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Natural Convection Boundary Layer Flow Past a Sphere with Constant Heat Flux in Viscoelastic Fluid

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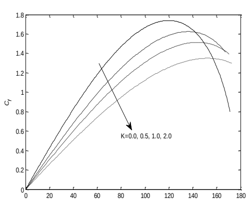
Graphical abstract



Abstract

The steady natural convection boundary layer flow of a viscoelastic fluid over a solid sphere with constant heat flux is studied in this paper. The boundary layer equations of viscoelastic fluid are an order higher than those for the Newtonian (viscous) fluid. The adherence boundary conditions are insufficient to determine the solution of these equations completely. Thus, the augmentation an extra boundary condition is needed to perform the numerical computational. The governing boundary layer equations are first transformed into non-dimensional form by using special dimensionless variables and then solved by using an implicit finite difference scheme known as Keller box method. Numerical results for the velocity and temperature profiles, wall temperature, as well as skin friction are shown graphically for different values of viscoelastic parameters and Prandtl number. It is found that, when the viscoelastic parameter increased, the values of skin friction decreased while the values of wall temperature are increased.

Keywords: Natural convection; boundary layer flow; viscoelastic fluid; sphere, constant heat flux

Abstrak

Aliran lapisan sempadan olakan semula jadi mantap bagi bendalir likat kenyal melepasi sfera dengan fluks haba dikaji dalam artikel ini. Persamaan-persamaan lapisan sempadan bagi bendalir likat kenyal terjana berperingkat lebih tinggi berbanding bendalir Newtonan (likat). Pematuhan syarat sempadan adalah tidak mencukupi untuk menentukan penyelesaian persamaan ini dengan selengkapnya. Oleh itu, penambahan syarat sempadan tambahan diperlukan untuk membolehkan pengiraan secara berangka dilakukan. Persamaan sempadan menakluk diubah ke dalam bentuk tak bermatra dengan menggunakan pemboleh ubah tak berdimensi yang khas dan diselesaikan menggunakan skim beza terhingga tersirat yang efektif dikenali sebagai kaedah kotak-Keller. Hasil kajian yang diperolehi bagi profil halaju dan suhu, suhu dinding, dan pekali geseran kulit dipersembahkan secara grafik untuk pelbagai nilai parameter likat kenyal dan nombor Prandtl. Didapati bahawa apabila parameter bendalir likat kenyal meningkat, nilai geseran kulit menurun tetapi nilai suhu dinding meningkat.

Kata kunci: Olakan semula jadi; aliran lapisan sempadan; bendalir likat kenyal; sfera; fluk haba

1.0 INTRODUCTION

The problem on natural convection flows under boundary layer analysis are fundamental theoretical and have many practical interest. Many researchers all over the world have investigated these type of flows in different geometries and different type of fluids (Newtonian or non Newtonian) due to wide practical applications in engineering. The flow and heat transfer phenomena over sphere have received a considerable attention during these recent years due to its practical needs in numerous engineering applications including solving the cooling problems in turbine blades, electronic systems and manufacturing processes.[1] Prhashanna and Chhabra,[2] reported that the flow of fluids past a sphere and heat transfer from it represents a classical model configuration to elucidate the nature of the underlying physical processes that will improve on our fundamental understanding. Extensive studies on the topic of natural convection specifically on sphere have been conducted by several researchers for the last few decades. For example, Chiang *et al.*[3] studied an exact analysis of the laminar free convection from a sphere by considering prescribed surface temperature and surface heat flux. The work has been continued by Huang and Chen,[4] by considering the effects of blowing and suction. Further, Nazar *et al.*[5-6] considered the problem of free convection boundary layer on an isothermal sphere in micropolar fluids by considered two cases which are constant wall temperature and constant heat flux. The theory boundary layer problem of viscoelastic fluids has gained a lot of interest, and become important in recent years because of their applications in several industrial-manufacturing processes involving petroleum drilling, manufacturing of foods and paper. In engineering applications, it is possible to use viscoelastic fluids to reduce frictional drag on the hulls of ships and submarines. Literature survey indicated there has been an extensive research available regarding the viscoelastic fluid. Thomas and walters,[7] presented the unsteady motion of a sphere in a viscoelastic liquid where they considered the unsteady motion of a sphere moving under a constant force. Verma,[8] derived the boundary layer equations near a body of revolution in a uniform stream and a case of the boundary layer over the surface of sphere and found that the increase in the elasticity of the liquid causes a shift in the point of separation towards the forward stagnation point. Carew *et al.*[9] have considered the problem of a sphere move along the axis of vertical cylindrical tube containing a viscoelastic fluid. On the other hand, viscoelastic flows also has the application in chemical engineering systems which arise in numerous processes. Such flows possess both viscous and elastic properties and can exhibit normal stresses and relaxation effects. Recently, Chang *et al.*[10] conducted a numerical study of transient free convective mass transfer in a Walters-B viscoelastic flow with wall suction. Velocity was found to increase with a rise in viscoelasticity parameter with both time and distances close to the plate surface. The differential governing equations of the viscoelastic fluid problems are an order higher than those for the Newtonian (viscous) fluid and the adherence boundary conditions are insufficient to determine the numerical solution completely. Therefore, a boundary condition is needed in addition to the usual adherence boundary conditions,[11-12]. Very recently, Kasim *et al.*[13] investigated the problem of viscoelastic over a sphere by considering the different boundary condition called Newtonian heating. Motivated by the work above, this paper aims to investigate the problem of natural convection boundary layer flow of viscoelastic fluid on solid sphere with constant heat flux. The full governing boundary layer equations are first transformed into a system of non-dimensional equations via the non-dimensional variables, and then into non-similar equations before they are solved numerically by the Keller-box method as described in the books by Na,[14] and Cebeci and Bradshaw,[15]. Results are presented for the skin friction coefficient and the wall temperature as well as the velocity and temperature profiles. To the best of our knowledge, this problem has not been considered before, so that the reported results are new and original.

2.0 MATHEMATICAL FORMULATION

The problem studied is this article is steady natural convection boundary layer flow past a sphere placed in a viscoelastic fluid. Figure 1 illustrates the geometry of the problem and the corresponding coordinate system. The problem was considered as a heated sphere of radius a, which is immersed in a viscous and incompressible fluid of ambient temperature, T_∞. It is assumed that the constant heat flux of the surface of the cylinder is q_w.

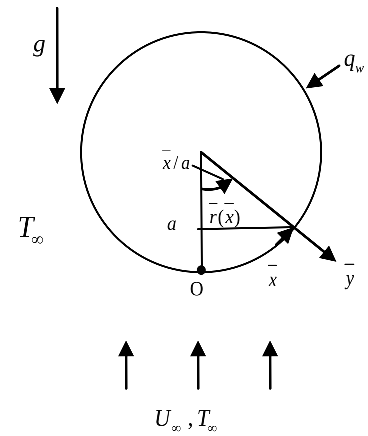


Figure 1 Physical model and coordinate system

Under the usual Boussinesq and boundary layer approximations, the equations for mass continuity, momentum and energy took the following form,

$$\frac{\partial}{\partial \bar{x}}\left(\bar{r}\,\bar{u}\right)+\frac{\partial}{\partial \bar{y}}\left(\bar{r}\,\bar{v}\right)=0, \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}=\upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+\frac{k_0}{\rho}\left[\frac{\partial}{\partial \bar{x}}\left(\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)+\bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}+\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right] \tag{2}$$
$$+g\beta(T-T_\infty)\sin(\bar{x}/a),$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}}+\bar{v}\frac{\partial T}{\partial \bar{y}}=\alpha\frac{\partial^2 T}{\partial \bar{y}^2}, \tag{3}$$

which are subjected to the boundary conditions;

$$\bar{u}=\bar{v}=0, \quad \frac{\partial T}{\partial \bar{y}}=-\frac{q_w}{k} \quad \text{on} \quad \bar{y}=0,$$
$$\bar{u}=0, \quad \frac{\partial \bar{u}}{\partial \bar{y}}=0, \quad T=T_\infty, \quad \text{as } \bar{y}\to\infty, \tag{4}$$

where ρ, g, β, μ, k_0, α, and T are the density, gravitational acceleration, coefficient of thermal expansion, dynamic viscosity, vortex viscosity and thermal diffusivity of the fluid and local temperature respectively. In this problem, we assume $\bar{r}(\bar{x})=a\sin\left(\bar{x}/a\right)$, $\bar{u}$ and $\bar{v}$ are the velocity components along $\bar{x}$ and $\bar{y}$ direction respectively.

Then, the following non-dimensional variables are introduce,

$$x=\bar{x}/a,\quad y=\mathrm{Gr}^{1/4}\left(\bar{y}/a\right),\quad r(x)=\bar{r}(\bar{x})/a,\quad u=\frac{a}{\upsilon}\mathrm{Gr}^{-1/2}\bar{u},$$
$$v=\frac{a}{\upsilon}\mathrm{Gr}^{-1/4}\bar{v},\quad \theta=\mathrm{Gr}^{1/4}\left(T-T_\infty\right)/\left(q_w a/k\right), \tag{5}$$

where $\mathrm{Gr}=\dfrac{g\beta(T_w-T_\infty)a^3}{\nu^2}$ is the Grashof number.

Substitution of Equation (5) into Equations (1) to (3), led to the following non-dimensional equations:

$$\frac{\partial}{\partial x}(ur)+\frac{\partial}{\partial y}(rv)=0, \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\mathrm{K}\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2}\right] +\theta\sin(x), \quad (7)$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}, \quad (8)$$

and the boundary conditions (4) becomes

$$u=v=0 \quad \theta'=-1 \;\; on \quad y=0,$$
$$u=0,\;\; \frac{\partial \bar{u}}{\partial y}=0,\;\; \theta=0,\;\; y\to\infty, \quad (9)$$

where $\mathrm{K}=\dfrac{k_0\mathrm{Gr}^{5/2}}{a^2}$ represent the viscoelastic parameter.

■3.0 SOLUTION PROCEDURES

In order to solve Equations (6) to (8) according to the boundary conditions (9), the following variables were assumed:

$$\psi = xr(x)f(x,y), \qquad \theta=\theta(x,y), \quad (10)$$

where ψ is the stream function defined as

$$u=\frac{1}{r}\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{1}{r}\frac{\partial\psi}{\partial x}, \quad (11)$$

which satisfies the continuity equation (6), thus equation (7) and (8) becomes

$$\frac{\partial^3 f}{\partial y^3}-\left(\frac{\partial f}{\partial y}\right)^2+\left(1+x\frac{\cos x}{\sin x}\right)f\frac{\partial^2 f}{\partial y^2} +\mathrm{K}\left[2\frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial y^3}-\left(1+x\frac{\cos x}{\sin x}\right)\left(f\frac{\partial^4 f}{\partial y^4}+\left(\frac{\partial^2 f}{\partial y^2}\right)^2\right)\right]+\theta\frac{\sin x}{x}$$
$$=\mathrm{K}x\left[\frac{\partial f}{\partial x}\frac{\partial^4 f}{\partial y^4}-\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^3 f}{\partial y^3}-\frac{\partial f}{\partial y}\frac{\partial^4 f}{\partial x\partial y^3}+\frac{\partial^2 f}{\partial y^2}\frac{\partial^3 f}{\partial x\partial y^2}\right] +x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \quad (12)$$

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}+\left(1+x\frac{\cos x}{\sin x}\right)f\frac{\partial\theta}{\partial y}=x\left(\frac{\partial f}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial\theta}{\partial y}\right), \quad (13)$$

with respect to the following boundary conditions:

$$f=0,\quad \frac{\partial f}{\partial y}=0 \quad \theta'=-1 \;\; on \quad y=0,$$
$$\frac{\partial f}{\partial y}=0,\;\; \frac{\partial^2 f}{\partial y^2}=0,\;\; \theta=0 \;\; on \quad y\to\infty. \quad (14)$$

At the lower stagnation point of the cylinder, $x\approx 0$, Equations (12) and (13) was reduced to the following ordinary differential equation:

$$f'''+2ff''-f'^2+\theta+2\mathrm{K}(ff''''-f'f'''+f''^2)=0, \quad (15)$$

$$\frac{1}{\Pr}\theta''+f\theta'=0, \quad (16)$$

with the boundary conditions

$$f(0)=f'(0)=0, \quad \theta'(0)=-1$$
$$f'(\infty)=0, \quad f''(\infty)=0, \quad \theta(\infty)=0, \quad (17)$$

where primes denote the differentiation with respect to y.

In practical application the physical quantities of principal interest are heat transfer which can be written in non-dimensional form as:

$$C_f = x\frac{\partial^2 f}{\partial y^2}(x,0), \qquad \theta_w(x)=\theta(x,0), \quad (18)$$

where $C_f=\tau_w/(\rho U_\infty{}^2)$ is the skin friction and wall sheer stress is donated as

$$\tau_w=\mu\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_{\bar{y}=0}+k_0\left(\bar{u}\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{y}}+\bar{v}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+2\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{y}}\right)_{\bar{y}=0}. \quad (19)$$

■4.0 RESULTS AND DISCUSSION

The systems of Equations (12) and (13) together with Equations (15) and (16) with respected to boundary conditions (14) and (17) respectively were solved numerically for some values of the viscoelastic parameter, K and Prandtl number, Pr using the implicit finite-difference method known as Keller-box method. In order to ensure the accuracy and convergence of the numerical solution to exact solution, the step sizes Δy have been optimized and the results presented are independent of the step sizes at least up to the sixth decimal places. The convergence criterions were based on the relative difference between the current and previous iteration values of the velocity and temperature gradients at wall. When the difference reaches less than for the flow fields, the solutions assumed to converge, and the iterative process is terminated. The present results for local wall temperature $\theta_w(0)$ were compared with those of Nazar *et al.*[6] and Salleh *et al.*[16] in order to validate the numerical results obtained. The comparison shows that the numerical solutions (see Table 1) obtained by the present authors concurs very well with those of previous authors.

Table 1 Numerical values of local wall temperature $\theta_w(x,0)$ for different position x of surface at Pr = 0.7

x	Nazar *et al.*[6]	Salleh *et al.* [16]	Present
0.0	1.8960	1.8692	1.868927
$\pi/9$	1.8795	1.8794	1.878951
$\pi/6$	1.8924	1.8922	1.891233
$\pi/3$	1.9653	1.9651	1.964208
$\pi/2$	2.1038	2.0469	2.100469
$2\pi/3$	2.3475	2.3444	2.345508

Figures 2 and 3 illustrate the behavior of skin friction and wall temperature for various values of viscoelastic parameter, K at Prandtl number 0.7. It shows that, when the viscoelastic parameter K increased, it reduced the values of skin friction and increased on the values of wall temperature. For industrial field, there is importance application since the power expenditure will decrease in increasing the value of viscoelastic parameter, K. This particular result also reported by a few researchers such as Rajagopal *et al.*[17], Subhas and Veena,[18] and also Veena *et al.*[19]

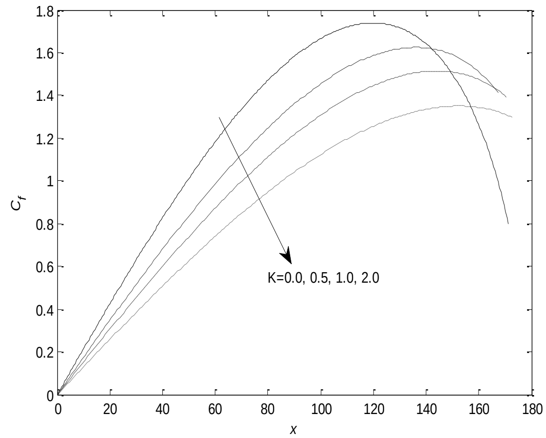


Figure 2 Variation of the skin friction coefficient with x for Pr = 0.7 with various values of K.

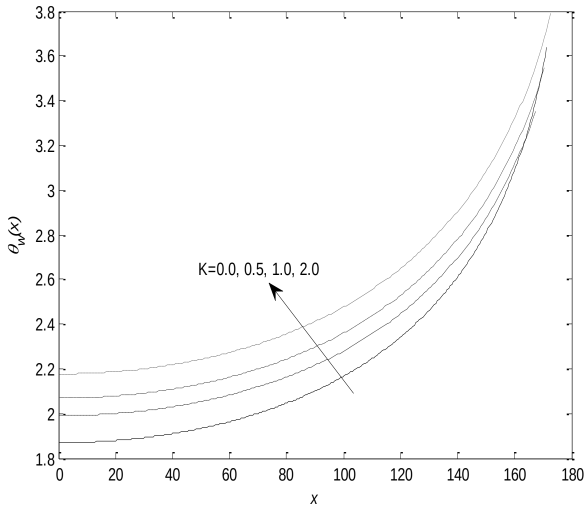


Figure 3 Variation of wall temperature with x for Pr = 0.7 with various values of K.

In Table 2, we can see the numerical values of skin friction and wall temperature for the various values of viscoelastic parameter K. It shows that, as the viscoelastic parameter K increases, the values of skin friction are decreased while the values of wall temperature are increased.

Table 2 Values of f'' (0) and θ_w (0) for variousvalues of K when Pr = 0.7,

K	f'' (0)	θ_w (0)
0.1	1.137651	1.912281
0.2	1.067937	1.948131
0.3	1.011718	1.978923
0.4	0.964914	2.006050
0.5	0.925029	2.030382
1.0	0.786504	2.125697
2.0	0.639894	2.252308
3.0	0.557493	2.341101
4.0	0.502402	2.410692
5.0	0.462058	2.468448

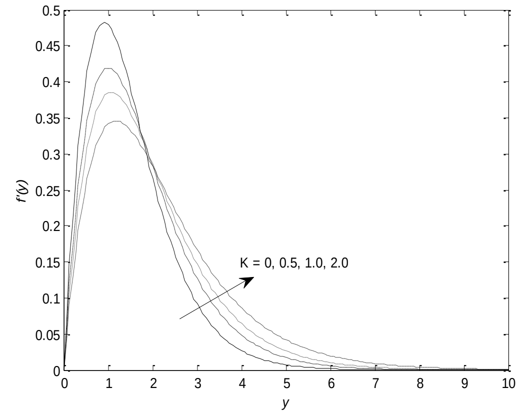


Figure 4 Velocity profile for various values of K with Pr = 0.7

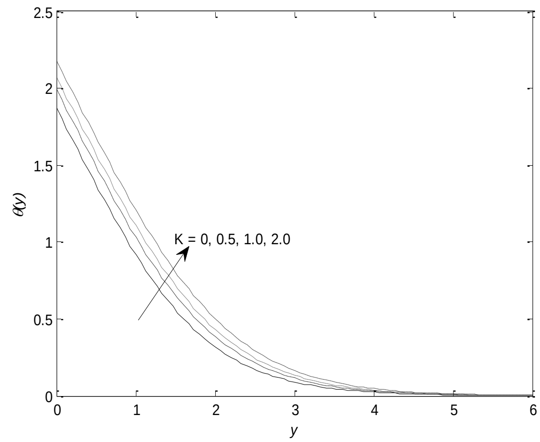


Figure 5 Temperature profile for various values of K with Pr = 0.7

The effects of viscoelastic parameter on velocity and temperature profile at the lower stagnation point at Pr = 0.7 are illustrated in Figures 4 and 5. Based on Figure 4, it is noticed that, the velocity distributions decreased when the value of viscoelastic parameter, K is increased until one point (y =1.92), then we can see the profile of velocity distribution increase with the increase of values viscoelastic parameter. The values of these profiles are lower for a viscoelastic fluid than for a Newtonian fluid (K=0) for the range values of boundary layer thickness $0<y<1.92$. Therefore, we can say that, the thickness of the velocity boundary layer for a viscoelastic fluid is higher than for a Newtonian fluid. From Figure 5, we can say that, an increase on the value of viscoelastic parameter leads to the increment in temperature distribution. This behavior reflects the coupling of the energy equation to the momentum equation through the temperature dependent body forces.

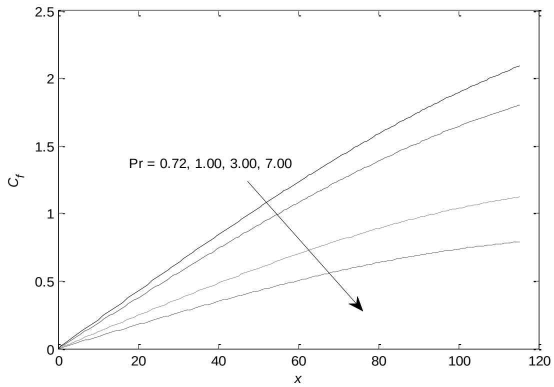


Figure 6 Variation of the skin friction coefficient with x For K= 1.0 with various values of Pr

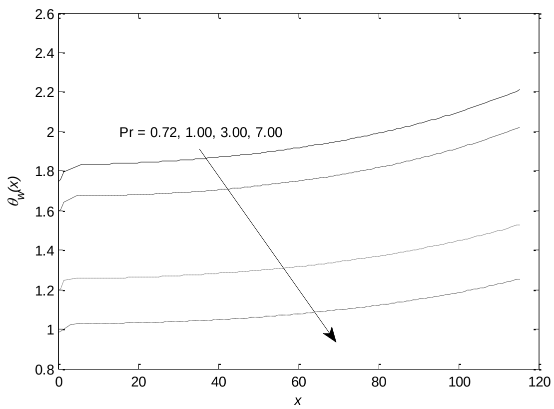


Figure 7 Variation of wall temperature with x for K = 1.0 with various values of Pr

The graphical illustration on the values skin friction and wall temperature against the position of x for the different values of Prandtl number are shown in Figures 6 and 7 respectively. From these figures, we can see that both skin friction and wall temperature decrease as the value of Prandtl number increase.

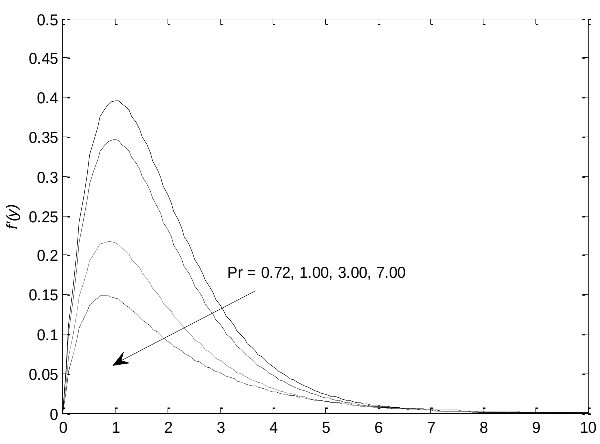


Figure 8 Velocity profile for various values of Pr with K=1

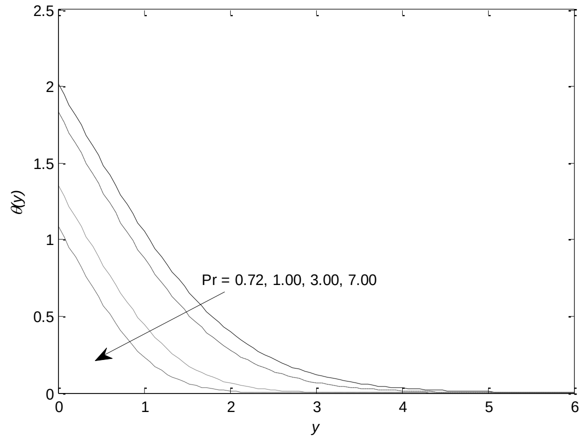


Figure 9 Temperature profile for various values of Pr with K=1.0

The effect of Prandtl number, Pr on the velocity and temperature profile were illustrated by Figures 8 and 9 respectively where the computation is running on Pr = 0.7, 1.0 and 7.0 at the fixed value of viscoelastic parameter equal to 1. It is found that as Pr increases, both velocity and temperature profiles decrease. This is because for small values of the Prandtl number the fluid is highly conductive. Physically, if Pr increases, the thermal diffusivity decreases and this phenomenon lead to the decreasing manner of the energy transfer ability that reduces the thermal boundary layer.

■5.0 CONCLUSION

In this paper we studied detail on the problem of steady natural convection boundary layer flow of a viscoelastic fluid on solid sphere with constant heat flux. The governing boundary layer equations were transformed into a non-dimensional form and the resulting nonlinear system of partial differential equations was solved numerically using the Keller-box method. From the investigation it revealed how the parameter K, and the Prandtl number, Pr affect the flow and heat transfer characteristics. From the present findings, we can conclude that,

- when the viscoelastic parameter K increased, it reduced the values of skin friction and increased the value of wall temperature.
- the velocity distributions decreased when the value of viscoelastic parameter K is increased until certain point at the fixed value of Prandtl number, Pr.
- as the values of viscoelastic parameter K increase, it will leads to the increment in temperature distribution
- Both values of velocity and temperature profiles decrease as increasing in the values of Prandtl number, Pr.

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