Jurnal Teknologi

Full paper

The Exterior Squares of Some Crystallographic Groups

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Article history

Received :18 March 2013
Received in revised form :
26 April 2013
Accepted :17 May 2013

Graphical abstract

$$G \wedge G = {}^{G \otimes G}\!\Big/_{\nabla(G)}.$$

Abstract

A crystallographic group is a discrete subgroup G of the set of isometries of Euclidean space E^n, where the quotient space E^n/G is compact. A specific type of crystallographic groups is called Bieberbach groups. A Bieberbach group is defined to be a torsion free crystallographic group. In this paper, the exterior squares of some Bieberbach groups with abelian point groups are computed. The exterior square of a group is the factor group of the nonabelian tensor square with the central subgroup of the group.

Keywords: Crystallographic groups; Bieberbach groups; exterior squares

Abstrak

Kumpulan kristalografi adalah suatu subkumpulan diskret G bagi set isometrik bagi ruang Euklidan E^n, di mana ruang hasil bahagi E^n/G adalah padat. Satu jenis kumpulan kristalografi yang spesifik dipanggil kumpulan Bieberbach. Kumpulan Bieberbach ditakrifkan sebagai suatu kumpulan kristalografi bebas kilasan. Dalam kertas kerja ini, kuasa dua peluaran bagi beberapa kumpulan Bieberbach dengan kumpulan titik abelan ditentukan. Kuasa dua peluaran bagi satu kumpulan adalah kumpulan faktor kuasa dua tensor tak abelan dengan subkumpulan pusat bagi kumpulan tersebut.

Kata kunci: Kumpulan kristalografi; kumpulan Bieberbach; kuasa dua peluaran

■1.0 INTRODUCTION

Farkas and Hiller in [1, 2] discussed on the mathematical approach regarding the pattern of a crystal. In the approach, the crystal pattern is replaced by the group G of rigid motions of Euclidean space E^n that preserve it. The group G is known as the space group or crystallographic group. As a continuation to the research involving the crystallographic groups, Plesken and Schulz [3] calculated the number of crystallographic groups in dimension five and six while Cid and Schulz in [4] constructed and classified five and six dimensional torsion free crystallographic groups which are also known as Bieberbach groups. A Bieberbach group is a torsion free crystallographic group which is an extension of a free abelian group L of finite rank by a finite group P. The group P is known as the point group while the dimension of the Bieberbach group is the rank of L. Both results of [3] and [4] deal with Crystallographic, Algorithms and Tables (CARAT) package. In this package, all Bieberbach groups up to dimension six are listed. Using the Bieberbach groups with cyclic point group of order two, C_2 and point group $C_2 \times C_2$ found in CARAT, the polycyclic presentations of some of these groups have been constructed by Rohaidah in 2009 [5] in order to compute their nonabelian tensor squares. By denoting the i th Bieberbach group with dimension j as $B_i(j)$, the polycyclic presentation for the Bieberbach groups with point group C_2 found in [5] are:

1. $B_1(2) = \left\langle a, l_1, l_2 \,\middle|\, a^2 = l_2, {}^a l_1 = l_1^{-1}, {}^a l_2 = l_2, {}^{l_1} l_2 = l_2 \right\rangle$,
2. $B_2(3) = \left\langle a, l_1, l_2, l_3 \,\middle|\, \begin{array}{l} a^2 = l_3, {}^a l_1 = l_2, {}^a l_2 = l_1, {}^a l_3 = l_3, \\ {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_2} l_3 = l_3 \end{array} \right\rangle$,
3. $B_3(3) = \left\langle a, l_1, l_2, l_3 \,\middle|\, \begin{array}{l} a^2 = l_3, {}^a l_1 = l_1^{-1}, {}^a l_2 = l_2^{-1}, {}^a l_3 = l_3, \\ {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_2} l_3 = l_3 \end{array} \right\rangle$,

4. $$B_4(5)=\left\langle a,l_1,l_2,l_3,l_4,l_5 \;\middle|\; \begin{array}{l} a^2=l_1, {}^{a}l_1=l_1, {}^{a}l_2=l_5, \\ {}^{a}l_3=l_4, {}^{a}l_4=l_3, {}^{a}l_5=l_2, \\ {}^{l_1}l_2=l_2, {}^{l_1}l_3=l_3, {}^{l_2}l_3=l_3, \\ {}^{l_1}l_4=l_4, {}^{l_2}l_4=l_4, {}^{l_3}l_4=l_4, \\ {}^{l_1}l_5=l_5, {}^{l_2}l_5=l_5, {}^{l_3}l_5=l_5, \\ {}^{l_4}l_5=l_5 \end{array} \right\rangle.$$

In addition, the polycylic presentation for the Bieberbach groups with point group $C_2\times C_2$ considered in [5] are as follows:

1. $$B_5(5)=\left\langle a,b,l_1,l_2,l_3,l_4,l_5 \;\middle|\; \begin{array}{l} a^2=l_1^{-1}l_2^{-1}l_3^{-1}, {}^{a}l_1=l_1, \\ {}^{a}l_2=l_2, {}^{a}l_3=l_3, {}^{a}l_4=l_5, \\ {}^{a}l_5=l_4, b^2=l_1^{-1}, {}^{b}l_1=l_1, \\ {}^{b}l_2=l_2, {}^{b}l_3=l_3^{-1}, {}^{b}l_4=l_4, \\ {}^{b}l_5=l_5, {}^{a}b=bl_3^{-1}, {}^{l_1}l_2=l_2, \\ {}^{l_1}l_3=l_3, {}^{l_2}l_3=l_3, {}^{l_1}l_4=l_4, \\ {}^{l_2}l_4=l_4, {}^{l_3}l_4=l_4, {}^{l_1}l_5=l_5, \\ {}^{l_2}l_5=l_5, {}^{l_3}l_5=l_5, {}^{l_4}l_5=l_5 \end{array} \right\rangle,$$

2. $$B_6(5)=\left\langle a,b,l_1,l_2,l_3,l_4,l_5 \;\middle|\; \begin{array}{l} a^2=l_1^{-1}l_3^{2}l_5^{-1}, {}^{a}l_1=l_1, \\ {}^{a}l_2=l_2, {}^{a}l_3=l_3, \\ {}^{a}l_4=l_3l_4^{-1}l_5^{-1}, {}^{a}l_5=l_5, \\ b^2=l_4l_5, {}^{b}l_1=l_1, \\ {}^{b}l_2=l_2, {}^{b}l_3=l_5, \\ {}^{b}l_4=l_3^{-1}l_4l_5, {}^{b}l_5=l_3, \\ {}^{a}b=bl_3^{2}l_4^{-1}l_5^{-2}, {}^{l_1}l_2=l_2, \\ {}^{l_1}l_3=l_3, {}^{l_2}l_3=l_3, {}^{l_1}l_4=l_4, \\ {}^{l_2}l_4=l_4, {}^{l_3}l_4=l_4, {}^{l_1}l_5=l_5, \\ {}^{l_2}l_5=l_5, {}^{l_3}l_5=l_5, {}^{l_4}l_5=l_5 \end{array} \right\rangle.$$

This research extends the results found in [5] to find the exterior squares of the Bieberbach groups with point group C_2 and $C_2\times C_2$.

The exterior square of a group G, denoted as $G\wedge G$ is one of the homological functors of the group, defined as the factor group $G\otimes G\big/\nabla(G)$ where $G\otimes G$ is the nonabelian tensor square of G and $\nabla(G)$ is the central subgroup of G. The nonabelian tensor square of a group G is generated by $g\otimes h$, for all $g,h\in G$ subjects to the relations

$$gg'\otimes h=\left({}^{g}g'\otimes {}^{g}h\right)(g\otimes h) \text{ and } g\otimes hh'=(g\otimes h)\left({}^{h}g\otimes {}^{h}h'\right)$$

where the action is taken to be conjugation.

The nonabelian tensor square of a group has been introduced by Brown and Loday in 1987 [6]. Throughout the years, many researches involving the nonabelian tensor squares of various groups have been conducted. Kappe *et al.* in [7] investigated on two-generator two-groups of class two and computed the nonabelian tensor squares of these groups. The computations of the nonabelian tensor squares of infinite metacyclic groups and free nilpotent groups of finite rank have been discussed in [8] and [9] respectively.

The exterior squares have been computed for finite $p-$ groups of nilpotency class two, infinite nonabelian 2-generator groups of nilpotency class two and symmetric groups of order six in [10], [11] and [12] respectively. Besides, in 2008, Eick and Nickel [13] determined the exterior squares of polycyclic groups.

2.0 PRELIMINARIES

In this section, some basic concepts and preparatory results that are used in the computations of the exterior squares of some Bieberbach groups with point groups C_2 and $C_2\times C_2$ are included.

Definition 2.1 [14] Let G be a group with presentation $\langle \mathcal{G}|\mathfrak{R}\rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi: g\to g^{\varphi}$ for all $g\in G$. The group $\nu(G)$ is defined to be

$$\nu(G)=\left\langle \mathcal{G},\mathfrak{G} \;\middle|\; \begin{array}{l} \mathfrak{R},\mathfrak{R}^{\varphi}, {}^{x}\left[g,h^{\varphi}\right]=\left[{}^{x}g,\left({}^{x}h\right)^{\varphi}\right]={}^{x^{\varphi}}\left[g,h^{\varphi}\right], \\ \forall x,g,h\in G \end{array} \right\rangle.$$

Theorem 2.1 [15] Let G be a group. The map $\sigma: G\otimes G\to\left[G,G^{\varphi}\right]\lhd\nu(G)$ defined by $\sigma(g\otimes h)=\left[g,h^{\varphi}\right]$ for all g,h in G is an isomorphism.

Proposition 2.1 [16] If G is polycylic then $\nu(G)$ is polycyclic.

Lemma 2.1 [16] Let G be a polycyclic group with subgroups A and B having polycyclic generating sets $a_1,\ldots,a_n$ and $b_1,\ldots,b_m$ respectively. If $G=\langle A,B\rangle$ then $[A,B]$ is generated by $\left[a_i^{\varepsilon},b_j^{\delta}\right]$, where $1\le i\le n$, $1\le j\le m$,

$$\varepsilon=\begin{cases} 1 & if \quad |a_i|<\infty; \\ \pm 1 & if \quad |a_i|=\infty, \end{cases} \text{ and } \delta=\begin{cases} 1 & if \quad |b_j|<\infty; \\ \pm 1 & if \quad |b_j|=\infty. \end{cases}$$

Lemma 2.2 [17] Let G be a group such that the abelianisation of the group, G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_iG'\rangle$, for $i=1,\ldots,s$ and set $E(G)$ to be $\left\langle \left[x_i,x_j^{\varphi}\right] \middle| i<j\right\rangle\left[G,G'^{\varphi}\right]$. Then, $\left[G,G^{\varphi}\right]=\nabla(G)E(G)$.

Theorem 2.2 [17] Assume that G^{ab} is finitely generated. If either G^{ab} has no element of order two or that G' has a complement in G, then

$$\nabla(G)\cong\nabla\left(G^{ab}\right) \text{ and } G\otimes G\cong\nabla(G)\times(G\wedge G).$$

Theorem 2.3 [18] Let G be a group. Then there exists a commutator mapping $\kappa: G\otimes G\to G'$ defined by $\kappa(g\otimes h)=[g,h]$. The kernel of κ is the center of $G\otimes G$.

Based on Theorem 2.1, since there is an isomorphism from $G\otimes G$ to $\left[G,G^{\varphi}\right]$, then all tensor computations can be translated into commutator computations within $[G,G^{\varphi}]$. Therefore, by Lemma 2.2 and Theorem 2.2, if the abelianisation of the group is finitely generated and it has no element of order two, then we have $G\wedge G$ to be $E(G)$. The commutators that are used in this paper are listed as follows:

$$[x,yz]=[x,y]^{y}[x,z]; \qquad (2.1)$$

$$[x,y^{-1}]={}^{y^{-1}}[x,y]^{-1}=[y^{-1},[x,y]^{-1}][x,y]^{-1}; \qquad (2.2)$$

$$^{z}[x,y]=[^{z}x,{}^{z}y]. \qquad (2.3)$$

The following lemmas and corollary record some basic identities used in this paper.

Lemma 2.3 [16] Let G be a group. Then the following hold in $\nu(G)$:

(i) $^{[g_3,g_4^{\varphi}]}[g_1,g_2^{\varphi}]={}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ and $^{[g_3^{\varphi},g_4]}[g_1,g_2^{\varphi}]={}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ for all g_1,g_2,g_3,g_4 in G;

(ii) $[g_1^{\varphi},g_2,g_3]=[g_1,g_2,g_3^{\varphi}]=[g_1^{\varphi},g_2,g_3^{\varphi}]=[g_1,g_2^{\varphi},g_3]=[g_1^{\varphi},g_2^{\varphi},g_3]=[g_1,g_2^{\varphi},g_3^{\varphi}]$ for all g_1,g_2,g_3 in G;

(iii) $[g_1,[g_2,g_3]^{\varphi}]=[g_2,g_3,g_1^{\varphi}]^{-1}$ for all g_1,g_2,g_3 in G;

(iv) $[g,g^{\varphi}]$ is central in $\nu(G)$ for all g in G;

(v) $[g_1,g_2^{\varphi}][g_2,g_1^{\varphi}]$ is central in $\nu(G)$ for all g_1,g_2 in G;

(vi) $[g,g^{\varphi}]=1$ for all g in G'.

Corollary 2.1 [17] Let G be any group. Then $[Z(G),(G')^{\varphi}]=1$.

Lemma 2.4 [16, 5] Let g and h be elements of G such that $[g,h]=1$. Then in $\nu(G)$,

(i) $[g^{n},h^{\varphi}]=[g,h^{\varphi}]^{n}=[g,(h^{\varphi})^{n}]$ for all integers n;

(ii) $[g^{n},(h^{m})^{\varphi}][h^{m},(g^{n})^{\varphi}]=([g,h^{\varphi}][h,g^{\varphi}])^{nm}$;

(iii) $[g,h^{\varphi}]$ is in the center of $\nu(G)$.

Lemma 2.5 [5] Let G and H be groups and let $g\in G$. Suppose ϕ is a homomorphism from G onto H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of $\phi(g)$ equals the order of g.

Lemma 2.6 [19] Let A, B and C be abelian groups. Consider the ordinary tensor product of two abelian groups. Then,

(i) $C_0\otimes A\cong A$,

(ii) $C_0\otimes C_0\cong C_0$,

(iii) $C_n\otimes C_m\cong C_{\gcd(n,m)}$, for $n,m\in$, and

(iv) $A\otimes(B\times C)=(A\otimes B)\times(A\otimes C)$

where C_0 is the infinite cyclic group.

Theorem 2.4 [18] Let G and H be groups such that there is an epimorphism $\varepsilon:G\to H$. Then there exists an epimorphism $\alpha:G\otimes G\to H\otimes H$ defined by $\alpha(g\otimes h)=\varepsilon(g)\otimes\varepsilon(h)$.

The following theorems by Rohaidah in [5] show the abelianisation of some Bieberbach groups with point groups C_2 and $C_2\times C_2$.

Theorem 2.5 [5] Let $B_i(j)$ denotes the i th Bieberbach group with point group C_2 with dimension j. Then,

$$B_1(2)^{ab}=\langle aB_1(2)',\ l_1B_1(2)'\rangle\cong C_0\times C_2,$$

$$B_2(3)^{ab}=\langle aB_2(3)',\ l_1B_2(3)'\rangle\cong C_0\times C_0,$$

$$B_3(3)^{ab}=\langle aB_3(3)',\ l_1B_3(3)',l_2B_3(3)'\rangle\cong C_0\times C_2\times C_2$$

and

$$B_4(5)^{ab}=\langle aB_4(5)',\ l_2B_4(5)',l_3B_4(5)'\rangle\cong C_0\times C_0\times C_0.$$

Theorem 2.6 [5] Let $B_i(j)$ denotes the i th Bieberbach group with point group $C_2\times C_2$ with dimension j. Then,

$$B_5(5)^{ab}=\langle aB_5(5)',\ bB_5(5)',\ l_5B_5(5)'\rangle\cong C_0\times C_0\times C_0$$

and

$$B_6(5)^{ab}=\langle aB_6(5)',\ bB_6(5)',\ l_2B_6(5)'\rangle\cong C_0\times C_0\times C_0.$$

3.0 COMPUTATIONS OF THE EXTERIOR SQUARES

In [5], there are six Bieberbach groups with point group C_2 and $C_2\times C_2$ as listed in Theorem 2.5 and Theorem 2.6. However, in this paper, the exterior squares are computed only for the groups where their abelianisations are finitely generated and have no element of order two namely $B_2(3)$, $B_4(5)$, $B_5(5)$ and $B_6(5)$. The results are shown in the following theorems.

Theorem 3.1 The exterior square of $B_2(3)$ is

$$B_2(3)\wedge B_2(3)=\langle[a,l_1^{\varphi}],[a,l_2^{\varphi}],[l_1,(l_2l_1^{-1})^{\varphi}]\rangle\cong C_0\times C_0\times C_2.$$

Proof:

By Theorem 2.5, $B_2(3)^{ab}=\langle aB_2(3)',\ l_1B_2(3)'\rangle\cong C_0\times C_0.$

Since $B_2(3)^{ab}$ has no element of order two, we have $E(B_2(3))$ to be $B_2(3)\wedge B_2(3)$. Using Lemma 2.2, set $E(B_2(3))$ to be $\langle[a,l_1^{\varphi}]\rangle[B_2(3),B_2(3)'^{\varphi}]$. Next, we determine $[B_2(3),B_2(3)'^{\varphi}]$.

From the relations of $B_2(3)$, $[a,l_1]=al_1a^{-1}l_1^{-1}=l_2l_1^{-1}\neq 1$ and $[a,l_2]=al_2a^{-1}l_2^{-1}=l_1l_2^{-1}=[a,l_1]^{-1}\neq 1$. Since $B_2(3)$ is a polycyclic group generated by polycylic generating sequence $a,\ l_1,\ l_2$ and l_3 where $[a,l_3]=1,\ [l_1,l_2]=1,\ [l_1,l_3]=1$ and $[l_2,l_3]=1,$ then $B_2(3)'=\langle l_2l_1^{-1}\rangle$. By Proposition 2.1, $\nu(B_2(3))$ is polycyclic. Hence, the subgroups $B_2(3)$ and $B_2(3)'^{\varphi}$ have polycyclic generating sequence $a,\ l_1,\ l_2,\ l_3$ and $(l_2l_1^{-1})^{\varphi}$. Thus, by Lemma 2.1, we have

$$\left[B_2(3),B_2(3)'^{\varphi}\right]=\left\langle \begin{matrix}\left[a,(l_2l_1^{-1})^{\varphi}\right],\left[l_1,(l_2l_1^{-1})^{\varphi}\right],\left[l_2,(l_2l_1^{-1})^{\varphi}\right],\\ \left[l_3,(l_2l_1^{-1})^{\varphi}\right]\end{matrix}\right\rangle.$$

However, some of these generators can be expressed as products of power of other generators. By Corollary 2.1, $\left[l_3,(l_2l_1^{-1})^{\varphi}\right]=1$. Next, by Lemma 2.4 (iii), since l_1 commutes with l_2, then $[l_1,l_2^{\varphi}]$ is in the center of $\nu(B_2(3))$. Hence, by Lemma 2.3 and 2.4 we have

$$\left[l_1,(l_2l_1^{-1})^{\varphi}\right]=[l_1,l_2^{\varphi}][l_1,l_1^{-\varphi}] \qquad \text{by commutator relations}$$
$$=[l_1,l_2^{\varphi}][l_1,l_1^{\varphi}]^{-1} \qquad \text{by Lemma 2.4 (i)}$$

and similarly

$$\left[l_2,(l_2l_1^{-1})^{\varphi}\right]=[l_2,l_2^{4}][l_2,l_1^{\varphi}]^{-1}.$$

However, ${}^{a}l_2=l_1$, then $[l_1,l_1^{\varphi}]={}^{a}[l_2,l_2^{\varphi}]=[l_2,l_2^{\varphi}]$ since $[l_2,l_2^{\varphi}]$ is in the center of $\upsilon(B_2(3))$. Hence

$$\left[l_1,(l_2l_1^{-1})^{\varphi}\right]=[l_1,l_2^{\varphi}][l_2,l_2^{\varphi}]^{-1}=\left[l_2,(l_2l_1^{-1})^{\varphi}\right]^{-1}.$$

Therefore,

$$\left[B_2(3),B_2(3)'^{\varphi}\right]=\left\langle\left[a,(l_2l_1^{-1})^{\varphi}\right],\left[l_1,(l_2l_1^{-1})^{\varphi}\right]\right\rangle.$$

Thus, $E(B_2(3))=\left\langle[a,l_1^{\varphi}],\left[a,(l_2l_1^{-1})^{\varphi}\right],\left[l_1,(l_2l_1^{-1})^{\varphi}\right]\right\rangle$.

Next, the orders of each of the three generators of $E(B_2(3))$ are computed. The derived subgroup of $B_2(3)$, $B_2(3)'=\langle l_2l_1^{-1}\rangle$. Then, since $[l_1,l_1^{\varphi}]=[l_2,l_2^{\varphi}]$ and $[l_1,l_2^{\varphi}]=[l_2,l_1^{\varphi}]$, then,

$$\left(\left[l_1,(l_2l_1^{-1})^{\varphi}\right]\right)^2=\left([l_1,l_2^{\varphi}][l_1,l_1^{-\varphi}]\right)^2 \qquad \text{by commutator relations}$$
$$=\left([l_1,l_2^{\varphi}][l_1,l_1^{\varphi}]^{-1}\right)^2 \qquad \text{by Lemma 2.4 (i)}$$
$$=[l_1,l_2^{\varphi}][l_1,l_2^{\varphi}][l_1,l_1^{\varphi}]^{-1}[l_1,l_1^{\varphi}]^{-1}$$
$$=\left([l_1,l_1^{\varphi}][l_1,l_1^{\varphi}][l_1,l_2^{\varphi}]^{-1}[l_1,l_2^{\varphi}]^{-1}\right)^{-1}$$
$$=\left([l_2,l_2^{\varphi}][l_1,l_1^{\varphi}][l_1,l_2^{\varphi}]^{-1}[l_2,l_1^{\varphi}]^{-1}\right)^{-1}$$
$$=\left([l_2,l_2^{\varphi}][l_1,l_1^{\varphi}][l_1^{-1},l_2^{\varphi}][l_2,l_1^{-\varphi}]\right)^{-1}$$
$$=\left([l_1^{-1},l_2^{\varphi}][l_1^{-1},l_1^{-\varphi}][l_2,l_2^{\varphi}][l_2,l_1^{-\varphi}]\right)^{-1}$$
$$=\left([l_1^{-1},l_2^{\varphi}]^{l_2}[l_1^{-1},l_1^{-\varphi}][l_2,l_2^{\varphi}]^{l_2}[l_2,l_1^{-\varphi}]\right)$$
$$=\left(\left[l_1^{-1},(l_2l_1^{-1})^{\varphi}\right]\left[l_2,(l_2l_1^{-1})^{\varphi}\right]\right)^{-1}$$
$$\left(\left[l_1,(l_2l_1^{-1})^{\varphi}\right]\right)^2=\left({}^{l_2}\left[l_1^{-1},(l_2l_1^{-1})^{\varphi}\right]\left[l_2,(l_2l_1^{-1})^{\varphi}\right]\right)^{-1}$$
$$=\left(\left[l_2l_1^{-1},(l_2l_1^{-1})^{\varphi}\right]\right)^{-1}$$
$$=1 \qquad \text{since } (l_2l_1^{-1})\in B_2(3)'.$$

Therefore $\left[l_1,(l_2l_1^{-1})^{\varphi}\right]$ has order 2. Next, by commutator relations,

$$\left[a,(l_2l_1^{-1})^{\varphi}\right]=[a,l_2^{\varphi}]\left[l_2l_1^{-1},[a,l_1]^{-\varphi}\right][a,l_1^{\varphi}]^{-1}$$
$$=[a,l_2^{\varphi}]\left[l_2l_1^{-1},(l_2l_1^{-1})^{-\varphi}\right][a,l_1^{\varphi}]^{-1} \quad \text{since } {}^{a}l_1=l_2$$
$$=[a,l_2^{\varphi}][a,l_1^{\varphi}] \qquad \text{since } (l_2l_1^{-1})\in B_2(3)'.$$

Therefore $\left[a,(l_2l_1^{-1})^{\varphi}\right]$ can be written as a product of the generators $[a,l_1^{\varphi}]$ and $[a,l_2^{\varphi}]$. The mapping κ from $B_2(3)\otimes B_2(3)$ to $B_2(3)'$ defined in Theorem 2.3 gives $\kappa([a,l_1^{\varphi}])=[a,l_1]=l_2l_1^{-1}\neq 1$ and $\kappa([a,l_2^{\varphi}])=[a,l_2]=l_1l_2^{-1}$ $=[a,l_1]^{-1}\neq 1$ in $B_2(3)'$ has infinite order, it follows from Lemma 2.5 that $[a,l_1^{\varphi}]$ and $[a,l_2^{\varphi}]$ have infinite order. Hence, we have $B_2(3)\wedge B_2(3)=\left\langle[a,l_1^{\varphi}],[a,l_2^{\varphi}],\left[l_1,(l_2l_1^{-1})^{\varphi}\right]\right\rangle\cong C_0\times C_0\times C_2$.

Theorem 3.2 The exterior square of $B_4(5)$ is

$$B_4(5)\wedge B_4(5)=\left\langle\begin{matrix}[a,l_2^{\varphi}],[a,l_3^{\varphi}],[l_2,l_3^{\varphi}],\left[a,(l_5l_2^{-1})^{\varphi}\right],\\ \left[a,(l_4l_3^{-1})^{\varphi}\right],\left[l_2,(l_5l_2^{-1})^{\varphi}\right],\left[l_3,(l_5l_2^{-1})^{\varphi}\right],\\ \left[l_2,(l_4l_3^{-1})^{\varphi}\right],\left[l_3,(l_4l_3^{-1})^{\varphi}\right]\end{matrix}\right\rangle.$$

Proof:

By Theorem 2.5, $B_4(5)^{ab}=\left\langle aB_4(5)',\ l_2B_4(5)',l_3B_4(5)'\right\rangle\cong$ $C_0\times C_0\times C_0$. Thus, from Lemma 2.2 and Theorem 2.2,

$$B_4(5)\wedge B_4(5)=\left\langle [a,l_2^{\varphi}],[a,l_3^{\varphi}],[l_2,l_3^{\varphi}]\right\rangle\left[B_4(5),B_4(5)'^{\varphi}\right].$$

Now we determine $\left[B_4(5),B_4(5)'^{\varphi}\right]$. From the relations of $B_4(5)$, we have $[a,l_2]=l_5l_2^{-1}\neq 1$, $[a,l_3]=l_4l_3^{-1}\neq 1$, $[a,l_4]=l_3l_4^{-1}=[a,l_3]^{-1}\neq 1$ and $[a,l_5]=l_2l_5^{-1}=[a,l_2]^{-1}\neq 1$. Since $B_4(5)$ is a polycylic group generated by polycyclic generating sequence $a,\ l_1,\ l_2,\ l_3,\ l_4,\ l_5$ where $[a,l_1]=1$, $[l_i,l_j]=1$ for $i,j=1,2,3,4,5$, then $B_4(5)'=\langle l_5l_2^{-1},l_4l_3^{-1}\rangle$. By Proposition 2.1, $\nu(B_4(5))$ is also polycylic. Therefore, the subgroups $B_4(5)$ and $B_4(5)'^{\varphi}$ have polycylic generating sequence $a,\ l_1,\ l_2,\ l_3,\ l_4,\ l_5,\ (l_5l_2^{-1})^{\varphi}$ and $(l_4l_3^{-1})^{\varphi}$. Then, by Lemma 2.1

$$\left[B_4(5),B_4(5)'^{\varphi}\right]=\left\langle\begin{array}{l}\left[a,(l_5l_2^{-1})^{\varphi}\right],\left[l_1,(l_5l_2^{-1})^{\varphi}\right],\left[l_2,(l_5l_2^{-1})^{\varphi}\right],\\ \left[l_3,(l_5l_2^{-1})^{\varphi}\right],\left[l_4,(l_5l_2^{-1})^{\varphi}\right],\left[l_5,(l_5l_2^{-1})^{\varphi}\right],\\ \left[a,(l_4l_3^{-1})^{\varphi}\right],\left[l_1,(l_4l_3^{-1})^{\varphi}\right],\left[l_2,(l_4l_3^{-1})^{\varphi}\right],\\ \left[l_3,(l_4l_3^{-1})^{\varphi}\right],\left[l_4,(l_4l_3^{-1})^{\varphi}\right],\left[l_5,(l_4l_3^{-1})^{\varphi}\right]\end{array}\right\rangle.$$

However, some of these generators can be expressed as products of other generators. By Corollary 2.1, $\left[l_1,(l_5l_2^{-1})^{\varphi}\right]=1$ and $\left[l_1,(l_4l_3^{-1})^{\varphi}\right]=1$. Since $l_1,\ l_2,\ l_3,\ l_4$ and l_5 commute with each other, then $\left[l_2,(l_5l_2^{-1})^{\varphi}\right]$, $\left[l_3,(l_4l_3^{-1})^{\varphi}\right]$, $\left[l_3,(l_5l_2^{-1})^{\varphi}\right]$ and $\left[l_2,(l_4l_3^{-1})^{\varphi}\right]$ are in the center of $\nu(B_4(5))$. Then, by (2.3) and relations of $B_4(5)$, we have $\left[l_2,(l_5l_2^{-1})^{\varphi}\right]={}^{a}\left[l_2,(l_5l_2^{-1})^{\varphi}\right]=\left[{}^{a}l_2,{}^{a}(l_5l_2^{-1})^{\varphi}\right]=\left[l_5,({}^{a}l_5\,{}^{a}l_2^{-1})^{\varphi}\right]=\left[l_5,(l_5l_2^{-1})^{\varphi}\right]^{-1}$. Similarly,

$$\left[l_3,(l_4l_3^{-1})^{\varphi}\right]={}^{a}\left[l_3,(l_4l_3^{-1})^{\varphi}\right]=\left[l_4,(l_4l_3^{-1})^{\varphi}\right]^{-1},$$

$$\left[l_3,(l_5l_2^{-1})^{\varphi}\right]={}^{a}\left[l_3,(l_5l_2^{-1})^{\varphi}\right]=\left[l_4,(l_5l_2^{-1})^{\varphi}\right]^{-1},$$

$$\left[l_2,(l_4l_3^{-1})^{\varphi}\right]={}^{a}\left[l_2,(l_4l_3^{-1})^{\varphi}\right]=\left[l_5,(l_4l_3^{-1})^{\varphi}\right]^{-1}.$$

Hence, we obtain

$$\left[B_4(5),B_4(5)'^{\varphi}\right]=\left\langle\begin{array}{l}\left[a,(l_5l_2^{-1})^{\varphi}\right],\left[a,(l_4l_3^{-1})^{\varphi}\right],\left[l_2,(l_5l_2^{-1})^{\varphi}\right],\\ \left[l_3,(l_5l_2^{-1})^{\varphi}\right],\left[l_2,(l_4l_3^{-1})^{\varphi}\right],\left[l_3,(l_4l_3^{-1})^{\varphi}\right]\end{array}\right\rangle.$$

Therefore,

$$B_4(5)\wedge B_4(5)=\left\langle\begin{array}{l}[a,l_2^{\varphi}],[a,l_3^{\varphi}],[l_2,l_3^{\varphi}],\left[a,(l_5l_2^{-1})^{\varphi}\right],\\ \left[a,(l_4l_3^{-1})^{\varphi}\right],\left[l_2,(l_5l_2^{-1})^{\varphi}\right],\left[l_3,(l_5l_2^{-1})^{\varphi}\right],\\ \left[l_2,(l_4l_3^{-1})^{\varphi}\right],\left[l_3,(l_4l_3^{-1})^{\varphi}\right]\end{array}\right\rangle.$$

Now, the order of each generators of $B_4(5)$ are determined. Here,

$$\begin{aligned}\left[l_2,(l_5l_2^{-1})^{\varphi}\right]^2&=\left[l_2,(l_2l_5^{-1})^{-\varphi}\right]^2\\ &=\left(\left[l_2,l_2^{\varphi}\right]^{l_2}\left[l_2,l_5^{-\varphi}\right]\right)^2\\ &=\left(\left[l_2,l_2^{\varphi}\right]\left[l_2,l_5^{\varphi}\right]^{-1}\right)^2\\ &=\left[l_5l_2^{-1},l_5^{\varphi}\right]^{l_5}\left[l_5l_2^{-1},l_2^{-\varphi}\right] && \text{by Lemma 2.4}\\ &=\left[l_5l_2^{-1},l_5l_2^{-\varphi}\right] && \text{by (2.1)}\\ &=1 && \text{by Lemma 2.3(vi).}\end{aligned}$$

Similarly we have $\left[l_3,(l_4l_3^{-1})^{\varphi}\right]^2=1$. Therefore, $\left[l_2,(l_5l_2^{-1})^{\varphi}\right]$ and $\left[l_3,(l_4l_3^{-1})^{\varphi}\right]$ have order dividing two.

The mapping κ from $B_4(5)\otimes B_4(5)$ to $B_4(5)'$ defined in Theorem 2.3 gives $\kappa([a,l_2^{\varphi}])=[a,l_2]$, $\kappa([a,l_3^{\varphi}])=[a,l_3]$, $\kappa\left(\left[a,(l_5l_2^{-1})^{\varphi}\right]\right)=\left[a,(l_5l_2^{-1})\right]=(l_5l_2^{-1})^{-2}$ and $\kappa\left(\left[a,(l_4l_3^{-1})^{\varphi}\right]\right)=\left[a,(l_4l_3^{-1})\right]=(l_4l_3^{-1})^{-2}$ have infinite order in $B_4(5)'$, then it follows from Lemma 2.5 that $[a,l_2^{\varphi}]$, $[a,l_3^{\varphi}]$, $\left[a,(l_5l_2^{-1})^{\varphi}\right]$ and $\left[a,(l_4l_3^{-1})^{\varphi}\right]$ have infinite order.

Next, to show that $[l_2,l_3^{\varphi}]$, $\left[l_2,(l_4l_3^{-1})^{\varphi}\right]$ and $\left[l_3,(l_5l_2^{-1})^{\varphi}\right]$ also have infinite order. The following conditions would lead these three generators to have finite order. Since l_2 and l_3 commute with one another in $B_4(5)$, then we can move any powers in $\nu(B_4(5))$. Then, $\left[l_2^m,(l_3^n)^{\varphi}\right]=\left[l_2,l_3^{\varphi(mn)}\right]=\left[l_2,l_3^{\varphi}\right]^{mn}$ for any integer m,n. However, $B_4(5)$ is torsion free and so l_2 and l_3 have infinite order. Next, one power of $[l_2,l_3^{\varphi}]$ give an element in $Z(B_4(5))$ and the other in $B_4(5)'$ by Corollary 2.1. However, this is a contradiction since there is no power of either l_2 or l_3 are in $Z(B_4(5))$. As there are no other relations can be used in $B_4(5)$ in order to get finiteness, therefore generator $[l_2,l_3^{\varphi}]$ has infinite order. Using similar arguments, the generators $\left[l_2,(l_4l_3^{-1})^{\varphi}\right]$ and $\left[l_3,(l_5l_2^{-1})^{\varphi}\right]$ also have infinite order.

Therefore,

$$B_4(5)\wedge B_4(5)=\left\langle\begin{array}{l}[a,l_2^{\varphi}],[a,l_3^{\varphi}],[l_2,l_3^{\varphi}],\left[a,(l_5l_2^{-1})^{\varphi}\right],\\ \left[a,(l_4l_3^{-1})^{\varphi}\right],\left[l_2,(l_5l_2^{-1})^{\varphi}\right],\left[l_3,(l_5l_2^{-1})^{\varphi}\right],\\ \left[l_2,(l_4l_3^{-1})^{\varphi}\right],\left[l_3,(l_4l_3^{-1})^{\varphi}\right]\end{array}\right\rangle.$$

Theorem 3.3 The exterior square of $B_5(5)$ is

$$B_5(5)\wedge B_5(5)=\left\langle \begin{array}{l}[a,b^{\varphi}],[a,l_5^{\varphi}],[b,l_5^{\varphi}],[a,l_3^{\varphi}],\\ [b,l_3^{\varphi}],[a,(l_4l_5^{-1})^{\varphi}],[b,(l_4l_5^{-1})^{\varphi}],\\ [l_4,(l_4l_5^{-1})^{\varphi}]\end{array}\right\rangle$$
$$\cong C_0^5\times C_2^3.$$

Proof:

Using similar arguments as in Theorem 3.1 and Theorem 3.2, since $B_5(5)^{ab}=\langle aB_5(5)',\ bB_5(5)',\ l_5B_5(5)'\rangle\cong C_0\times C_0\times C_0$, then,

$$B_5(5)\wedge B_5(5)=\langle [a,b^{\varphi}],[a,l_5^{\varphi}],[b,l_5^{\varphi}]\rangle[B_5(5),B_5(5)'^{\varphi}].$$

From the relations of $B_5(5)$, $[a,l_4]=l_5l_4^{-1}\neq 1$, $[a,l_5]=l_4l_5^{-1}=[a,l_4]^{-1}\neq 1$, $[a,b]=bl_3^{-1}b^{-1}=l_3\neq 1$ and $[b,l_3]=l_3^{-2}\neq 1$. Since $B_5(5)$ is a polycylic group generated by polycyclic generating sequence a, b, l_1, l_2, l_3, l_4 and l_5. Then, $B_5(5)'=\langle l_3,l_4l_5^{-1}\rangle$.

Since $\nu(B_5(5))$ is polycyclic then the subgroups $B_5(5)$ and $B_5(5)'^{\varphi}$ have polycylic generating sequence $a, b, l_1, l_2, l_3, l_4, l_5, l_3^{\varphi}$ and $(l_4l_5^{-1})^{\varphi}$. Then, by Lemma 2.1

$$[B_5(5),B_5(5)'^{\varphi}]=\left\langle \begin{array}{l}[a,l_3^{\varphi}],[b,l_3^{\varphi}],[l_1,l_3^{\varphi}],[l_2,l_3^{\varphi}],[l_3,l_3^{\varphi}],\\ [l_4,l_3^{\varphi}],[l_5,l_3^{\varphi}],[a,(l_4l_5^{-1})^{\varphi}],[b,(l_4l_5^{-1})^{\varphi}],\\ [l_1,(l_4l_5^{-1})^{\varphi}],[l_2,(l_4l_5^{-1})^{\varphi}],[l_3,(l_4l_5^{-1})^{\varphi}],\\ [l_4,(l_4l_5^{-1})^{\varphi}],[l_5,(l_4l_5^{-1})^{\varphi}]\end{array}\right\rangle.$$

However, some of these generators can be expressed as products of powers of other generators. By Corollary 2.1, $[l_1,l_3^{\varphi}]=1$, $[l_2,l_3^{\varphi}]=1$, $[l_1,(l_4l_5^{-1})^{\varphi}]=1$ and $[l_2,(l_4l_5^{-1})^{\varphi}]=1$. Besides, by Lemma 2.3 (vi), $[l_3,l_3^{\varphi}]=1$.

By Lemmas 2.3 and 2.4, since ${}^{a}l_4=l_5$ and ${}^{a}l_5=l_4$, we have $[l_5,l_3^{\varphi}]={}^{a}[l_5,l_3^{\varphi}]=[l_4,l_3^{\varphi}]$ and $[l_5,(l_4l_5^{-1})^{\varphi}]={}^{a}[l_5,(l_4l_5^{-1})^{\varphi}]=[l_4,(l_4l_5^{-1})^{\varphi}]^{-1}$. Since b commutes with l_4 and l_5, then $[b,l_4^{\varphi}]$ and $[b,l_5^{\varphi}]$ are in the center of $\nu(B_4(5))$. Thus,

$$[b,(l_4l_5^{-1})^{\varphi}]=[b,l_4^{\varphi}][b,l_5^{\varphi}]^{-1}=[b,l_4^{\varphi}]\,{}^{a}[b,l_5^{\varphi}]^{-1}=[b,l_4^{\varphi}][bl_3^{-1},l_4^{\varphi}]^{-1}$$

and

$$[b,(l_4l_5^{-1})^{\varphi}]={}^{a}[b,(l_4l_5^{-1})^{\varphi}]=[bl_3^{-1},(l_5l_4^{-1})^{\varphi}].$$

However, since $[l_3,l_4^{\varphi}]$ and $[l_3^{-1},(l_5l_4^{-1})^{\varphi}]$ are in the center of $\nu(B_5(5))$, then

$$\begin{aligned}[bl_3^{-1},l_4^{\varphi}]^{-1}&=\left({}^{b}[l_3^{-1},l_4^{\varphi}][b,l_4^{\varphi}]\right)^{-1}\\ &=[b,l_4^{\varphi}]^{-1}[l_3,l_4^{\varphi}]^{-1}\\ &=[b,l_4^{\varphi}]^{-1}[l_3^{-1},l_4^{\varphi}]\\ &=[b,l_4^{\varphi}]^{-1}\,{}^{b}[l_3^{-1},l_4^{\varphi}]=[l_3,l_4^{\varphi}][b,l_4^{\varphi}]^{-1}\end{aligned}$$

and

$$\begin{aligned}[bl_3^{-1},(l_5l_4^{-1})^{\varphi}]&={}^{b}[l_3^{-1},(l_5l_4^{-1})^{\varphi}][b,(l_5l_4^{-1})^{\varphi}]\\ &=[l_3,(l_5l_4^{-1})^{\varphi}][b,(l_5l_4^{-1})^{\varphi}]^{-1}.\end{aligned}$$

Hence, we have $[b,(l_4l_5^{-1})^{\varphi}]=[l_3,l_4^{\varphi}]=[l_4,l_3^{\varphi}]^{-1}$ since $l_3\in B_5(5)'$. Also, $[b,(l_4l_5^{-1})^{\varphi}]^2=[l_3,(l_5l_4^{-1})^{\varphi}]$. Therefore,

$$[B_5(5),B_5(5)'^{\varphi}]=\left\langle \begin{array}{l}[a,l_3^{\varphi}],[b,l_3^{\varphi}],[a,(l_4l_5^{-1})^{\varphi}],\\ [b,(l_4l_5^{-1})^{\varphi}],[l_4,(l_4l_5^{-1})^{\varphi}]\end{array}\right\rangle.$$

Thus, we obtain

$$B_5(5)\wedge B_5(5)=\left\langle \begin{array}{l}[a,b^{\varphi}],[a,l_5^{\varphi}],[b,l_5^{\varphi}],[a,l_3^{\varphi}],\\ [b,l_3^{\varphi}],[a,(l_4l_5^{-1})^{\varphi}],[b,(l_4l_5^{-1})^{\varphi}],\\ [l_4,(l_4l_5^{-1})^{\varphi}]\end{array}\right\rangle.$$

Next the order of each generators of $B_5(5)\wedge B_5(5)$ are computed. Since $B_5(5)'=\langle l_3,(l_4l_5^{-1})\rangle$, then by Corollary 2.1 and Lemma 2.3 (iv),

$$[a,l_3^{\varphi}]^2=[a^2,l_3^{\varphi}]=[l_1^{-1}l_2^{-1}l_3^{-1},l_3^{\varphi}]=1,$$
$$[b,(l_4l_5^{-1})^{\varphi}]^2=[b^2,(l_4l_5^{-1})^{\varphi}]=[l_1^{-1},(l_4l_5^{-1})^{\varphi}]=1,$$

and

$$\begin{aligned}[l_4,(l_4l_5^{-1})^{\varphi}]^2&=([l_4,l_4^{\varphi}]\cdot[l_4,l_5^{-\varphi}])^2\\ &=[l_4,l_4^{\varphi}]\cdot[l_4,l_5^{-\varphi}]\cdot{}^{a}[l_4,l_4^{\varphi}]\cdot{}^{a}[l_4,l_5^{-\varphi}]\\ &=[l_4,l_4^{\varphi}]\cdot[l_5,l_5^{\varphi}]\cdot[l_4,l_5^{-\varphi}][l_5^{-1},l_4^{\varphi}]\\ &=[l_4l_5^{-1},(l_4l_5^{-1})^{\varphi}]\\ &=1.\end{aligned}$$

Therefore, $[a,l_3^{\varphi}]$, $[b,(l_4l_5^{-1})^{\varphi}]$ and $[l_4,(l_4l_5^{-1})^{\varphi}]$ have order two.

Next $[a,(l_4l_5^{-1})]=al_4l_5^{-1}a^{-1}l_5l_4^{-1}=(l_5l_4^{-1})^2\neq 1$. The mapping $\kappa: B_5(5)\otimes B_5(5)\to B_5(5)'$ defined in Theorem 2.3 gives $\kappa([a,(l_4l_5^{-1})^{\varphi}])=[a,(l_4l_5^{-1})]$, $\kappa([a,l_5^{\varphi}])=[a,l_5]$, $\kappa([b,l_3^{\varphi}])=[b,l_3]$ and $\kappa([a,b^{\varphi}])=[a,b]$. Since $[a,(l_4l_5^{-1})]=(l_5l_4^{-1})^2\neq 1$, $[a,l_5]=l_4l_5^{-1}\neq 1$, $[b,l_3]=l_3^{-2}\neq 1$ and $[a,b]=bl_3^{-1}b^{-1}=l_3^{-1}\neq 1$ in

$B_5(5)'$ have infinite order, it follows from Lemma 2.5 that $\left[a,\left(l_4 l_5^{-1}\right)^{\varphi}\right]$, $\left[a,l_5^{\varphi}\right]$, $\left[b,l_3^{\varphi}\right]$ and $\left[a,b^{\varphi}\right]$ have infinite order.

Next we denote the abelianisation of $B_5(5)$ by $B_5(5)^{ab}$ with natural homomorphism

$$\varepsilon : B_5(5) \to B_5(5)^{ab}.$$

By Theorem 2.4 there is an epimorphism

$$\begin{aligned}\alpha &: B_5(5)\otimes B_5(5) \to B_5(5)^{ab}\otimes B_5(5)^{ab}\\ &\cong \left(C_0\times C_0\times C_0\right)\otimes\left(C_0\times C_0\times C_0\right)\\ &\cong C_0^9\end{aligned}$$

(by Lemma 2.6). By Theorem 2.6 the group $B_5(5)^{ab}$ is generated by $\varepsilon(a)$, $\varepsilon(b)$ and $\varepsilon(l_5)$ of infinite order. Lemma 2.6 gives

$$\left\langle \varepsilon(b)\otimes\varepsilon(l_5)\right\rangle \cong C_0.$$

Therefore, the image $\alpha(b\otimes l_5)=\varepsilon(b)\otimes\varepsilon(l_5)$ has infinite order. Hence, by Lemma 2.5, $b\otimes l_5$ has infinite order. Then, $\left[b,l_5^{\varphi}\right]$ has infinite order as needed.

Finally, we have

$$\begin{aligned}B_5(5)\wedge B_5(5) &= \left\langle \begin{matrix}\left[a,b^{\varphi}\right],\left[a,l_5^{\varphi}\right],\left[b,l_5^{\varphi}\right],\left[a,l_3^{\varphi}\right],\left[b,l_3^{\varphi}\right],\\ \left[a,\left(l_4l_5^{-1}\right)^{\varphi}\right],\left[b,\left(l_4l_5^{-1}\right)^{\varphi}\right],\left[l_4,\left(l_4l_5^{-1}\right)^{\varphi}\right]\end{matrix}\right\rangle\\ &\cong C_0\times C_0\times C_0\times C_2\times C_0\times C_0\times C_2\times C_2.\end{aligned}$$

Theorem 3.4 The exterior square of $B_6(5)$ is

$$B_6(5)\wedge B_6(5) = \left\langle \begin{matrix}\left[a,l_2^{\varphi}\right],\left[b,l_2^{\varphi}\right],\left[a,b^{\varphi}\right],\left[b,l_4^{\varphi}\right],\\ \left[l_3,l_4^{\varphi}\right],\left[b,\left(l_3l_5^{-1}\right)^{\varphi}\right]\end{matrix}\right\rangle.$$

Proof:
The proof is omitted since the method is similar to the previous three theorems

4.0 CONCLUSION

In this paper, the exterior squares have been computed for some Bieberbach groups with abelian point groups where their abelianisations are finitely generated and have no element of order two. The results are shown in Theorem 3.1 to Theorem 3.4.

Acknowledgement

The researchers would like to acknowledge the Ministry of Higher Education (MOHE) Malaysia for the financial funding of this research through Fundamental Research Grant Scheme (FRGS), Vote No: 2011-0068-101-02 from Research Management Centre (RMC) UPSI and Research University Grant (RUG), Vote No. Q.J130000.2626.05J46 from RMC Universiti Teknologi Malaysia (UTM) Johor Bahru. The first author is also indebted to MOHE for her MyPhD Scholarship.

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