Jurnal Teknologi

Full paper

Multiscale Element-Free Galerkin Method with Penalty for 2D Burgers' Equation

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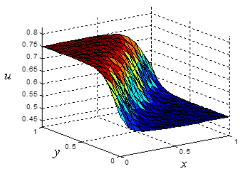
Article history

Received :18 March 2013
Received in revised form :
26 April 2013
Accepted :17 May 2013

Graphical abstract



Abstract

In this paper, a new numerical method which is based on the coupling between multiscale method and meshless method with penalty is developed for 2D Burgers' equation. The advantage of meshless method over the finite element method (FEM) is that remeshing process is not required. This is because the meshless method approximation is constructed entirely in terms of a set of nodes. Since the moving least squares (MLS) shape function does not satisfy the Kronecker delta property, so penalty method is adopted to enforce the essential boundary conditions in this paper. In order to obtain the fine scale approximation, the local enrichment basis is applied. The local enrichment basis may adopt the polynomial basis functions or any other analytical basis functions. Here, the polynomial basis functions are chosen as local enrichment basis. This multiscale meshless method with penalty will provide a more accurate result especially in the critical region which requires higher accuracy. It is believed that this proposed method is an attractive approach for solving more general problems which involve large deformation.

Keywords: Burgers' equation; multiscale method; meshless method; penalty method; critical area

Abstrak

Dalam kertas kerja ini, satu kaedah baru berangka yang berdasarkan gandingan antara kaedah multiscalar dan kaedah meshless dengan penalti telah dibangunkan bagi persamaan 2D Burgers'. Kelebihan kaedah meshless lebih kaedah unsur terhingga (FEM) adalah bahawa proses remeshing tidak diperlukan. Ini adalah kerana kaedah penghampiran meshless dibina sepenuhnya dari segi set nod. Sejak dataran kurangnya bergerak (MLS) rangkap bentuk tidak memenuhi delta Kronecker, jadi kaedah penalti dipakai untuk menguatkuasakan syarat sempadan penting dalam kertas ini. Dalam usaha untuk mendapatkan penghampiran skala halus, asas pengayaan tempatan digunakan. Asas pengayaan tempatan boleh menerima pakai fungsi asas polinomial atau mana-mana fungsi analisis asas yang lain. Di sini, fungsi asas polinomial yang dipilih sebagai asas pengayaan tempatan. Multiscalar kaedah meshless dengan penalti ini akan memberikan hasil yang lebih tepat terutamanya di rantau genting yang memerlukan ketepatan yang lebih tinggi. Adalah dipercayai bahawa kaedah yang dicadangkan adalah satu pendekatan yang menarik untuk menyelesaikan masalah yang lebih umum yang melibatkan ubah bentuk yang besar.

Kata kunci: Persamaan Burgers'; kaedah multiscalar; kaedah meshless; kaedah penalti; kawasan kritikal

1.0 INTRODUCTION

The Burgers' equation was first introduced by Bateman [1] who studied for the weather problem in 1915. Later, Burgers treated this equation as a mathematical model for free turbulence and shock wave [2]. After the extensive works of Burgers, it is widely to be called as Burgers' equation. This Burgers' equation is a nonlinear partial differential equation of second order. It is occurs in various areas of applied mathematics such as modelling of gas dynamics, traffic and aerofoil flow theory, acoustic transmission, turbulence problems and boundary layer behaviour. Burgers' equation is a nonlinear equation which contains a convection term, a viscosity term and a time-dependent term. If the viscous term is included, Burgers' equation is parabolic, else it is hyperbolic. This clearly shown that Burgers equation behaves "mixed" property. Due to the complex geometry and complicated initial and boundary conditions of Burgers equation caused the exact solutions for the practical applications are very restricted. As a result, many researchers proposed various kinds of numerical methods to obtain the solution of Burgers' equation. These solution methodologies are finite difference method (FDM) [3, 4], finite element method (FEM) [5, 6] and the boundary element method (BEM) [7, 8]. But, there are some problems arise such as time consuming of mesh generation in FDM and FEM and the complicated singular integrals of BEM. Researchers faced a challenging task in development of a robust numerical method for seeking accurate solutions of Burgers' equation.

Recently, a kind of numerical method called meshless or meshfree methods have attract the researchers' attentions. To approximate the solution, this method only use a set of nodes scattered within the problem domain and no element is involved. Therefore, the mesh generation time will be saved. Thus, it is very convenient for adding particles in the desired region to refine the solution. In addition, meshfree methods are a modern approach to deal with challenging problems such as large deformation, nonlinearity or high gradient. Nowadays, there exist many meshfree methods such as element-free Galerkin method (EFGM) [9, 10, 11], reproducing kernel particle method (RKPM) [10] and meshless local Petrov–

Galerkin (MLPG) method [11]. There are some papers which applying meshfree methods to solve Burgers' equation such as Ouyang *et al*. [12] combined the characteristic Galerkin (CG) method with EFG to solve 1D and 2D Burgers' equation, Young *et al*. [13] have used the Eulerian–Lagrangian method of fundamental solutions to solve the two-dimensional unsteady Burgers' equations, Xie *et al*. [14] used reproducing kernel particle method (RKPM) to solve one-dimensional Burgers' equation and Zhang *et al*. [15] employed multiscale element-free Galerkin method for solving 2D Burgers' equation.

In the mid-90s, Hughes revisited the origins of the stabilization schemes from a variational multiscale approach and presented the variational multiscale method [16, 17]. In this method, different stabilization techniques appear as special cases of the underlying sub-grid scale modelling concept. The new formulation, termed as the Hughes variational multiscale (HVM) method was proposed. The starting point of this method is the decomposition of the scalar field into the coarse and the fine scales. Masud and coworkers developed multiscale/stabilized formulations for the linearized incompressible Navier–Stokes equations [18], the Darcy flow equation [19] and the advection-diffusion equation [20]. Besides, there are some researches also combined the variational multiscale with the meshfree methods. For example, Zhang and coworkers employed variational multiscale element free Galerkin method for the water wave problems [21], 2D Burgers' equation [15] and Stokes problem [22]. Jeoung and Sung [23] employed variational multiscale with meshfree approximation for efficient analysis of elastoplastic deformation.

An outline of the paper is as follows: Section 2 presents the fundamental principle of element free Galerkin method (EFGM). Emphasis in the paper is the description of the multiscale element free Galerkin method in Section 3. Section 4 presents the results and discussions and conclusions are drawn in Section 5.

2.0 FUNDAMENTAL PRINCIPAL OF THE EFGM

According to the moving least square (MLS) interpolant [9, 10], a local approximation $u^h(\boldsymbol{x})$ to the function $u(\boldsymbol{x})$ is given by

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{m} p_i(\boldsymbol{x}) a_i(\boldsymbol{x}) = \boldsymbol{p}^T(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) \tag{1}$$

where m is the number of basis functions, $p_i(\boldsymbol{x})$ are monomial basis functions, and $a_i(\boldsymbol{x})$ are their unknown coefficients. A commonly used linear basis is provided as

$$\boldsymbol{p}^T = (1, x) \quad \text{in 1D} \qquad \boldsymbol{p}^T = (1, x, y) \quad \text{in 2D}$$

and the quadratic basis,

$$\boldsymbol{p}^T = (1, x, x^2) \quad \text{in 1D} \qquad \boldsymbol{p}^T = (1, x, y, xy, x^2, y^2) \quad \text{in 2D}$$

The unknown coefficient $\boldsymbol{a}(\boldsymbol{x})$ is obtained at any point by minimizing the following weighted, discrete error norm

$$J = \sum_{I=1}^{N} w(\boldsymbol{x}-\boldsymbol{x}_I)[\boldsymbol{p}^T(\boldsymbol{x}_I)\boldsymbol{a}(\boldsymbol{x}) - u_I]^2 \tag{2}$$

where $w(\boldsymbol{x}-\boldsymbol{x}_I)$ is a weight function of compact support, N is the number of nodes in the neighbourhood of x for which the weight function $w(\boldsymbol{x}-\boldsymbol{x}_I) \neq 0$. In the present work, the following cubic spline is chosen as the weight function

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3, & r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3, & \frac{1}{2} < r \le 1 \\ 0, & r > 1 \end{cases}$$

where $r = \frac{|\boldsymbol{x}-\boldsymbol{x}_I|}{d_{mI}}$, d_{mI} is the size of the domain of influence of Node I. The minimum of J in (2) with respect to $\boldsymbol{a}(\boldsymbol{x})$ yields the following linear equations

$$\boldsymbol{A}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}$$

If A is invertible, the coefficient $\boldsymbol{a}(\boldsymbol{x})$ can be expressed as

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}$$

where $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are matrices defined by

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{\boldsymbol{I}=1}^{\boldsymbol{N}} w(\boldsymbol{x}-\boldsymbol{x}_I)\boldsymbol{p}(\boldsymbol{x}_I)\boldsymbol{p}^T(\boldsymbol{x}_I)$$

$$\boldsymbol{B}(\boldsymbol{x}) = [w(\boldsymbol{x}-\boldsymbol{x}_1)\boldsymbol{p}(\boldsymbol{x}_1), w(\boldsymbol{x}-\boldsymbol{x}_2)\boldsymbol{p}(\boldsymbol{x}_2), \cdots, w(\boldsymbol{x}-\boldsymbol{x}_N)\boldsymbol{p}(\boldsymbol{x}_N)]$$

$$\boldsymbol{u}^T = [u_1, u_2, ..., u_N]$$

Substitute $\boldsymbol{a}(\boldsymbol{x})$ into (1); the approximation $u^h(\boldsymbol{x})$ can be expressed as

$$u^h(\boldsymbol{x}) = \boldsymbol{p}^T(\boldsymbol{x})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} = \phi^T(\boldsymbol{x})\boldsymbol{u},$$

where the shape functions of EFGM is defined by

$$\phi^T(\boldsymbol{x}) = \boldsymbol{p}^T(\boldsymbol{x})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})$$

The moving least squares (MLS) approximation does not pass through the data used to fit the curve. Therefore, the MLS shape functions lack of Kronecker delta property, $\phi_i(\boldsymbol{x}_j) \neq \delta_{ij}$. This leads to the imposition of essential boundary conditions can be awkward. The inequality is

$$u^h(\boldsymbol{x}_I) \neq u_I$$

Due to unfulfillment of this property, researchers encounter problems in exerting Dirichlet boundary condition. Various strategies have been proposed to solve this problem. The general techniques are penalty methods [10], Lagrange multiplier approaches [10], modified variational principles [24], perturbed Lagrangian [25] and direct collocation method [26]. In this paper, penalty method has been selected to enforce essential boundary conditions.

3.0 THE VARIATIONAL MULTISCALE ELEMENT FREE GALERKIN METHOD

3.1 The 2D Burgers' Equation

Consider two-dimensional coupled nonlinear viscous Burgers' equations is defined over the domain $\Omega=[0,1]\times[0,1]$ given by the following equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{3}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{4}$$

where u and v are the velocity along x-axis and y-axis, Re is the Reynolds number. The initial conditions are:

$$u(x,y,0)=\frac{3}{4}-\frac{1}{4\left[1+e^{\frac{(-4x+4y)Re}{32}}\right]}\quad,\quad v(x,y,0)=\frac{3}{4}+\frac{1}{4\left[1+e^{\frac{(-4x+4y)Re}{32}}\right]}$$

the boundary conditions for u velocity are

$$u(0,y,t)=\frac{3}{4}-\frac{1}{4\left[1+e^{\frac{(4y-t)Re}{32}}\right]}\quad,\quad u(1,y,t)=\frac{3}{4}-\frac{1}{4\left[1+e^{\frac{(-4+4y-t)Re}{32}}\right]}$$

$$\frac{\partial u}{\partial n}(x,0,t)=\frac{1}{2}\frac{Re\left[e^{\frac{1}{32}(-4x-t)Re}\right]}{\left[4+4e^{\frac{1}{32}(-4x-t)Re}\right]^2}\quad,\quad \frac{\partial u}{\partial n}(x,1,t)=\frac{1}{2}\frac{Re\left[e^{\frac{1}{32}(-4x+4-t)Re}\right]}{\left[4+4e^{\frac{1}{32}(-4x+4-t)Re}\right]^2}$$

while the boundary conditions for v velocity are

$$v(x,0,t)=\frac{3}{4}+\frac{1}{4\left[1+e^{\frac{(-4x-t)Re}{32}}\right]}\quad,\quad v(x,1,t)=\frac{3}{4}+\frac{1}{4\left[1+e^{\frac{(-4x+4-t)Re}{32}}\right]}$$

$$\frac{\partial v}{\partial n}(0,y,t)=\frac{1}{2}\frac{Re\left[e^{\frac{1}{32}(4y-t)Re}\right]}{\left[4+4e^{\frac{1}{32}(4y-t)Re}\right]^2}\quad,\quad \frac{\partial v}{\partial n}(1,y,t)=\frac{1}{2}\frac{Re\left[e^{\frac{1}{32}(-4+4y-t)Re}\right]}{\left[4+4e^{\frac{1}{32}(-4+4y-t)Re}\right]^2}$$

3.2 The Standard Weak Form

We derive the discretization by developing a weak formulation. Let $V\subset H^1(\Omega)\cap C^0(\Omega)$ denote the space of trial solutions and weighting functions for the unknown variables. Multiplying both sides of eq. (3) by an admissible weighting function and integrating it over the domain yields

$$\int w\left[u_t+uu_x+vu_y\right]d\Omega=\int w\frac{1}{Re}\nabla^2 u\,d\Omega$$

Using Green's theorem, we get

$$\int w\left[u_t+uu_x+vu_y\right]d\Omega=\frac{1}{Re}\int_\Gamma w\nabla u\cdot n\,ds-\frac{1}{Re}\int_\Omega \nabla w\cdot\nabla u\,d\Omega \tag{5}$$

So, the functional $I(u)$ is obtained as

$$I(u)=\int u\left[u_t+uu_x+vu_y\right]d\Omega+\frac{1}{Re}\int_\Omega \nabla u\cdot\nabla u\,d\Omega-\frac{1}{Re}\int_\Gamma u\nabla u\cdot n\,ds \tag{6}$$

To enforce the essential boundary conditions, penalty method is used. Therefore, the functional $I(u)$ becomes

$$I(u)=\int u\left[u_t+uu_x+vu_y\right]d\Omega+\frac{1}{Re}\int_\Omega \nabla u\cdot\nabla u\,d\Omega-\frac{1}{Re}\int_\Gamma u\nabla u\cdot n\,ds+\frac{1}{Re}\int_{\Gamma_u}\frac{\alpha}{2}(u-u_E)^2 ds \tag{7}$$

where u_E is the u velocity on essential boundary. Taking variation of Eq. (7), it reduces to

$$\begin{aligned}\delta I(u)=&\int \delta u\left[u_t+uu_x+vu_y\right]d\Omega\\&+\int u\left[(\delta u)_t+\delta u u_x+u(\delta u)_x+v(\delta u)_y\right]d\Omega\\&+\frac{1}{Re}\int_\Omega \nabla\delta u\cdot\nabla u\,d\Omega+\frac{1}{Re}\int_\Omega \nabla u\cdot\nabla\delta u\,d\Omega-\frac{1}{Re}\int_\Gamma \delta u\nabla u\cdot n\,ds\\&-\frac{1}{Re}\int_\Gamma u\nabla\delta u\cdot n\,ds+\frac{1}{Re}\int_{\Gamma_u}\alpha(u-u_E)\delta u\,ds=0\end{aligned}$$

Since

$$(\delta u)_t+\delta u u_x+u(\delta u)_x+v(\delta u)_y=\frac{1}{Re}\nabla^2\delta u$$

Finally, we get the following weak form for u velocity and v velocity

$$\begin{aligned}&(\delta u,u_t)+(\delta u,\boldsymbol{a}\cdot\nabla u)+\left(\nabla\delta u,\frac{1}{Re}\nabla u\right)+\frac{\alpha}{Re}\int_{\Gamma_u}u\delta u\,ds\\&=\frac{1}{Re}\int_{\Gamma_u}\delta u\nabla u\cdot n\,ds+\frac{1}{Re}\int_{\Gamma_t}\delta u\nabla u\cdot n\,ds+\frac{\alpha}{Re}\int_{\Gamma_u}u_E\delta u\,ds\end{aligned} \tag{8}$$

$$\begin{aligned}&(\delta v,v_t)+(\delta v,\boldsymbol{a}\cdot\nabla v)+\left(\nabla\delta v,\frac{1}{Re}\nabla v\right)+\frac{\alpha}{Re}\int_{\Gamma_v}v\delta v\,ds\\&=\frac{1}{Re}\int_{\Gamma_v}\delta v\nabla v\cdot n\,ds+\frac{1}{Re}\int_{\Gamma_t}\delta v\nabla v\cdot n\,ds+\frac{\alpha}{Re}\int_{\Gamma_v}v_E\delta v\,ds\end{aligned} \tag{9}$$

where w is the weighting function for u, $\boldsymbol{a}=(u,v)$, and $(\cdot,\cdot)=\int_\Omega(\cdot)d\Omega$ is the $L_2(\Omega)$ inner product space.

3.3 Multiscale Decomposition of the Classical Weak Formulation

Assume that the scalar field can be decomposed into coarse scale and fine scale,

$$u = \bar{u} + \hat{u} \qquad (10)$$

where $\bar{u}$ is coarse scale and $\hat{u}$ is fine scale. Meanwhile, it is also assumed that there is a linearity between $\bar{u}$ and $\hat{u}$. The trial function spaces of each scale are defined as

$$\begin{aligned} \bar{U} &= \{u \mid u \in H^1(\Omega),\, u = g \text{ at } \Gamma_u\} \quad \bar{u} \in \bar{U} \\ \hat{U} &= \{u \mid u \in H^1(\Omega),\, u = 0 \text{ at } \Gamma_u\} \quad \hat{u} \in \hat{U} \\ U &= \bar{U} \oplus \hat{U} \end{aligned}$$

where function g represents displacement boundary condition prescribed on smooth boundary Γ_u and equals to zero. Besides, the test function can also be decomposed into coarse and fine scale components indicated as $\bar{w}$ and $\hat{w}$, given by

$$w = \bar{w} + \hat{w} \qquad (11)$$

The test function spaces of each scale can be shown as follows:

$$\begin{aligned} \bar{V} &= \{w \mid w \in H^1(\Omega),\, w = 0 \text{ at } \Gamma_u\} \quad \bar{w} \in \bar{V} \\ \hat{V} &= \{w \mid w \in H^1(\Omega),\, w = 0 \text{ at } \Gamma_u\} \quad \hat{w} \in \hat{V} \\ W &= \bar{W} \oplus \hat{W} \end{aligned}$$

3.4 The Multiscale Variational Problem

Employing backward Euler as time discretization for equation (8) and linearization, the trial solutions (10) and the weighting functions (11) are substituted into the standard variational form to obtain

$$\begin{aligned} &\left(\bar{w}+\hat{w}+\frac{\bar{u}^{n+1}+\hat{u}^{n+1}}{\Delta t}\right)+\left(\bar{w}+\hat{w},\boldsymbol{a}^n\cdot\nabla(\bar{u}^{n+1}+\hat{u}^{n+1})\right) \\ &+\left(\nabla(\bar{w}+\hat{w}),\frac{1}{Re}\nabla(\bar{u}^{n+1}+\hat{u}^{n+1})\right)+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w}+\hat{w},\bar{u}^{n+1}+\hat{u}^{n+1}\right)ds \\ &=\left(\bar{w}+\hat{w},\frac{u^n}{\Delta t}\right)+\frac{1}{Re}\int_{\Gamma_u}\left(\bar{w}+\hat{w},\nabla u^{n+1}\cdot n\right)ds \\ &+\frac{1}{Re}\int_{\Gamma_t}\left(\bar{w}+\hat{w},\nabla u^{n+1}\cdot n\right)ds+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w}+\hat{w},u_E\right)ds \qquad (12) \end{aligned}$$

where n and n+1 represent two adjacent time points with time-step $\Delta t = t^{n+1} - t^n, \boldsymbol{a}^n = (u^n, v^n), u^n = \bar{u}^n + \hat{u}^n$. Assuming that the coarse scale and fine scale are linearly independent, therefore equation (12) can be split into the coarse and the fine scale parts, the two sub-problems becomes:

coarse scale $\bar{W}$ sub-problem:

$$\begin{aligned} \bar{W}:&\left(\bar{w},\frac{\bar{u}^{n+1}+\hat{u}^{n+1}}{\Delta t}\right)+\left(\bar{w},\boldsymbol{a}^n\cdot\nabla(\bar{u}^{n+1}+\hat{u}^{n+1})\right) \\ &+\left(\nabla\bar{w},\frac{1}{Re}\nabla(\bar{u}^{n+1}+\hat{u}^{n+1})\right)+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w},\bar{u}^{n+1}+\hat{u}^{n+1}\right)ds \\ =&\left(\bar{w},\frac{u^n}{\Delta t}\right)+\frac{1}{Re}\int_{\Gamma_u}\left(\bar{w},\nabla u^{n+1}\cdot n\right)ds+\frac{1}{Re}\int_{\Gamma_t}\left(\bar{w},\nabla u^{n+1}\cdot n\right)ds \\ &+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w},u_E\right)ds \qquad (13) \end{aligned}$$

fine scale $\hat{W}$ sub-problem:

$$\begin{aligned} \hat{W}:&\left(\hat{w},\frac{\bar{u}^{n+1}+\hat{u}^{n+1}}{\Delta t}\right)+\left(\hat{w},\boldsymbol{a}^n\cdot\nabla(\bar{u}^{n+1}+\hat{u}^{n+1})\right) \\ &+\left(\nabla\hat{w},\frac{1}{Re}\nabla(\bar{u}^{n+1}+\hat{u}^{n+1})\right)+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\hat{w},\bar{u}^{n+1}+\hat{u}^{n+1}\right)ds \\ =&\left(\hat{w},\frac{u^n}{\Delta t}\right)+\frac{1}{Re}\int_{\Gamma_u}\left(\hat{w},\nabla u^{n+1}\cdot n\right)ds+\frac{1}{Re}\int_{\Gamma_t}\left(\hat{w},\nabla u^{n+1}\cdot n\right)ds \\ &+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\hat{w},u_E\right)ds \qquad (14) \end{aligned}$$

After we rearranged equation (13) and (14) respectively, we get

$$\begin{aligned} \bar{W}:&\left(\bar{w},\frac{\bar{u}^{n+1}}{\Delta t}\right)+\left(\bar{w},\boldsymbol{a}^n\cdot\nabla\bar{u}^{n+1}\right)+\left(\nabla\bar{w},\frac{1}{Re}\nabla\bar{u}^{n+1}\right)+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w},\bar{u}^{n+1}\right)ds \\ =&\left(\bar{w},\frac{u^n}{\Delta t}\right)+\frac{1}{Re}\int_{\Gamma_u}\left(\bar{w},\nabla u^{n+1}\cdot n\right)ds+\frac{1}{Re}\int_{\Gamma_t}\left(\bar{w},\nabla u^{n+1}\cdot n\right)ds \\ &+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w},u_E\right)ds-\left(\bar{w},\frac{\hat{u}^{n+1}}{\Delta t}\right)-\left(\bar{w},\boldsymbol{a}^n\cdot\nabla\hat{u}^{n+1}\right) \\ &-\left(\nabla\bar{w},\frac{1}{Re}\nabla\hat{u}^{n+1}\right)-\frac{\alpha}{Re}\int_{\Gamma_u}\left(\bar{w},\hat{u}^{n+1}\right)ds \end{aligned}$$

$$\hat{W}:\left(\hat{w},\frac{\hat{u}^{n+1}}{\Delta t}\right)+\left(\hat{w},\boldsymbol{a}^{n}\cdot\nabla\hat{u}^{n+1}\right)+\left(\nabla\hat{w},\frac{1}{Re}\nabla\hat{u}^{n+1}\right)+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\hat{w},\hat{u}^{n+1}\right)ds$$
$$=\left(\hat{w},\frac{u^{n}}{\Delta t}\right)+\frac{1}{Re}\int_{\Gamma_u}\left(\hat{w},\nabla u^{n+1}\cdot n\right)ds+\frac{1}{Re}\int_{\Gamma_t}\left(\hat{w},\nabla u^{n+1}\cdot n\right)ds$$
$$+\frac{\alpha}{Re}\int_{\Gamma_u}\left(\hat{w},u_E\right)ds-\left(\hat{w},\frac{\bar{u}^{n+1}}{\Delta t}\right)-\left(\hat{w},\boldsymbol{a}^{n}\cdot\nabla\bar{u}^{n+1}\right)$$
$$-\left(\nabla\hat{w},\frac{1}{Re}\nabla\bar{u}^{n+1}\right)-\frac{\alpha}{Re}\int_{\Gamma_u}\left(\hat{w},\bar{u}^{n+1}\right)ds \tag{16}$$

3.5 The Solution of the Fine Scale Problem

The theory about partition of unity (PU) is used [27-30] in order to obtain the fine scale approximation. The product of the PU functions and the local approximation functions is the space of functions that is used for the approximation. The approximation of the displacement field at point $\boldsymbol{x}$ is given by

$$u^{h}(\boldsymbol{x})=\sum_{i}\sum_{V_i^j\in v_i}\phi_i V_i^j(\boldsymbol{x})u_{i,j}.$$

Local enrichment basis V_i^j may employ the polynomial basis functions or any other analytical basis functions. Some polynomial basis functions are shown as the following [27]:

First order (p=1): the influence function has only one term

$$\{V_i^j\}=\{V_i^1\}=1 \qquad \text{in } 1D$$
$$\{V_i^j\}=\{V_i^1\}=1 \qquad \text{in } 2D$$

Second order (p=2):

$$\{V_i^j\}=\{V_i^1,V_i^2\}=\{1,(x-x_i)^2\} \qquad \text{in } 1D$$
$$\{V_i^j\}=\{V_i^1,V_i^2,V_i^3\}=\{1,(x-x_i)^2,(y-y_i)^2\} \qquad \text{in } 2D$$

Third order (p=3):

$$\{V_i^j\}=\{V_i^1,V_i^2,V_i^3\}=\{1,(x-x_i)^2,(x-x_i)^3\} \qquad \text{in } 1D.$$

$$\{V_i^j\}=\{V_i^1,V_i^2,V_i^3,V_i^4,V_i^5,V_i^6,V_i^7\}=\{1,(x-x_i)^2,(y-y_i)^2,(x-x_i)^3,(x-x_i)^2(y-y_i),(x-x_i)(y-y_i)^2,(y-y_i)^3\} \qquad \text{in } 2D.$$

Second order polynomial basis functions are used in this work. So, we obtain

$$u^{h}(\boldsymbol{x})=\sum_{i=1}^{N+M}\phi_i u_{i,0}+\sum_{j=1}^{M}\phi_j(x-x_j)^2u_{j,1}+\sum_{j=1}^{M}\phi_j(y-y_j)^2u_{j,2} \tag{17}$$

In the coarse scale region, the approximation using classical EFG method as follows:

$$\bar{u}=\sum_{i=1}^{N+M}\phi_i u_{i,0} \tag{18}$$

While the two added terms are considered as fine scale approximation which is

$$\hat{u}=\sum_{j=1}^{M}\phi_j(x-x_j)^2u_{j,1}+\sum_{j=1}^{M}\phi_j(y-y_j)^2u_{j,2} \tag{19}$$

Thereafter, we substitute equations (18), (19) into equations (15), (16). The matrix form for $\overline{W}$ and $\hat{W}$ are obtained, respectively.

$$\overline{W}:\left[K_1^{n+1}\right]\left\{\bar{u}^{n+1}\right\}=\left\{F_1^{n}\right\}+\left\{F_2(\hat{u}^{n+1})\right\}, \tag{20}$$
$$\hat{W}:\left[K_2^{n+1}\right]\left\{\hat{u}^{n+1}\right\}=\left\{F_3^{n}\right\}+\left\{F_4(\bar{u}^{n+1})\right\}, \tag{21}$$

Let

$$(w,q)^{*}=\left(w,\frac{q}{\Delta t}\right)+\left(w,\boldsymbol{a}^{n}\cdot\nabla q\right)+\left(\nabla w,\frac{1}{Re}\nabla q\right)+\frac{\alpha}{Re}\int_{\Gamma_u}wq\,ds$$
$$(w,q)^{**}=\frac{1}{Re}\int_{\Gamma_u}w\nabla u^{n+1}\cdot n\,ds+\frac{1}{Re}\int_{\Gamma_t}w\nabla u^{n+1}\cdot n\,ds+\frac{\alpha}{Re}\int_{\Gamma_u}wu_E\,ds$$
$$-\left(w,\frac{q}{\Delta t}\right)-\left(w,\boldsymbol{a}^{n}\cdot\nabla q\right)-\left(\nabla w,\frac{1}{Re}\nabla q\right)-\frac{\alpha}{Re}\int_{\Gamma_u}wq\,ds$$

We obtain

$$\left[K_1^{n+1}\right]=\begin{bmatrix}\vdots & \vdots & \vdots\\ \vdots & (\phi_i,\phi_j)^{*} & \vdots\\ \vdots & \vdots & \vdots\end{bmatrix}_{(N+M)\times(N+M)}$$

$$\left[K_2^{n+1}\right]=\begin{bmatrix}\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \left(\phi_i(x-x_i)^2,\phi_j(x-x_j)^2\right)^{*} & \vdots & \vdots & \left(\phi_i(x-x_i)^2,\phi_j(y-y_j)^2\right)^{*} & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \left(\phi_i(y-y_i)^2,\phi_j(x-x_j)^2\right)^{*} & \vdots & \vdots & \left(\phi_i(y-y_i)^2,\phi_j(y-y_j)^2\right)^{*} & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}_{2M\times 2M}$$

$$\{\bar{u}^{n+1}\}=\begin{bmatrix}\vdots\\ u_{j,0}^{n+1}\\ \vdots\end{bmatrix}_{(N+M)\times 1}, \quad \{\hat{u}^{n+1}\}=\begin{bmatrix}\vdots\\ u_{j,1}^{n+1}\\ \vdots\\ \vdots\\ u_{j,2}^{n+1}\\ \vdots\end{bmatrix}_{2M\times 1}$$

$$\{F_1^{n}\}=\begin{bmatrix}\vdots\\ \left(\phi_i,\frac{u^{n}}{\Delta t}\right)\\ \vdots\end{bmatrix}_{(N+M)\times 1}, \quad \{F_2(\hat{u}^{n+1})\}=\begin{bmatrix}\vdots\\ \left(\phi_i,\hat{u}^{n+1}\right)^{**}\\ \vdots\end{bmatrix}_{(N+M)\times 1}$$

$$\{F_3^{n}\}=\begin{bmatrix}\vdots\\ \left(\phi_i(x-x_i)^2,\frac{u^{n}}{\Delta t}\right)\\ \vdots\\ \vdots\\ \left(\phi_i(y-y_i)^2,\frac{u^{n}}{\Delta t}\right)\\ \vdots\end{bmatrix}_{2M\times 1}, \quad \{F_4(\bar{u}^{n+1})\}=\begin{bmatrix}\vdots\\ \left(\phi_i(x-x_i)^2,\bar{u}^{n+1}\right)^{**}\\ \vdots\\ \vdots\\ \left(\phi_i(y-y_i)^2,\bar{u}^{n+1}\right)^{**}\\ \vdots\end{bmatrix}_{2M\times 1}$$

N represents the number of coarse scale nodes,
$i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$

M represents the number of fine scale nodes,
$i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, M$

3.6 Numerical Iteration

The iteration procedures are as following:

(1) Let $\bar{u}^{n+1,0} = u^n$;

(2) Replace the coarse variable $\bar{u}^{n+1}$ in the right-hand side with $\bar{u}^{n+1,i}$ to determine $\hat{u}^{n+1,i+1}$

$$\hat{W} : \left[K_2^{n+1,i+1}\right]\left\{\hat{u}^{n+1,i+1}\right\} = \left\{F_3^n\right\} + \left\{F_4(\bar{u}^{n+1,i})\right\}$$

(3) Use $\hat{u}^{n+1,i+1}$ from (2) to solve the coarse scale problem to determine $\bar{u}^{n+1,i+1}$,

$$\bar{W} : \left[K_1^{n+1,i+1}\right]\left\{\bar{u}^{n+1,i+1}\right\} = \left\{F_1^n\right\} + \left\{F_2(\hat{u}^{n+1,i+1})\right\}$$

(4) Employ $u^h(\mathbf{x}) = \sum_{i=1}^{41} \phi_i u_{i,0}$ for coarse nodes and employ

$$u^h(\mathbf{x}) = \sum_{i=1}^{41} \phi_i u_{i,0} + \sum_{j=42}^{46} \phi_j (x - x_j)^2 u_{j,1} + \sum_{j=47}^{51} \phi_j (y - y_j)^2 u_{j,2}$$

for fine nodes to get $u_{\text{real}}^{n+1,i}$ and $u_{\text{real}}^{n+1,i+1}$,

(5) Next, calculate error = max $| u_{\text{real}}^{n+1,i+1} - u_{\text{real}}^{n+1,i} |$;

(6) If error $<10^{-5}$, then let $u^{n+1,i+1} \to u^n, n+1 \to n$ and go to (1); else go to (2).

Table 1 The representation of global index numbering.

Approximation, $u^h(x)$	Index number, i
$\sum_{i=1}^{41} \phi_i u_{i,0}$	$i = 1 \ldots 36$ Coarse Nodes $i = 37 \ldots 41$ Fine Nodes
$\sum_{j=42}^{46} \phi_j (x - x_j)^2 u_{j,1}$	$j = 42 \ldots 46$ Fine Nodes
$\sum_{j=47}^{51} \phi_j (y - y_j)^2 u_{j,2}$	$j = 47 \ldots 51$ Fine Nodes

Table 1 shows the representation of global index numbering. For the approximation of the classical EFG method, the index number from 1 to 36 represent coarse nodes while index number from 37 to 41 indicate fine nodes. For the approximation of polynomial basis function, the index number which start from 42 to 46 for $(x - x_j)$ and index number from 47 to 51 for $(y - y_j)$ are defined as fine nodes.

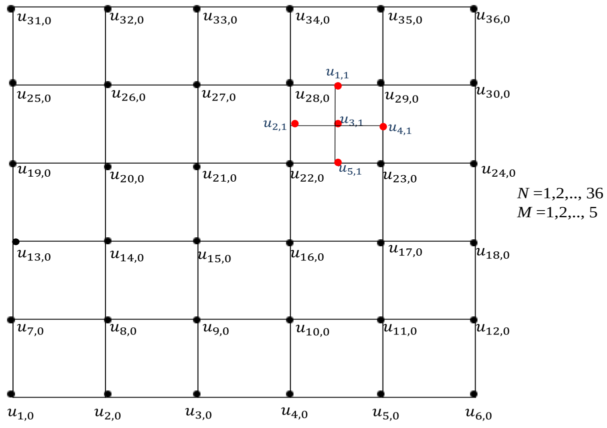


Figure 1 Coarse nodes and fine nodes in 2D Burgers' equation
Figure 1 shows the geometry for coarse nodes and fine nodes in 2D Burgers' problem. N indicates the number of coarse nodes while M represents the number of fine nodes

4.0 RESULTS AND DISCUSSIONS

4.1 The 2D Burgers' Equation

The computing results of multiscale EFGM with penalty for 2D Burgers' equation are presented as below

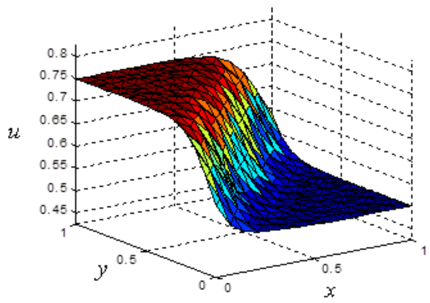


Figure 2 A numerical illustration of approximation solutions u velocity by multiscale method at t=0.2 on 15×15 nodes when Re=100

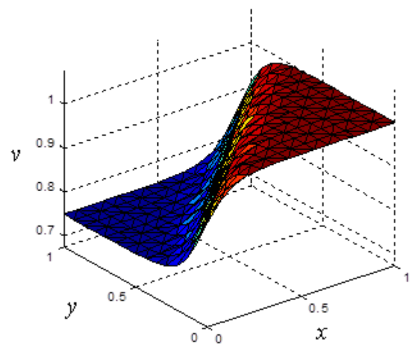


Figure 3 A numerical illustration of approximation solutions v velocity by multiscale method at t=0.2 on 15×15 nodes when Re=100

Table 2 The comparison of maximum error of *u* velocity and *v* velocity by multiscale method for time *t*=0.4 and *t*=0.8 with different node size for *Re*=100

	t=0.4			*t*=0.8		
	Node Size			Node Size		
	10×10	15×15	20×20	10×10	15×15	20×20
u velocity	0.050232	0.006986	0.001204	0.041652	0.003643	0.002136
v velocity	0.022614	0.005418	0.001861	0.017284	0.004472	0.002554

Table 2 shows the comparison of the maximum error of *u* velocity and *v* velocity for 10×10, 15×15 and 20×20 nodes. Two different times, *t*=0.4 and *t*=0.8 have been used in this table. From the table 2, we can observe that the maximum error reduce with the increment of nodes for multiscale methods with penalty for 2D Burgers' equation. This shows that the accuracy increase with the increment of nodes.

Table 3 The comparison errors of *u* velocity for multiscale area nodes by without multiscale method and with multiscale method at *t*=0.4 and *t*=0.8 on 15x15 nodes for *Re*=100

	t=0.4		*t*=0.8	
Critical Nodes	Without multiscale	Multiscale	Without multiscale	Multiscale
Node 49	0.003523	0.002249	0.001801	0.000994
Node 50	0.001739	0.001094	0.002880	0.002525
Node 64	0.001427	0.000923	0.001940	0.001085
Node 65	0.000984	0.000130	0.003329	0.002743

From the table 3 above, node 49, node 50, node 64 and node 65 represent the nodes that are close to the multiscale area. It can be observed that the errors of these *u* velocity nodes by multiscale method are smaller than without multiscale method at time *t*=0.4 and *t*=0.8. The errors in table 3 indicate that the multiscale method produce higher accuracy comparing to without multiscale method.

■5.0 CONCLUSION

In this paper, a new numerical method called multiscale element-free Galerkin method with penalty is developed for 2D Burgers' equation. In this proposed method, the velocity field is decomposed into coarse scales and fine scales, $u = \bar{u} + \hat{u}$. This will enable the fine scale information can be captured. According to the numerical results obtained, the accuracy of multiscale EFGM with penalty is improved with the increment of nodes. Besides, the results of this proposed multiscale method is better than without multiscale method based on the errors of *u* velocity nodes which are close to the multiscale area. Another advantageous of this method is less computational time to be used comparing with the conventional multiscale meshless method. This is because the multiscale method will only be applied in the critical area which requires higher accuracy. This brings convenience for adding or deleting nodes in the desired regions as the mesh generation is not needed in this method. So, this proposed method is suitable to deal with high gradient, nonlinear and large deformation problems.

Acknowledgement.

The financial support from MyBrain15 (MyPhD), Kementerian Pengajian Tinggi Malaysia and Research University Grant (GUP) (Q.J.130000. 2626.08J01) are fully acknowledged.

References

[1] Bateman, H. 1915. Some Recent Researches on the Motion of Fluids. *MonWeather Rev*. 43: 163–70.
[2] Burgers, J. M. 1948. A Mathematical Model Illustrating the Theory of Turbulence. *Advanced in Applied Mechanics*. 1: 171–199.
[3] Bahadir, A. R. 2003. A Fully Implicit Finite-difference Scheme for Two Dimensional Burgers' Equation. *Applied Mathematics and Computation*. 137: 131–137.
[4] Kutluay, S., A. R. Bahadir, A. Ozdes. 1999. Numerical Solution of One Dimensional Burgers Equation: Explicit and Exact-explicit Finite Difference Methods. *Journal of Compuational and Applied Math*. 103: 251–261.
[5] Froncioni, A. M., P. Labbe, A. Garon, and R. Camarero. 1997. Interpolation-free Space-time Remeshing for the Burgers Equation. *Communications in Numerical Methods Engineering*. 13: 875–884.
[6] Dogan, A. 2004. A Galerkin Finite Element Approach to Burgers' Equation. *Apply Math. Comput*. 157: 331–346.
[7] Chino, E., N. Tosaka. 1998. Dual Reciprocity Boundary Element Analysis of Time-independent Burgers' Equation. *Engineering Analysis with Boundary Elements*. 21: 261–270.
[8] Kakuda, K., N. Tosaka. 1990. The Generalized Boundary Element Approach to Burgers' Equation. *Int J Numer Methods Eng*. 29: 245–261.
[9] Belytschko, T., Y. Y. Lu, and L. Gu. 1994. Element-free Galerkin Methods. *Int. J. Numer. Methods Eng*. 37(2): 229–256.
[10] Belytschko, T., Y. Krongauz, D. Organ, and M. Fleming. 1996. Meshless Method: An Over View and Recent Developments. *Comput Methods Appl Mech Eng*. 139: 3–47.
[11] Liu, G. R., Y. T. Gu. 2005. *An Introduction to Meshfree Methods and Their Programming*. Berlin and NewYork: Springer.
[12] Ouyang, J., X. H. Zhang, and L. Zhang. 2009. Element-free Characteristic Galerkin Method for Burgers' Equation. *Engineering Analysis with Boundary Elements*. 33: 356–362.
[13] Young, D. L., C. M. Fan, S. P. Hu, and S. N. Atluri. 2008. The Eulerian–Lagrangian Method of Fundamental Solutions for Two-Dimensional Unsteady Burgers' Equations. *Engineering Analysis with Boundary Elements*. 32: 395–412.
[14] Xie, S. S., S. Heo, S. Kim, G. Woo, and S. Yi. 2008. Numerical Solution of One-dimensional Burgers' Equation Using Reproducing Kernel Function. *Journal of Computational and Applied Mathematics*. 214(2): 417–434.
[15] Zhang, L., J. Ouyang, X. X. Wang, and H. H. Zhang. 2010. Variational Multiscale Element-free Galerkin Method for 2D Burgers' Equation. *Journal of Computational Physics*. 229: 7147–7161.
[16] Hughes, T. J. R. 1995. Multiscale Phenomena: Green's Function, the Dirichlet-to-Neumann Formulation, Subgrid Scale Models, Bubbles and the Origins of Stabilized Methods. *Comput. Methods Appl. Mech. Eng*. 127: 387–401.
[17] Hughes, T. J. R, G. R. Feijoo, M. Luca, and Q. Jean-Baptiste. 1998. The Variational Multiscale Method-A Paradigm for Computational Mechanics. *Comput. Mehtods Appl. Mech. Eng*. 166(1–2): 3–24.
[18] Masud, A. 2002. On a stabilized finite element formulation for incompressible Navier-Stokes equations. *Proceedings of the Fourth US–Japan Conference on Computational Fluid Dynamics*. 28–30.
[19] Masud, A., T. J. R. Hughes. 2002. A stabilized mixed finite element method for Darcy flow. *Comput. Methods Appl. Mech. Eng*. 191(39–40): 4341–4370.
[20] Masud, A., R. A. Khurram. 2004. A Multiscale/Stabilized Finite Element Method for the Advection-diffusion Equation. *Comput. Methods Appl. Mech. Eng*. 193(21–22): 1997–2018.
[21] Zhang, L., J. Ouyang, T. Jiang, and C. Ruan. 2011. Variational Multiscale Element Free Galerkin Method for the Water Wave Problems. *Journal of Computational Physics*. 230(2011): 5045–5060.
[22] Zhang, L., J. Ouyang, X. H. Zhang, and W. B. Zhang. 2008. On a Multi-scale Element-free Galerkin Method for the Stokes Problem. *Applied Mathematics and Computation*. 203: 745–753.
[23] Jeoung, H. Y., K. Y. Sung. 2008. Variational Multiscale Analysis of Elastoplastic Deformation Using Meshfree Approximation. *International Journal of Solids and Structures*. 45: 4709–4724.
[24] Lu, Y. Y., T. Belytschko, and L. Gu. 1994. A New Implementation of the Element-free Galerkin Method. *Comput. Meth. Appl. Mech. Engrg*. 113: 397–414.

[25] Chu, Y. A., B. Moran. 1995. A Computational Model for Nucleation of the Solid-Solid Phase Transformations, Modell. *Simul. Mater. Sci. Engrg*. 3: 455–471.
[26] Mukherjee, Y. X., S. Mukherjee. 1997. On Boundary Conditions in the Element Free Galerkin Method. *Comput. Mech*. 19(4): 264–270.
[27] Fan, S. C., X. Liu, and C. K. Lee. 2004. Enriched Partition of Unity Finite Element Method for Stress Intensity Factors at Crack Tips. *Computers & Structures*. 82(4–5): 445–461.
[28] Hazard, L., P. Bouillard. 2007. Structural Dynamics of Viscoelastic Sandwich Plates by the Partition of Unity Finite Element Method. *Comput. Methods in Appl. Mechanics and Eng*. 196(41–44): 4101–4116.
[29] Li, S. C., S. C. Li, Y. M. Cheng. 2005. Enriched Meshless Manifold Method for Two-Dimensional Crack Modelling. *Theor. Appl. Fract. Mec*. 44(3): 234–248.
[30] Melenk, J. M., I. Babuška. 1996. The Partition of Unity Finite Element Method: Basic Theory And Applications. *Comput. Methods Appl. Eng.* 139: 289–314.