the foundation plate's size to the profile and magnitude of soil displacement under centric loading. Images of the deformation occurred in the soil sample were captured at every 10 seconds and later analysed using MatPIV.

Prior to the MatPIV analysis, the deformation of the soil sample could only be visualized by observing the major failure surface extended to the ground surface as shown in Figure 12. The pattern and profile of soil displacement could neither be clearly viewed nor measured manually using bare eyes observations. Thus, in order to envisage the mode of displacement, MatPIV is adapted to analyse the displacement and visualize the changes in the soil deformation when the continuous loading was applied to the foundation plate.

The MatPIV analysis on the soil sample under 5 mm diameter loading rod placed centrically on 40 × 60 mm foundation plate is used to describe the changes of the profile and magnitude of the displacement and the strain in a soil sample for time, $t_{initial}$ = 60s and t_{final} = 300s and the images are displayed respectively in Figure 13.

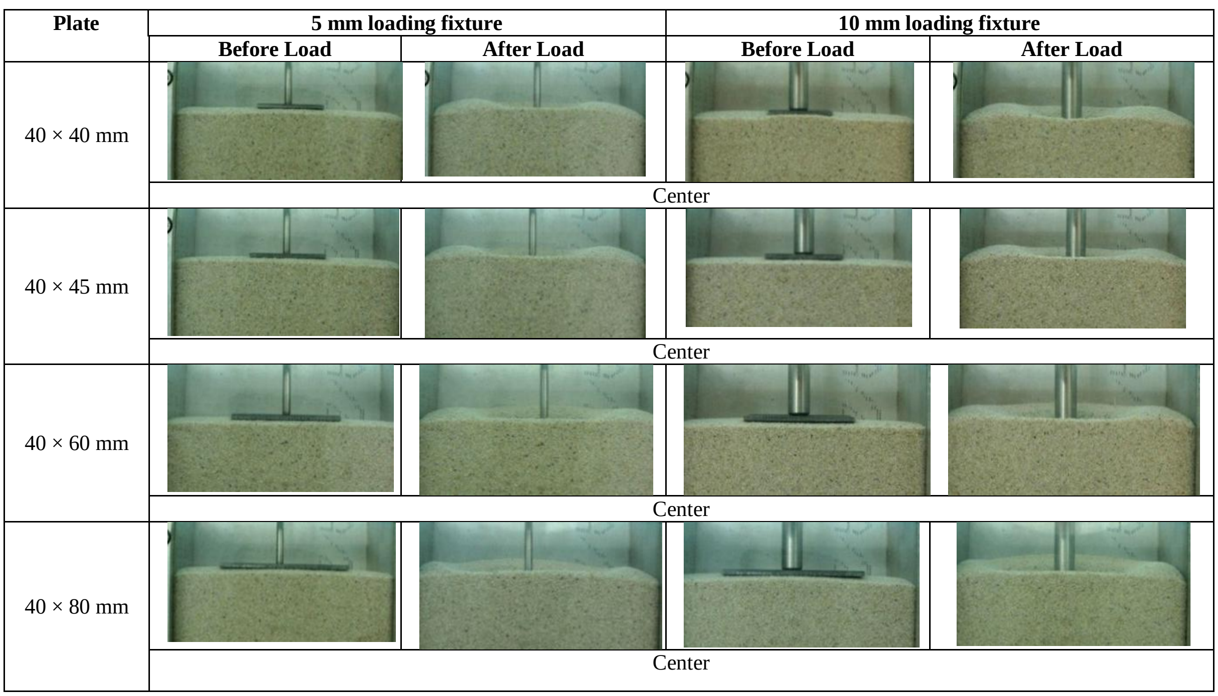


Figure 12 Profile of soil displacement for centric loaded foundation plate

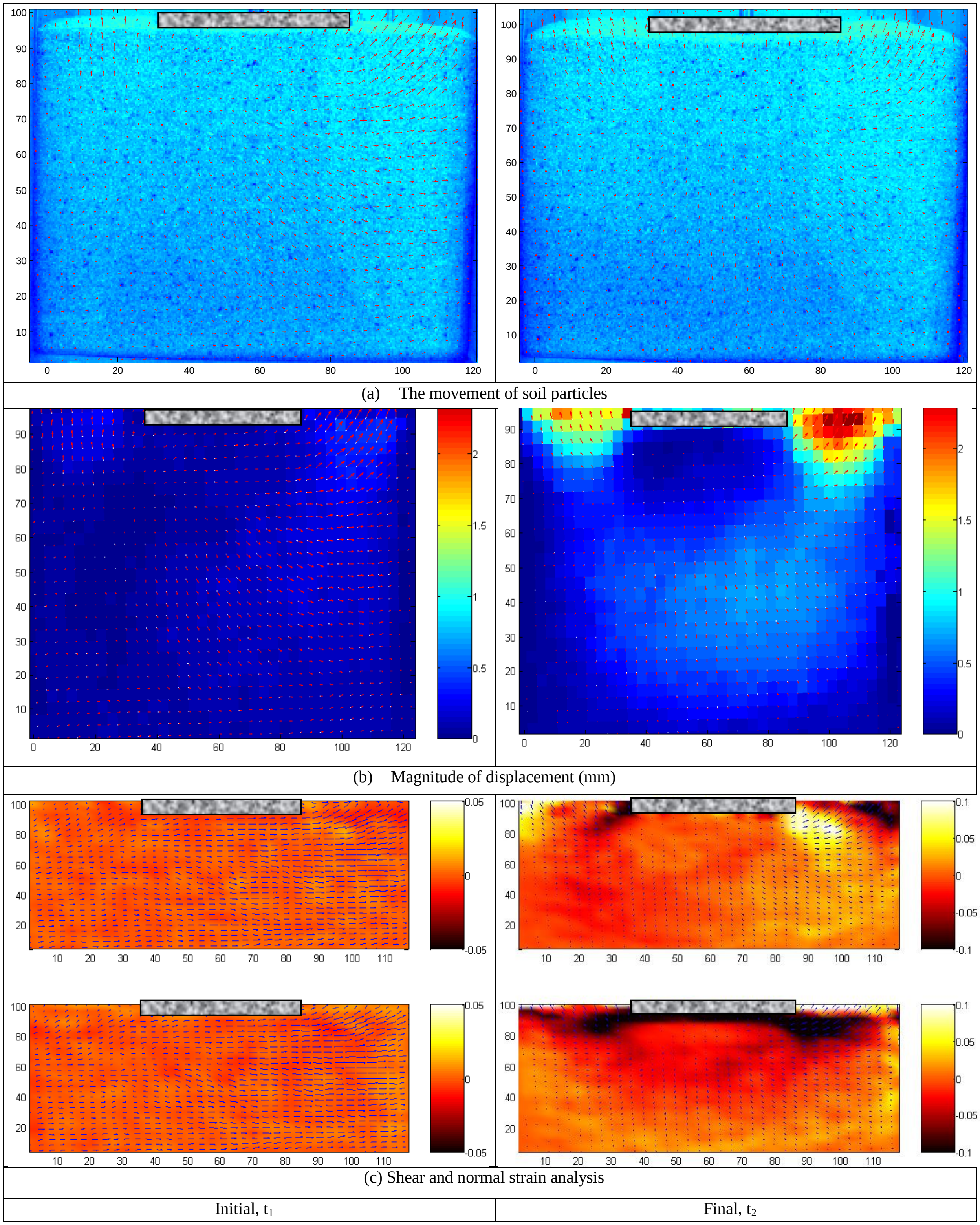


Figure 13 5 mm diameter loading fixture center loaded on 40 × 60 mm foundation plate

Figure 13(a) shows movement of the soil particles prior and after the structural loading. The arrows in the figures illustrated the movement of the soil particles only. Magnitude of displacement is shown in Figure 13(b). In the initial stage when $t = t_{initial}$, the maximum magnitude of displacement was measured at 0.25 mm. When the loading was increased to $t = t_{final}$, the soil was stressed further causing the maximum soil displacement to increase to approximately 2.5 mm.

Figure 13(c) shows the shear strain and the normal strain of the soil sample under centered loading for time $t_{initial}$ and t_{final}. The shear strain analysis was carried out by measuring the difference of horizontal movement of the soil particle at $t = t_{initial}$

and t = t_{final}. The sign convention for the shear strain analysis in soil is positive when soil moved to the right and negative when the soil moved to the left. On the other hand, the normal strain analysis was carried out by measuring the difference of vertical movement of the soil particle at $t_{initial}$ and t_{final}. The sign convention for the normal strain analysis in soil is positive when soil protruded upward and negative when it was pressed downward. Figure 13(c) shows that when the structural loading is increased, the shear strain and the normal strain increased from ± 0.01 % to ± 0.1 % in the direction as discussed.

■4.0 CONCLUSIONS & FUTURE WORKS

MatPIV, a toolbox for image processing analysis could be utilised in geotechnical engineering displacement analysis with moderate modifications to the source code to suit its usage in a simple soil structure interaction as presented in this research. Image processing using the MatPIV toolbox provides comprehensive information on analysing the profile of soil settlement where graphs of displacement vectors and magnitude of displacements could be produced. The toolbox also produces strain values of the soil particles' displacement.

The PIV analyses in this paper show agreeable findings where the diameter of the loading rod and the size of the foundation plate, (width and length) do affect the load-displacement analysis, mode and magnitude of displacement and the shear and normal strains of a foundation. On the effects of loading rod diameter to the load-displacement graph of the soil sample, it was shown that when the size of the loading rod increased, the pressure acting on the surface of the foundation plate increased and vice versa. On the effects of the size of the foundation plate, it was shown that when the size of the foundation plate increased under a single size loading rod, the loading was dispersed to the wider area of the soil surface, thus reducing the bearing stress acting on the soil sample and vice versa. PIV analysis provides information in agreement with fundamental understanding in shallow foundation theory.

MatPIV can be used with full confidence as it has been shown in the study that the mean error for the displacement was -0.088 for a 95% confidence interval with accuracy up to 98.8%. With excellent accuracy value, the reading from the MatPIV is considered acceptable.

Further works have been planned for this research especially in performing the PIV analysis on shallow foundation in different soil conditions. A laboratory module shall be initiated to introduce the technology to university students.

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