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Fuzzy Generalized Bi-Γ-Ideals of Type (λ, θ) In Ordered Γ-Semigroups

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Graphical abstract

$$\left(\bigcap_{i\in I} F_i\right)((x\alpha y\beta z)\vee\lambda = \bigwedge_{i\in I} F_i((x\alpha y\beta z)\vee\lambda \geq \bigwedge_{i\in I}(F_i(x)\wedge F_i(z)\wedge\theta)$$
$$=\left(\bigwedge_{i\in I}\left(F_i(x)\wedge\theta\right)\wedge\bigwedge_{i\in I}\left(F_i(z)\wedge\theta\right)\right)$$
$$=\left(\bigcap_{i\in I} F_i\right)(x)\wedge\left(\bigcap_{i\in I} F_i\right)(z)\wedge\theta.$$

Abstract

Subscribing to the Zadeh's idea on fuzzy sets, many researchers strive to identify the key attributes of these sets for new finding in mathematics. In this perspective, we introduce a new concept of fuzzy generalized bi-Γ-ideal of an ordered Γ-semigroup G called a (λ, θ)-fuzzy generalized bi-Γ-ideal of G. Fuzzy generalized bi-Γ-ideals of type (λ, θ) are the generalization of ordinary fuzzy generalized bi-Γ-ideals of an ordered Γ-semigroup G. A new classification of ordered Γ-semigroups in terms of a (λ, θ)-fuzzy generalized bi-Γ-ideal is given. Furthermore, we proved that $U(\mu, t)$ is a generalized bi-Γ-ideal if and only if the fuzzy subset μ is a (λ, θ)-fuzzy generalized bi-Γ-ideal of G for all $t \in (\lambda, \theta]$. Similarly, A is a generalized bi-Γ-ideal if and only if the characteristic function μ_A of A is a (λ, θ)-fuzzy generalized bi-Γ-ideal of G. Finally, the relationship between ordinary fuzzy generalized bi-Γ-ideal and (λ, θ)-fuzzy generalized bi-Γ-ideal is discussed.

Keywords: Ordered Γ-semigroups; fuzzy generalized bi-Γ-ideals; (λ, θ)-fuzzy generalized bi-Γ-ideals; level set; characteristic function

Abstrak

Untuk mempraktikkan idea Zadeh dalam set kabur, ramai penyelidik telah berusaha untuk mengenal pasti ciri-ciri utama set-set tersebut sebagai penemuan baru dalam matematik. Dalam perspektif ini, kami memperkenalkan satu konsep baru tentang bi-Γ-ideal kabur teritlak bagi Γ-semikumpulan G tertib, iaitu bi-Γ-ideal (λ, θ)- kabur teritlak bagi G. Bi-Γ-ideal kabur teritlak jenis (λ, θ) adalah suatu generalisasi untuk bi-Γ-ideal kabur teritlak bagi Γ-semikumpulan G tertib yang biasa. Satu klasifikasi baru untuk Γ-semikumpulan tertib dalam sebutan bi-Γ-ideal (λ, θ)- kabur teritlak telah diberikan. Sebagai tambahan, kami juga membuktikan bahawa $U(\mu, t)$ adalah satu bi-Γ-ideal teritlak jika dan hanya jika subset kabur μ ialah satu bi-Γ-ideal (λ, θ)- kabur teritlak bagi G untuk semua $t \in (\lambda, \theta]$. Begitu juga, A adalah satu bi-Γ-ideal teritlak jika dan hanya jika fungsi cirian μ_A bagi A ialah satu bi-Γ-ideal (λ, θ)- kabur teritlak bagi G. Akhirnya, hubungan di antara bi-Γ-ideal kabur teritlak yang biasa dan bi-Γ-ideal (λ, θ)- kabur teritlak dibincangkan.

Kata kunci: Γ-semikumpulan tertib; bi-Γ-ideal kabur teritlak; bi-Γ-ideal (λ, θ)- kabur teritlak; set oras; fungsi cirian

1.0 INTRODUCTION

Flourishing with an ever-expending long list of applications viz., formal languages, theoretical physics and fuzzy automata, fuzzification of algebraic structures have now become synonymous with other useful branches of contemporary mathematics [1]. Since the first pioneering report by Zadeh [2] in 1965, fuzzy sets have had a long, study history in the field of fuzzy algebra. A series of research papers and Ph D thesis yielded a large number of contributions, leading to the expansion of fuzzy sets in different branches of science and technology. In 1971, Rosenfeld [3] initiated the notion of fuzzy sets in algebraic structures. Further, Kehayopulu and Tsingelis [4] investigated fuzzy bi-ideals in ordered semigroups in terms of their fuzzy bi-ideals (also see [5, 6]).

Rosenfeld's idea of fuzzy subgroup [3] has been dominating the world of fuzzy algebra in an unprecedented manner in the last couple of decades, which in turn motivated the researchers to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. Understanding the underlying reasons Rosenfeld's approach, Sen and Saha [7] initiated the concept of a Γ-semigroup, and established a relation between regular Γ-semigroup and Γ-group (see also [8]). Furthermore, Kwon and Lee introduced the concept of Γ-ideals and weakly prime Γ-ideals in ordered Γ-semigroups [9], and established the basic properties of ordered Γ-semigroups in terms of weakly prime Γ-ideals. Iampan [10] proposed the concept of (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups, and give some characterizations of (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups (also see [11-14]). Similarly, Davvaz *et al.* [15] studied fuzzy Γ-hypernear-rings and investigated some important results in terms of this notion. Khan *et al.* [16] characterized ordered Γ-semigroups in terms of fuzzy interior Γ-ideals and discussed some interesting properties of fuzzy interior Γ-ideals in ordered Γ-semigroups.

The concept of fuzzy subfield with thresholds and a (λ,μ)-fuzzy subrings and a (λ,μ)-fuzzy ideals have been proposed by Yuan [17] and Yao [18], respectively. Subscribing to this view, many researchers studied (λ,μ)-fuzzy ideals in semigroups [19-21].

Here we describe a more generalized form of the fuzzy generalized bi Γ-ideals, and introduced the concept of a (λ,θ)-fuzzy generalized bi- Γ -ideal of *G*. We showed that every fuzzy generalized bi Γ-ideal is a (λ,θ)-fuzzy generalized bi Γ-ideals of *G*. We also discuss that the level subset $U(\mu;t)(\neq\phi)$ is a generalized bi Γ-ideal if and only if μ is a (λ,θ)-fuzzy generalized bi Γ-ideal of *G* for all $t\in(\lambda,\theta]$. We prove that a non-empty subset *A* of *G* is a generalized bi Γ-ideal if and only if a characteristic function μ_A of *A* is a (λ,θ)-fuzzy generalized bi Γ-ideal of *G*

2.0 PRELIMINARIES

Herein, we provide necessary basic concepts that are essential for this research.

A structure (G,Γ) is called a Γ-*semigroup* [7] if for $G=\{x,y,z,...\}$ and $\Gamma=\{\alpha,\beta,\gamma,...\}$ be any two non-empty sets. If there is a function $G\times\Gamma\times G\to G$ such that $(x\alpha y)\beta z=x\alpha(y\beta z)$ for all $x,y,z\in G$ and $\alpha,\beta\in\Gamma$. By an *ordered* Γ-*semigroup* we mean a Γ-semigroup *G* at the same time a poset $(G,\leq)$ satisfying the following condition:

$$a\leq b\Rightarrow a\alpha x\leq b\alpha x \text{ and } x\beta a\leq x\beta b$$
for all $a,b,x\in G$ and $\alpha,\beta\in\Gamma$.

For $A\subseteq G$, we denote $(A]:=\{t\in G\mid t\leq h \text{ for some } h\in A\}$. If $A=\{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A,B\subseteq G$, we denote

$$A\Gamma B:=\{a\alpha b\mid a\in A,b\in B,\alpha\in\Gamma\}.$$

Throughout the paper *G* will denote the ordered Γ-semigroup unless otherwise stated.

A non-empty subset *A* of an ordered Γ-semigroup *G* is called a *generalized bi Γ-ideal* [14] of *G* if it satisfies

(i) $(\forall a,b\in G)(\forall b\in A)(a\leq b\Rightarrow a\in A)$,
(ii) $A\Gamma G\Gamma A\subseteq A$.

Now we recall some fuzzy logic concepts.
A *fuzzy subset* μ of a universe *X* is a function from *X* into a unit closed interval $[0,1]$ of real numbers, i.e., $\mu : X\to[0,1]$.

For a non-empty subset *A* of *G*, the *characteristic function* μ_A of *A* is a fuzzy subset of *G* defined by

$$\mu_A=\begin{cases}1, & \text{if } x\in A,\\ 0, & \text{if } x\notin A.\end{cases}$$

If *A* is a non-empty subset of *G* and $a\in G$. Then,

$$A_a=\{(y,z)\in G\times G\mid a\leq y\alpha z \text{ where } \alpha\in\Gamma\}.$$

If μ and F are two fuzzy subsets of *G*. Then the product $\mu\circ F$ of μ and F is defined by:

$$\mu\circ F : G\to[0,1]\mid a\mapsto(\mu\circ F)(a)=\begin{cases}\bigvee_{(y,z)\in A_a}(\mu(y)\wedge F(z)) & \text{if } A_a\neq\phi\\ 0 & \text{otherwise.}\end{cases}$$

2.1 Definition

A fuzzy subset μ of an ordered Γ-semigroup $(G,\Gamma,\leq)$ is called a *fuzzy generalized* bi *Γ-ideal* of *G* if it satisfies

(i) $(\forall x,y,z\in G)(\alpha,\beta\in\Gamma)(\mu(x\alpha y\beta z)\geq\mu(x)\wedge\mu(z))$,
(ii) $(\forall x,y\in G)(x\leq y\Rightarrow\mu(x)\geq\mu(y))$.

2.2 Lemma [14]

If μ is a fuzzy subset of an ordered Γ-semigroup *G*, then the following conditions are equivalent

(i) *A* is a generalized bi Γ-ideal of *G*.
(ii) A characteristic function μ_A of *A* is a fuzzy generalized bi Γ-ideal of *G*.

For a fuzzy subset μ of *G* and $t\in(0,1]$, the *crisp set*

$$U(\mu;t):=\{x\in G\mid\mu(x)\geq t\}$$

is called the *level subset* of μ.

2.3 Theorem [14]

If μ is a fuzzy subset of an ordered Γ-semigroup *G*, then the following conditions are equivalent

(i) *A* is a generalized bi Γ-ideal of *G*.
(ii) A non-empty level subset $U(\mu;t)$ is a generalized bi Γ-ideal of *G* for all $t\in(0,1]$.

3.0 (λ,θ)-FUZZY GENERALIZED BI Γ-IDEAL

New types of fuzzy generalized bi Γ-ideal in ordered Γ-semigroups called (*λ*, *θ*)-fuzzy generalized bi Γ-ideals are introduced and different characterization theorems in terms of this notion are provided in this section.

In what follows, let $\lambda,\theta\in[0,1]$ be such that $0\leq\lambda<\theta\leq1$. Both λ and θ are arbitrary but fixed.

Now, we initiate the concept of a (*λ*, *θ*)-fuzzy generalized bi Γ-ideal of an ordered Γ-semigroup *G* in the following definition.

3.1 Definition

A fuzzy subset μ of *G* is called a (*λ*, *θ*)-fuzzy *generalized bi* Γ-ideal of *G* if it satisfies the following conditions:

(B1) $(\forall x,y\in G \text{ with } x\leq y)(\mu(x)\vee\lambda\geq\mu(y)\wedge\theta)$.
(B2) $(\forall x,y,z\in G)(\forall\alpha,\beta\in\Gamma)(\mu(x\alpha y\beta z)\vee\lambda\}\geq\mu(x)\wedge\mu(z)\wedge\theta)$

3.2 Theorem

If a fuzzy subset μ of an ordered Γ-semigroup G is a (λ, θ)-fuzzy generalized bi Γ-ideal of G, then the set $\mu_{\bar{\lambda}}$ is a generalized bi Γ-ideal of G.

Proof Assume that μ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. Let $x, y \in G$ such that $x \le y$, $y \in \mu_{\bar{\lambda}}$. Then $\mu(y) > \lambda$. Since μ is a (λ, θ)-fuzzy generalized bi Γ-ideal therefore $\mu(x) \vee \lambda \ge \mu(y) \wedge \theta > \lambda \wedge \theta = \lambda$. Hence $\mu(x) > \lambda$. It implies that $x \in \mu_{\bar{\lambda}}$. let $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ such that $x, z \in \mu_{\bar{\lambda}}$. Then $\mu(x) > \lambda$ and $\mu(z) > \lambda$. Since μ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G, then

$$\mu(x\alpha y\beta z) \vee \lambda \ge \mu(x) \wedge \mu(z) \wedge \theta > \lambda \wedge \lambda \wedge \theta = \lambda.$$

Hence $\mu(x\alpha y\beta z) > \lambda$. It shows that $x\alpha y\beta z \in \mu_{\bar{\lambda}}$ for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$. Hence $\mu_{\bar{\lambda}}$ is a generalized bi Γ-ideal of G.

3.3 Theorem

A non empty subset A of an ordered Γ-semigroup G is a generalized bi Γ-ideal of G if and only if a fuzzy subset μ of G defined as follows:

$$\mu(x) = \begin{cases} \ge \theta \text{ for all } x \in A, \\ \lambda \text{ for all } x \notin A, \end{cases}$$

is a (λ, θ)-fuzzy generalized bi Γ-ideal of G.

Proof Let A be a bi generalized Γ-ideal of G. If there exist $x, y \in G$ such that $x \le y$ and $y \in A$ then $x \in A$. Hence $\mu(x) \ge \theta$. Therefore

$$\mu(x) \vee \lambda \ge \theta = \mu(y) \wedge \theta.$$

If $y \notin A$, then $\mu(y) \wedge \theta = \lambda$. Thus

$$\mu(x) \vee \lambda \ge \lambda = \mu(y) \wedge \theta.$$

Assume that $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ such that $x, z \in A$. Then $\mu(x) \ge \theta$ and $\mu(z) \ge \theta$. Hence

$$\mu(x\alpha y\beta z) \vee \lambda \ge \theta = \mu(x) \wedge \mu(z) \wedge \theta.$$

If $x \notin A$ or $z \notin A$ then $\mu(x) \wedge \mu(z) \wedge \theta = \lambda$. Thus

$$\mu(x\alpha y\beta z) \vee \lambda \ge \lambda = \mu(x) \wedge \mu(z) \wedge \theta.$$

Consequently, μ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G.

3.4 Proposition

If $\{F_i : i \in I\}$ is a family of (λ, θ)-fuzzy generalized bi Γ -ideals of an ordered Γ-semigroup G, then $\bigcap_{i \in I} F_i$ is an (λ, θ)-fuzzy generalized Γ-ideal of G.

Proof Let $\{F_i\}_{i \in I}$ be a family of (λ, θ)-fuzzy generalized bi Γ-ideals of G. Let $x, y \in G$, $\alpha \in \Gamma$ and $x \le y$. Then

$$\left(\bigcap_{i \in I} F_i\right)(x) \vee \lambda = \bigwedge_{i \in I} F_i(x) \vee \lambda \ge \bigwedge_{i \in I} (F_i(y) \wedge \theta)$$

$$= \left(\bigcap_{i \in I} F_i\right)(y) \wedge \theta.$$

Let $x, y, z \in G$ and $\alpha, \beta \in \Gamma$. Then,

$$\left(\bigcap_{i \in I} F_i\right)((x\alpha y\beta z) \vee \lambda = \bigwedge_{i \in I} F_i((x\alpha y\beta z) \vee \lambda \ge \bigwedge_{i \in I} (F_i(x) \wedge F_i(z) \wedge \theta)$$

$$= \left(\bigwedge_{i \in I} (F_i(x) \wedge \theta) \wedge \bigwedge_{i \in I} (F_i(z) \wedge \theta)\right)$$

$$= \left(\bigcap_{i \in I} F_i\right)(x) \wedge \left(\bigcap_{i \in I} F_i\right)(z) \wedge \theta.$$

Thus $\bigcap_{i \in I} F_i$ is an (λ, θ)-fuzzy generalized bi Γ-ideal of G.

3.5 Theorem

A fuzzy subset μ of an ordered Γ-semigroup G is a (λ, θ)-fuzzy generalized bi Γ-ideal of G if and only if each non-empty level subset $U(\mu;t)$ is a generalized bi Γ-ideal of G for all $t \in (\lambda, \theta]$.

Proof Let μ be a (λ, θ)-fuzzy generalized bi Γ-ideal of G. If there exist $x, y \in G$, $t \in (\lambda, \theta]$ with $x \le y$ such that $y \in U(\mu;t)$.

Then $\mu(y) \ge t$, since μ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. Therefore

$$\mu(x) \vee \lambda \ge \mu(y) \wedge \theta = t \wedge \theta = t > \lambda.$$

It implies that $\mu(x) \ge t$. Thus $x \in U(\mu;t)$. Now assume $x, y, z \in G$, $\alpha, \beta \in \Gamma$ and $x, z \in U(\mu;t)$ where $t \in (\lambda, \theta]$. Then $\mu(x) \ge t$ and $\mu(z) \ge t$, since μ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. Therefore

$$\mu(x\alpha y\beta z) \vee \lambda \ge \mu(x) \wedge \mu(z) \wedge \theta = t \wedge t \wedge \theta = t > \lambda.$$

It implies that $\mu(x\alpha y\beta z) \ge t$. Therefore $x\alpha y\beta z \in U(\mu;t)$. Hence $U(\mu;t)$ is a generalized bi Γ-ideal of G for all $t \in (\lambda, \theta]$.

Conversely, assume that $U(\mu;t)$ is a generalized bi Γ-ideal of G for all $t \in (\lambda, \theta]$. If there exist $x, y \in G$ with $x \le y$ such that $\mu(x) \vee \lambda < \mu(y) \wedge \theta$, then there exists $t \in (\lambda, \theta]$ such that $\mu(x) \vee \lambda < t \le \mu(y) \wedge \theta$. This shows that $\mu(y) \ge t$ and $\mu(x) < t$ so $y \in U(\mu;t)$, since $U(\mu;t)$ is a generalized bi Γ-ideal of G. Therefore $x \in U(\mu;t)$, but this is a contradiction to $\mu(x) < t$. Thus $\mu(x) \vee \lambda \ge \mu(y) \wedge \theta$. Now let $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ such that $\mu(x\alpha y\beta z) \vee \lambda < \mu(x) \wedge \mu(z) \wedge \theta$, then there exists $t \in (\lambda, \theta]$ such that

$$\mu(x\alpha y\beta z) \vee \lambda < t \le \mu(x) \wedge \mu(z) \wedge \theta.$$

This shows that $\mu(x) \ge t$, $\mu(z) \ge t$ and $\mu(x\alpha y\beta z) < t$ so $x, z \in U(\mu;t)$, since $U(\mu;t)$ is a generalized bi Γ -ideal of G. Therefore $x\alpha y\beta z \in U(\mu;t)$, but this is a contradiction to $\mu(x\alpha y\beta z) < t$. Hence

$$\mu(x\alpha y\beta z) \vee \lambda \ge \mu(x) \wedge \mu(z) \wedge \theta$$

for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$. Consequently μ is a (λ, θ)-fuzzy generalized bi Γ -ideal of G.

3.6 Theorem

Let μ be a fuzzy subset of G. If μ is a (λ, θ)-fuzzy generalized bi Γ -ideal, then the following conditions hold:

(1) $(\forall x, y \in G)(\mu(x) \vee \lambda \ge \mu(y) \wedge \theta$ with $x \le y)$,

(2)($\forall x, y, z \in G$)
$(\forall \beta, \gamma \in \Gamma)$ ($\mu(x\beta y\gamma z) \vee \lambda \geq \mu(x) \wedge \mu(z) \wedge \theta$).

If $\lambda = 0$ and $\theta = 1$ then we have the following corollary.

3.7 Corollary

Every fuzzy generalized bi Γ-ideal μ of an ordered Γ-semigroup G is a (λ, θ)-fuzzy generalized bi Γ-ideal of G.

3.8 Lemma

A non-empty subset A of an ordered Γ-semigroup G is a generalized bi Γ-ideal if and only if a characteristic function μ_A of A is a (λ, θ)-fuzzy generalized bi Γ-ideal of G.

Proof Suppose that A is a generalized bi Γ-ideal of G and μ_A is a characteristic function of A. Let $x, y \in G$ with $x \leq y$.

If $y \in A$ then $\mu_A(y) = 1$. Since A is generalized bi Γ-ideal and $x \leq y$ such that $y \in A$ then $x \in A$. Therefore $\mu_A(x) = 1$. As $\lambda < \theta$ so we have

$$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \theta.$$

If $y \notin A$ then $\mu_A(y) = 0$. Hence $\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \theta$. If there exist $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ then we discuss the following Cases:

Case 1 if $x, z \in A$, then $\mu_A(x) = 1 = \mu_A(z)$. Since A is generalized bi Γ-ideal and $x, z \in A$, $\alpha, \beta \in \Gamma$. Therefore $x\alpha y\beta z \in A$. Hence $\mu_A(x\alpha y\beta z) = 1$. Also $\lambda < \theta$. Thus

$$\mu_A(x\alpha y\beta z) \vee \lambda \geq \mu_A(x) \wedge \mu_A(z) \wedge \theta.$$

Case 2 if $x, z \notin A$, then $\mu_A(x) = 0 = \mu_A(z)$. Hence

$$\mu_A(x\alpha y\beta z) \vee \lambda \geq \mu_A(x) \wedge \mu_A(z) \wedge \theta.$$

Case 3 if $x \in A$ and $z \notin A$ then $\mu_A(x) = 1$ and $\mu_A(z) = 0$.

Thus $\mu_A(x) \wedge \mu_A(z) \wedge \theta = 0$. Therefore

$$\mu_A(x\alpha y\beta z) \vee \lambda \geq \mu_A(x) \wedge \mu_A(z) \wedge \theta.$$

Similarly for $x \notin A$ and $z \in A$ then $\mu_A(x) = 0$ and $\mu_A(z) = 1$. It implies that $\mu_A(x) \wedge \mu_A(z) \wedge \theta = 0$. Thus in any case $\mu_A(x\alpha y\beta z) \vee \lambda \geq \mu_A(x) \wedge \mu_A(z) \wedge \theta$ for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$. Consequently, μ_A is a (λ, θ)-fuzzy generalized bi Γ-ideal of G.

Conversely, assume that μ_A is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. Suppose that $x, y \in G$ such that $x \leq y$ and $y \in A$. Then $\mu_A(y) = 1$, since μ_A is a (λ, θ)-fuzzy generalized bi Γ-ideal so

$$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \theta$$
$$= 1 \wedge \theta = \theta.$$

Since $\lambda < \theta$ therefore $\mu_A(x) \geq \theta$. Hence $x \in A$. Now let $x, y, z \in G$ and $\beta, \gamma \in \Gamma$ such that $x, z \in A$. Then $\mu_A(x) = 1 = \mu_A(z)$, since μ_A is a (λ, θ)-fuzzy generalized bi Γ-ideal so

$$\mu_A(x\beta y\gamma z) \vee \lambda \geq \mu_A(x) \wedge \mu_A(z) \wedge \theta$$
$$= 1 \wedge 1 \wedge \theta = \theta.$$

It implies that $\mu_A(x\beta y\gamma z) \geq \theta$. Hence $x\beta y\gamma z \in A$. Therefore A is a generalized bi Γ-ideal of G.

Further, we investigate some new notion μ_λ^θ of G. We showed that if μ is a (λ, θ)-fuzzy generalized bi Γ-ideals of G, then μ_λ^θ is a fuzzy bi Γ-ideal of G. We also characterize ordered Γ-semigroups by the properties of this new notion.

3.9 Definition

Let μ be a fuzzy subset of an ordered Γ-semigroup G, we define the fuzzy subset μ_λ^θ of G as follows:

$$\mu_\lambda^\theta(x) = \{\mu(x) \wedge \theta\} \vee \lambda \text{ for all } x \in G.$$

3.10 Definition

Let μ and F be fuzzy subsets of an ordered Γ-semigroup G. Then we define the fuzzy subsets $\mu \wedge_\lambda^\theta F, \mu \vee_\lambda^\theta F$ and $\mu \circ_\lambda^\theta F$ of G as follows:

$$(\mu \wedge_\lambda^\theta F)(x) = \{(\mu \wedge F)(x) \wedge \theta\} \vee \lambda,$$
$$(\mu \vee_\lambda^\theta F)(x) = \{(\mu \vee F)(x) \wedge \theta\} \vee \lambda,$$
$$(\mu \circ_\lambda^\theta F)(x) = \{(\mu \circ F)(x) \wedge \theta\} \vee \lambda,$$

for all $x \in G$.

3.11 Lemma

Let μ and F be fuzzy subsets of an ordered Γ-semigroup G. Then the following holds.

(1) $(\mu \wedge_\lambda^\theta F) = (\mu_\lambda^\theta \wedge F_\lambda^\theta)$

(2) $(\mu \vee_\lambda^\theta F) = (\mu_\lambda^\theta \vee F_\lambda^\theta)$

(3) $(\mu \circ_\lambda^\theta F) = (\mu_\lambda^\theta \circ F_\lambda^\theta)$.

Proof The proofs are straightforward.

3.12 Definition

If χ_A is the characteristic function of A, then $(\chi_A)_\lambda^\theta$ is defined as:

$$(\chi_A)_\lambda^\theta(x) := \begin{cases} \theta \text{ if } x \in A, \\ \lambda \text{ if } x \notin A. \end{cases}$$

3.13 Lemma

Let A and B be non-empty subsets of an ordered Γ-semigroup G. Then the following hold:

(i) $(\chi_A \wedge_\lambda^\theta \chi_B) = (\chi_{A\cap B})_\lambda^\theta$

(ii) $(\chi_A \vee_\lambda^\theta \chi_B) = (\chi_{A\cup B})_\lambda^\theta$

(iii) $(\chi_A \circ_\lambda^\theta \chi_B) = (\chi_{(A\Gamma B]})_\lambda^\theta$.

Proof (i) and (ii) are obvious.

For the proof of (iii) let $x \in (A\Gamma B]$. Then $\chi_{(A\Gamma B]}(x) = 1$ and hence

$$\left\{\chi_{(A\Gamma B]}(x) \wedge \theta\right\} \vee \lambda = \{1 \wedge \theta\} \vee \lambda = \theta.$$

It implies that $(\chi_{(A\Gamma B]})_\lambda^\theta(x)=\theta$. Since $x\in(A\Gamma B]$, we have $x\le a\alpha b$ for some $a\in A$, $b\in B$ and $\alpha\in\Gamma$. Then $(a,b)\in A_x$ and $A_x\ne\phi$. Thus

$$\begin{aligned}(\chi_A\circ_\lambda^\theta\chi_B)(x)&=\{(\chi_A\circ\chi_B)(x)\wedge\theta\}\vee\lambda\\&=\left[\left\{\bigvee_{(y,z)\in A_x}(\chi_A(y)\wedge\chi_B(z))\right\}\wedge\theta\right]\vee\lambda\\&\ge[\{(\chi_A(a)\wedge\chi_B(b))\}\wedge\theta]\vee\lambda.\end{aligned}$$

Since $a\in A$ and $b\in B$, we have $\chi_A(a)=1$ and $\chi_B(b)=1$ and so

$$\begin{aligned}(\chi_A\circ_\lambda^\theta\chi_B)(x)&\ge[\{(\chi_A(a)\wedge\chi_B(b))\}\wedge\theta]\vee\lambda\\&=[\{(1\wedge1)\}\wedge\theta]\vee\lambda\\&=[1\wedge\theta]\vee\lambda=\theta\vee\lambda=\theta.\end{aligned}$$

Thus,

$$(\chi_A\circ_\lambda^\theta\chi_B)(x)=\theta=(\chi_{(A\Gamma B]})_\lambda^\theta(x).$$

Let $x\notin(A\Gamma B]$, then $\chi_{(A\Gamma B]}(x)=0$ and hence,

$$\{\chi_{(A\Gamma B]}(x)\wedge\theta\}\vee\lambda=(0\wedge\theta)\vee\lambda=\lambda.$$

So $(\chi_{(A\Gamma B]})_\lambda^\theta(x)=\lambda$. Let $(y,z)\in A_x$. Then

$$\begin{aligned}(\chi_A\circ_\lambda^\theta\chi_B)(x)&=\{(\chi_A\circ\chi_B)(x)\wedge\theta\}\vee\lambda\\&=\left[\left\{\bigvee_{(y,z)\in A_x}(\chi_A(y)\wedge\chi_B(z))\right\}\wedge\theta\right]\vee\lambda\end{aligned}$$

Since $(y,z)\in A_x$, then $x\le y\beta z$ for $\beta\in\Gamma$. If $y\in A$ and $z\in B$, then $y\beta z\in A\Gamma B$ and so $x\in(A\Gamma B]$. This is a contradiction. If $y\notin A$ and $z\in B$, then

$$\left[\left\{\bigvee_{(y,z)\in A_x}(\chi_A(y)\wedge\chi_B(z))\right\}\wedge\theta\right]\vee\lambda=\left[\left\{\bigvee_{(y,z)\in A_x}(0\wedge1)\right\}\wedge\theta\right]\vee\lambda=\lambda.$$

Hence, $(\chi_A\circ_\lambda^\theta\chi_B)(x)=\lambda=(\chi_{(A\Gamma B]})_\lambda^\theta(x)$. Similarly, for $y\in A$ and $z\notin B$, we have $(\chi_A\circ_\lambda^\theta\chi_B)(x)=\lambda=(\chi_{(A\Gamma B]})_\lambda^\theta(x)$.

3.14 Theorem

The characteristic function $(\chi_A)_\lambda^\theta$ of A is a (λ, θ)-fuzzy generalized bi Γ-ideal if and only if A is a generalized bi Γ-ideal of G.

Proof Suppose that A is a generalized bi Γ-ideal of G. Then by Lemma 3.8, $(\chi_A)_\lambda^\theta$ is a (λ,θ)-fuzzy generalized bi Γ-ideal of G.

Conversely, assume that $(\chi_A)_\lambda^\theta$ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. Let $x,y\in G$, $x\le y$ be such that $y\in A$. It implies that $(\chi_A)_\lambda^\theta(y)=\theta$. Since $(\chi_A)_\lambda^\theta$ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. Therefore

$$\begin{aligned}(\chi_A)_\lambda^\theta(x)\vee\lambda&\ge(\chi_A)_\lambda^\theta(y)\wedge\theta\\&=\theta\wedge\theta=\theta.\end{aligned}$$

Since $\lambda<\theta$. Hence $(\chi_A)_\lambda^\theta(x)=\theta$. It shows that $x\in A$.

Now if there exist $x,y,z\in G$ and $\beta,\gamma\in\Gamma$ such that $x,z\in A$. Then $(\chi_A)_\lambda^\theta(x)=\theta$ and $(\chi_A)_\lambda^\theta(z)=\theta$. Since $(\chi_A)_\lambda^\theta$ is a (λ, θ)-fuzzy bi Γ-ideal of G. We have

$$\begin{aligned}(\chi_A)_\lambda^\theta(x\beta y\gamma z)\vee\lambda&\ge(\chi_A)_\lambda^\theta(x)\wedge(\chi_A)_\lambda^\theta(z)\wedge\theta\\&=\theta\wedge\theta\wedge\theta=\theta.\end{aligned}$$

Since $\lambda<\theta$. Hence $(\chi_A)_\lambda^\theta(x\beta y\gamma z)=\theta$. Therefore $x\beta y\gamma z\in A$. Consequently, A is a generalized bi Γ-ideal of G.

3.15 Proposition

If μ is a (λ, θ)-fuzzy generalized bi Γ-ideal, then μ_λ^θ is a fuzzy generalized bi Γ-ideal of G.

Proof Assume that μ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G. If there exist $x,y,z\in G$ and $\alpha,\beta\in\Gamma$, then

$$\begin{aligned}\mu_\lambda^\theta(x\alpha y\beta z)\vee\lambda&=(\{\mu(x\alpha y\beta z)\wedge\theta\}\vee\lambda)\vee\lambda\\&=\{\mu(x\alpha y\beta z)\wedge\theta\}\vee\lambda\\&=\{\mu(x\alpha y\beta z)\vee\lambda\}\wedge\{\theta\vee\lambda\}\\&=\{\mu(x\alpha y\beta z)\vee\lambda\}\wedge\theta\\&=\{(\mu(x\alpha y\beta z)\vee\lambda)\vee\lambda\}\wedge\theta\\&\ge\{(\mu(x)\wedge\mu(z)\wedge\theta)\vee\lambda\}\wedge\theta\\&=\{(\mu(x)\wedge\mu(z)\wedge\theta\wedge\theta)\vee\lambda\vee\lambda\}\wedge\theta\\&=\{\{(\mu(x)\wedge\theta)\vee\lambda\}\wedge\{(\mu(z)\wedge\theta)\vee\lambda\}\}\wedge\theta\\&=\{\mu_\lambda^\theta(x)\wedge\mu_\lambda^\theta(z)\}\wedge\theta\\&=\mu_\lambda^\theta(x)\wedge\mu_\lambda^\theta(z)\wedge\theta.\end{aligned}$$

By similar way we can show the remaining part of the proposition.

3.16 Corollary

If $\{\mu_i\ :\ i\in I\}$ is a family of (λ, θ)-fuzzy generalized bi Γ-ideal of an ordered Γ-semigroups G, then $\bigcap_{i\in I}(\mu_i)_\lambda^\theta$ is a (λ, θ)-fuzzy generalized bi Γ-ideal of G.

4.0 CONCLUSION

Ordered semigroups and related structures are most significant in that they help extend the range of application of pure mathematics in the fields of computer science, fuzzy automata, formal languages and coding theory. Thus even further away from the concept of level subset and its characterization are now-adays a central trunk for the engineers and researchers in the broader framework of fuzzy setting. In this perspective, we investigated the more generalized form of fuzzy generalized bi Γ-ideals of an ordered Γ-semigroup G and gave the concept of a (λ, θ)-fuzzy generalized bi Γ-ideal of G. We showed that μ is an ordinary fuzzy generalized bi Γ-ideal for $\lambda=0$ and $\theta=1$. Further, the necessary and sufficient conditions for both level subset $U(\mu;t)$ of μ and a characteristic function μ_A of A to be fuzzy generalized bi Γ-ideals of the type (λ, θ) are also provided. In future, this piece of work will provide a platform for researchers to apply this new concept in other branches of algebra, as well.

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