Jurnal
Teknologi

Full Paper

RESOLVING DELAY DIFFERENTIAL EQUATIONS WITH HOMOTOPY PERTURBATION AND SUMUDU TRANSFORM

Subashini Vilu*, Rokiah Rozita Ahmad, Ummul Khair Salma Din, Mohd Almie Alias

School of Mathematical Sciences, Faculty of Science & Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia

Article history
Received
31 July 2022
Received in revised form
21 December 2022
Accepted
27 February 2023
Published Online
19 April 2023

*Corresponding author
subashini12@yahoo.com

Graphical abstract

$$\mathbb{S}[D[W(\tau)]]+\mathbb{S}[L[W(\tau)]]+\mathbb{S}[N[W(\tau)]]=\mathbb{S}[g(\tau)]$$

⬇

$$\frac{1}{u^m}\mathbb{S}(W)+\frac{1}{u^m}W(0)-\frac{1}{u^{m-1}}W'(0)-\ldots-\frac{1}{u}W^{(m-1)}(0)=-\mathbb{S}\{L[W(\tau)]+N[W(\tau)]-g(\tau)\}$$

⬇

$$W(\tau)=\mathbb{S}^{-1}\left\{\left\{(u^m)\left\{\frac{1}{u^m}W(0)+\frac{1}{u^{m-1}}W'(0)+\ldots+\frac{1}{u}W^{(m-1)}(0)+\mathbb{S}[g(\tau)]\right\}\right\}-(u^m)\mathbb{S}\{L[W(\tau)]+N[W(\tau)]\}\right\},$$

⬇

$$\sum_{n=0}^{\infty}p^nW_n(\tau)=G(\tau)-p\left(\mathbb{S}^{-1}\left[(u^m)\mathbb{S}\left[L\sum_{n=0}^{\infty}p^nW_n(\tau)+N\sum_{n=0}^{\infty}p^nH_n(W)\right]\right]\right),$$

⬇

$$w=w_0+pw_1+p^2w_2+\ldots$$

⬇

$$W(\tau)=\lim_{p\to 1}W=w_0+w_1+w_2+\ldots$$

Abstract

A novel proposition has been introduced in this study for resolving delay differential equations (DDEs) of nature that is a composite in reference to Homotopy perturbation method (HPM) along with Sumudu transform. A rare transform called the Sumudu transform is used alongside the perturbation theory. Demonstration of this new methodology is shown by solving a few numerical cases. Reducing the complication of computational tasks associated to the conservative means is the objective of this research. Results display the amount of valuation being reduced and is as good as in the previous studies as well in comparison.

Keywords: Homotopy Perturbation Method, Delay Differential Equations, Sumudu Transform, Homotopy Perturbation Sumudu Transform Method

Abstrak

Proposisi baru telah diperkenalkan dalam kajian ini untuk menyelesaikan persamaan pembezaan tunda (PPT) alam semula jadi yang merupakan komposit merujuk kepada kaedah usikan Homotopy (HPM) bersama dengan transformasi Sumudu. Transformasi yang jarang dipanggil transformasi Sumudu digunakan bersama-sama teori gangguan. Demonstrasi metodologi baharu ini ditunjukkan dengan menyelesaikan beberapa kes berangka. Mengurangkan kerumitan tugas pengiraan yang dikaitkan dengan cara konservatif adalah objektif penyelidikan ini. Keputusan memaparkan jumlah penilaian yang dikurangkan dan sama baiknya dengan kajian terdahulu dan juga perbandingan.

Kata kunci: Kaedah Usikan Homotopy, Persamaan Pembezaan Tunda, Transformasi Sumudu, Kaedah Usikan Homotopy Transformasi Sumudu

1.0 INTRODUCTION

Delay differential equations (DDEs) has always been playing important roles in the history of many real-life occurrences in physics, engineering, biology, economics, medicine and many more. Recently, they are being used in many biological and population dynamics modelling extensively [9, 10, 12, 22, 23, 24, 25, 26]. DDEs are an exceptional class of functional equations where a particular finite interval of the immediate past is included in the discovery of the present. Introduction of delay into the ordinary differential equations enhances its vitality and gives a rather accurate depiction of the real events. Unlike ODEs, DDEs need a historical value or interval which is then used as an initial value or interval in its model. As such, the delay equations emerge as one of the multidimensional equations naturally that makes the analytical analysis of DDEs complex and as a result necessitates a numerical approach.

DDE's plain expression is,

$$(w')_i(\tau) = f(\tau, w_i(\tau), w_i(\tau - c)), \quad (1)$$

such that i = 1, 2, 3, …, *n* and τ a constant delay.

A small number of numerical approaches has been applied as a solution for DDEs in the recent times such as Anakira *et al*. engaged Optimal Homotopy Asymptotic Method (OHAM) for unravelling linear as well as non-linear DDE [11] whereas Alomari *et al.,* 2009 [1] employed Homotopy Analysis Method (HAM) in discovering formula in the interest of DDE. Across every applied method, one of it is Homotopy Perturbation Method (HPM). HPM came to be primarily introduced by Ji Huan He, a Chinese mathematician [6, 7, 8]. Implementation of HPM to differential equations more often than usual deals with introducing a homotopy parameter which simplifies the system of equations which generally gives a simple solution. HPM is efficient in decreasing the amount of calculations greatly in any case yet retaining sharp precision of the numerical solution. This shows in the various application of the HPM in the field of engineering and physics [2, 3, 4, 13].

The disadvantage of the HPM is determining the small parameter which seems a little complicated. Furthermore, an inapt choice of such parameter leads to an erroneous result. This shortcoming could be solved by coupling the HPM with Sumudu transform [5, 14]. The Sumudu transform, having unit preserving properties, might be employed to decipher issues devoid of falling back to the frequency domain. The indicated property enables the operation of the clarification to be effortless and develops the accuracy as well.

Fundamental stimulation in this study is lengthening the fresh well-founded stratagem in the direction of the unraveling of linear as well as nonlinear DDEs that occur to be in general perplexing to investigate on resulting from its multidimensional character and vast amplitude.

2.0 METHODOLOGY

2.1 The transform of Sumudu

In early 90's, Watugala [14] pioneered one fresh integral transform called Sumudu transform in addition to implementing the transform to resolving ODE in control engineering issues. Sumudu transform is described onto following set of functions

$$A = f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{(|t|/\tau_j)}, \text{ " if " } t \in (-1)^j \times [0, \infty) \quad (2)$$

by succeeding formulation

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut)\, e^{-t} dt, \quad u \in (-\tau_1, \tau_2) \quad (3)$$

or

$$\bar{f}(u) = \mathbb{S}[f(t)] = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)\, dt. \quad (4)$$

2.2 Combination of HPM and Sumudu transform (HPSTM)

To elucidate elementary notion of this mechanism, a common nonlinear DE is assessed with its primary condition of the model:

$$D[W(\tau)] + L[W(\tau)] + N[W(\tau)] = g(\tau), \quad (5)$$

$$W(0) = f(x), \quad (6)$$

whereupon D is the highest order linear differential operator (*DO*), $D = \frac{\partial^m}{\partial t^m}$, L is the linear *DO* of reduced order than D, N denotes general nonlinear *DO* whereas $g(\tau)$ is the historical term.

Implementing Sumudu transform on the two sides of Eq. (5), the following is obtained

$$\mathbb{S}[D[W(\tau)]] + \mathbb{S}[L[W(\tau)]] + \mathbb{S}[N[W(\tau)]] = \mathbb{S}[g(\tau)]. \quad (7)$$

Employing differential feature of Sumudu transform, we obtain

$$\frac{1}{u^m}\mathbb{S}(W) + \frac{1}{u^m}W(0) - \frac{1}{u^{m-1}}W'(0) - \ldots - \frac{1}{u}W^{(m-1)}(0) = -\mathbb{S}\{L[W(\tau)] + N[W(\tau)] - g(\tau)\}, \quad (8)$$

or

$$\mathbb{S}\{W\} = (u^m)\left\{\frac{1}{u^m}W(0) + \frac{1}{u^{m-1}}W'(0) + \ldots + \frac{1}{u}W^{(m-1)}(0)\right\} + (u^m)\,\mathbb{S}[g(\tau)] - (u^m)\,\mathbb{S}\{L[W(\tau)] + N[W(\tau)]\}. \quad (9)$$

Promptly employing inverse Sumudu transform on two sides of Eq. (9), the subsequent is obtained

$$W(\tau) = \mathbb{S}^{-1}\left\{\left\{(u^m)\left\{\frac{1}{u^m}W(0) + \frac{1}{u^{m-1}}W'(0) + \ldots + \frac{1}{u}W^{(m-1)}(0) + \mathbb{S}[g(\tau)]\right\}\right\} - (u^m)\,\mathbb{S}\{L[W(\tau)] + N[W(\tau)]\}\right\}. \quad (10)$$

or

$$W(\tau) = G(\tau) - \mathbb{S}^{-1}\{(u^m)\,\mathbb{S}\{L[W(\tau)] + N[W(\tau)]\}\} \quad (11)$$

where $G(\tau)$ characterizes the expression surfacing from the historical term and the formulated preliminary settings. Subsequently, HPM is implemented,

$$L[W(\tau)] = \sum_{n=0}^{\infty} p^n W_n(\tau) \quad (12)$$

and the nonlinear expression might as well be disintegrated as

$$N[W(\tau)] = \sum_{n=0}^{\infty} p^n H_n(W) \tag{13}$$

considering several He's polynomial $H_n(W)$ that are specified as

$$H_n(W_0, W_1, \ldots, W_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i W_i\right)\right]_{p=0}, \qquad n = 0,1,2,3,\ldots \tag{14}$$

Replacing Eq. (12), (13) into Eq. (11), the following is obtained

$$\sum_{n=0}^{\infty} p^n W_n(\tau) = G(\tau) - p\left(\mathbb{S}^{-1}\left[(u^m)\,\mathbb{S}\left[L\sum_{n=0}^{\infty} p^n W_n(\tau) + N\sum_{n=0}^{\infty} p^n H_n(W)\right]\right]\right), \tag{15}$$

which is the pairing of HPM and Sumudu by means of He's polynomials in which $p \in [0,1]$ is an embedding parameter.

Equating the coefficient of like powers of p, the subsequent estimates are obtained.

$$p^0: W_0(\tau) = G(\tau) \tag{16}$$

$$p^1: W_1(\tau) = -\mathbb{S}^{-1}\left[(u^m)\,\mathbb{S}\left[L\left(W_0(\tau)\right) + H_0(W)\right]\right] \tag{17}$$

$$p^2: W_2(t) = -\mathbb{S}^{-1}\left[(u^m)\,\mathbb{S}\left[L\left(W_1(\tau)\right) + H_1(W)\right]\right] \tag{18}$$

$$p^3: W_3(\tau) = -\mathbb{S}^{-1}\left[(u^m)\,\mathbb{S}\left[L\left(W_2(\tau)\right) + H_2(W)\right]\right] \tag{19}$$

$$\vdots$$

$$p^n: W_n(\tau) = -\mathbb{S}^{-1}\left[(u^m)\,\mathbb{S}\left[L\left(W_{n-1}(\tau)\right) + H_{n-1}(W)\right]\right] \tag{20}$$

Continuing in this alike routine, the remaining components, $W_n(\tau)$ can be acquired entirely and the series resolution is thus determined completely.

As a final point, we estimate the analytical solution

$$w = w_0 + pw_1 + p^2 w_2 + \ldots \tag{21}$$

Thus, an approximate solution is

$$W(\tau) = \lim_{p \to 1} W = w_0 + w_1 + w_2 + \ldots \tag{22}$$

Solution of these series mostly converge rapidly.

3.0 RESULTS & DISCUSSION

Over this fragment, implementation of the methodology aforementioned is employed to several nonlinear DDEs.

Example 3.1: Let us evaluate a nonlinear DDE of a first order,

$$w'(\tau) = 1 - 2w^2\left(\frac{\tau}{2}\right), \qquad w(0) = 0, \tag{23}$$

Precise resolution is noted to be $w(\tau) = \sin(\tau)$.

Implementing Sumudu transform on the two sides of Eq. (23), the following is obtained

$$\mathbb{S}[w'(\tau)] = \mathbb{S}\left[1 - 2w^2\left(\frac{\tau}{2}\right)\right], \tag{24}$$

Employing differential feature of Sumudu transform, we obtain

$$\frac{1}{u}\mathbb{S}(W) - \frac{1}{u}W(0) = \mathbb{S}\left[1 - 2w^2\left(\frac{\tau}{2}\right)\right], \tag{25}$$

$$\mathbb{S}\{W\} = (u)\left\{\frac{1}{u}W(0)\right\} + (u)\,\mathbb{S}\left\{1 - 2w^2\left(\frac{\tau}{2}\right)\right\}. \tag{26}$$

Promptly employing inverse Sumudu transform on two sides of Eq. (26), the subsequent is obtained

$$W(\tau) = \mathbb{S}^{-1}\left\{(u)\left[\left\{\frac{1}{u}W(0)\right\} + \mathbb{S}\left\{1 - 2w^2\left(\frac{\tau}{2}\right)\right\}\right]\right\}. \tag{27}$$

Subsequently, we implement the HPM,

$$\sum_{n=0}^{\infty} p^n W_n(\tau) = \mathbb{S}^{-1}\left\{(u)\left\{\frac{W(0)}{u}\right\}\right\} + p\left\{\mathbb{S}^{-1}\left\{(u)\left[\mathbb{S}\left\{1 - 2\sum_{n=0}^{\infty} p^n H_n(W)\right\}\right]\right\}\right\}, \tag{28}$$

which is the pairing of HPM and Sumudu by means of He's polynomials.

Equating coefficients of equivalent powers of p, the subsequent estimates are procured.

$$\begin{aligned}
p^0 &: w_0(\tau) = \mathbb{S}^{-1}\left\{(u)\left\{\frac{W(0)}{u}\right\}\right\}, \\
p^1 &: w_1(\tau) = \mathbb{S}^{-1}\left[(u)\,\mathbb{S}\left\{1 - 2H_0(w)\right\}\right], \\
p^2 &: w_2(\tau) = \mathbb{S}^{-1}\left[(u)\,\mathbb{S}\left\{-2H_1(w)\right\}\right], \\
p^3 &: w_3(\tau) = \mathbb{S}^{-1}\left[(u)\,\mathbb{S}\left\{-2H_2(w)\right\}\right], \\
&\vdots
\end{aligned} \tag{29}$$

Hence, comparing the coefficient of like powers of p as in Eq. (29), we actually get

$$\begin{aligned}
p^0 &: w_0(\tau) = 0, \\
p^1 &: w_1(\tau) = \tau, \\
p^2 &: w_2(\tau) = 0, \\
p^3 &: w_3(\tau) = -\frac{\tau^3}{6},
\end{aligned} \tag{30}$$

$p^4: w_4(\tau) = 0,$
$p^5: w_5(\tau) = \frac{\tau^5}{120},$
$\vdots$

Thus, an approximate solution will be

$$W(\tau) = \lim_{p\to 1} W = w_0 + w_1 + w_2 + \dots$$
$$= \tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} - \frac{\tau^7}{5040} + \dots$$
$$= \sin(\tau)$$

Results obtained inclines towards forming the Taylor series expansion of $\sin(\tau)$ which is the exact solution. In order to verify numerically whether the proposed methodology maintains the accuracy, numerical solutions of the approximate solution was evaluated and compared with one of the previous studies using the Laplace Variational Iteration Method (LVIM). The absolute value of LVIM and HPSTM are compared in Table 1 while Figure 1 shows the behavior of the error between these two methods in Example 3.1. Results obtained are found to be in good agreement with each other. Calculation of terms of the sequence obtained from HPSTM can be done using mathematical packages such as Maple, Mathematica and Matlab.

Table 1 Comparison of the absolute errors for Example 3.1

t	Absolute Error	
	Laplace VIM [21]	HPSTM
0.0	0	0
0.1	0.0980877	0.0980877
0.2	0.1951794	0.1951794
0.3	0.290284	0.290284
0.4	0.3824387	0.3824387
0.5	0.4707035	0.4707035
0.6	0.554168	0.554168
0.7	0.632003	0.632003
0.8	0.703398	0.703398
0.9	0.767623	0.767623
1.0	0.824018	0.824018

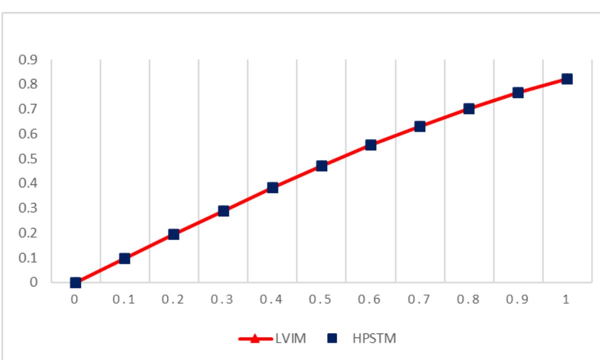

Figure 1 Graphs of the absolute errors for some values of t with LVIM and HPSTM for Example 3.1

Example 3.2: Let's consider the second order linear DDE.

$$w''(t) = \frac{3}{4}w(\tau) + w\left(\frac{\tau}{2}\right) - \tau^2 + 2, \qquad w(0) = 0, \qquad w'(0) = 0, \tag{31}$$

The exact resolution is noted as $w(\tau) = \tau^2$.

Implementing Sumudu transform on the two sides of Eq. (31), the following is obtained

$$\mathbb{S}[w''(\tau)] = \mathbb{S}\left[\frac{3}{4}w(\tau) + w\left(\frac{\tau}{2}\right) - \tau^2 + 2\right], \tag{32}$$

Employing differential feature of Sumudu transform, we obtain

$$\frac{1}{u^2}\mathbb{S}(W) - \frac{W(0)}{u^2} - \frac{W'(0)}{u} = \mathbb{S}\left[\frac{3}{4}w(\tau) + w\left(\frac{\tau}{2}\right) - \tau^2 + 2\right], \tag{33}$$

$$\mathbb{S}\{W\} = (u^2)\left\{\frac{W(0)}{u^2} + \frac{W'(0)}{u}\right\} + (u^2)\,\mathbb{S}\left[\frac{3}{4}w(\tau) + w\left(\frac{\tau}{2}\right) - \tau^2 + 2\right]. \tag{34}$$

Promptly employing inverse Sumudu transform on two sides of Eq. (34), the subsequent is obtained

$$W(\tau) = \mathbb{S}^{-1}\left\{(u^2)\left\{\frac{W(0)}{u^2} + \frac{W'(0)}{u}\right\} + (u^2)\,\mathbb{S}\left[\frac{3}{4}w(\tau) + w\left(\frac{\tau}{2}\right) - \tau^2 + 2\right]\right\}. \tag{35}$$

Subsequently, we implement the HPM,

$$\sum_{n=0}^{\infty} p^n W_n(\tau) = \mathbb{S}^{-1}\left\{(u^2)\left\{\frac{W(0)}{u^2} + \frac{W'(0)}{u}\right\}\right\} + p\left\{\mathbb{S}^{-1}\left\{(u^2)\,\mathbb{S}\left\{\frac{3}{4}\sum_{n=0}^{\infty} p^n W_n(\tau) + \sum_{n=0}^{\infty} p^n W_n\left(\frac{\tau}{2}\right) - \tau^2 + 2\right\}\right\}\right\}. \tag{36}$$

which is the pairing of HPM and Sumudu by means of He's polynomials. Equating coefficients of equivalent powers of p, the subsequent estimates are procured.

$$\begin{aligned} p^0&: w_0(\tau) = 0,\\ p^1&: w_1(\tau) = \tau^2 - \frac{\tau^4}{12},\\ p^2&: w_2(\tau) = \frac{\tau^4}{12} - \frac{13\tau^6}{5760},\\ p^3&: w_3(\tau) = \frac{13\tau^6}{5760} - \frac{91\tau^8}{2949120},\\ p^4&: w_4(\tau) = \frac{91\tau^8}{2949120} - \frac{17381\tau^{10}}{67947724800},\\ &\vdots \end{aligned} \tag{37}$$

Consequently, an approximate solution will be

$$W(\tau) = \lim_{p\to 1} W = w_0 + w_1 + w_2 + \dots$$
$$= \tau^2$$

Convergence of estimates, to the precise solution, τ^2 mayhap noted from the result. The absolute value of LVIM and HPSTM are compared in Table 2 while Figure 2 shows the behavior of the error between these two methods in Example 3.2. The values show that the results are in excellent agreement with those of the other methods.

Table 2 Comparison of the absolute errors for Example 3.2

t	Absolute Error	
	Laplace VIM [21]	**HPSTM**
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	3E-10	3E-10
0.6	1.6E-09	2.1E-09
0.7	7.3E-09	9.7E-09
0.8	2.78E-08	3.68E-08
0.9	9.01E-08	1.196E-07
1.0	2.585E-07	3.429E-07

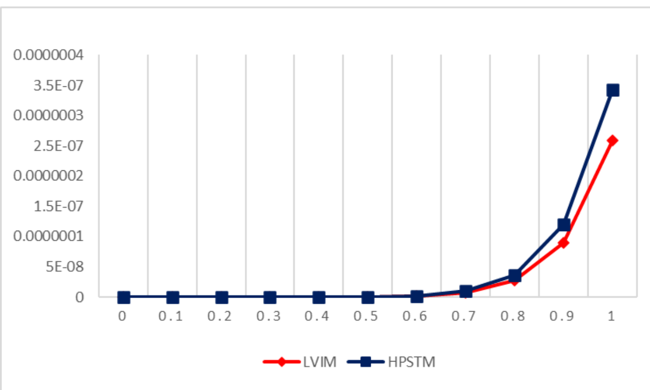

Figure 2 Graphs of the absolute errors for some values of t with LVIM and SVIM for Example 3.2

Example 3.3: Last but not least, let's evaluate the third order nonlinear DDE.

$$w'''(\tau) = -1 + 2w^2\left(\frac{\tau}{2}\right), \quad w(0) = 0, \quad w'(0) = 1, \quad w''(0) = 0, \tag{38}$$

The exact resolution is noted as $w(\tau) = \sin(\tau)$.

Implementing Sumudu transform on the two sides of Eq. (38), the following is obtained

$$\mathbb{S}[w'''(\tau)] = \mathbb{S}\left[-1 + 2w^2\left(\frac{\tau}{2}\right)\right], \tag{39}$$

Employing differential feature of Sumudu transform, we obtain

$$\frac{1}{u^3}\mathbb{S}(W) - \frac{W(0)}{u^3} - \frac{W'(0)}{u^2} - \frac{W''(0)}{u} = \mathbb{S}\left[-1 + 2w^2\left(\frac{\tau}{2}\right)\right], \tag{40}$$

$$\mathbb{S}\{W\} = (u^3)\left\{\frac{W(0)}{u^3} + \frac{W'(0)}{u^2} + \frac{W''(0)}{u}\right\} + (u^3)\,\mathbb{S}\left\{-1 + 2w^2\left(\frac{\tau}{2}\right)\right\}. \tag{41}$$

Promptly employing inverse Sumudu transform on two sides of Eq. (41), the subsequent is obtained

$$W(\tau) = \mathbb{S}^{-1}\left\{(u^3)\left\{\frac{W(0)}{u^3} + \frac{W'(0)}{u^2} + \frac{W''(0)}{u}\right\} + (u^3)\,\mathbb{S}\left[-1 + 2w^2\left(\frac{\tau}{2}\right)\right]\right\}. \tag{42}$$

Subsequently, we implement the HPM,

$$\sum_{n=0}^{\infty} p^n W_n(\tau) = \mathbb{S}^{-1}\left\{(u^3)\left\{\frac{W(0)}{u^3} + \frac{W'(0)}{u^2} + \frac{W''(0)}{u}\right\}\right\} + p\left(\mathbb{S}^{-1}\left\{(u^3)\,\mathbb{S}\left[-1 + 2\sum_{n=0}^{\infty} p^n H_n(W)\right]\right\}\right). \tag{43}$$

Following the progressions of the initial examples, we come by the subsequent consecutive approximations,

$$\begin{aligned} p^0 &: w_0(\tau) = \tau, \\ p^1 &: w_1(\tau) = -\frac{\tau^3}{6} + \frac{\tau^5}{120}, \\ p^2 &: w_2(\tau) = -\frac{\tau^7}{5040} + \frac{\tau^9}{967680}, \\ p^3 &: w_3(\tau) = \frac{\tau^9}{580608} - \frac{\tau^{11}}{39916800} + \frac{173\tau^{13}}{2125489766400}, \\ &\vdots \end{aligned} \tag{44}$$

Consequently, an approximate solution will be

$$W(\tau) = \lim_{p \to 1} W = w_0 + w_1 + w_2 + \dots$$

$$= \tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} - \frac{\tau^7}{5040} + \frac{\tau^9}{362880} - \frac{\tau^{11}}{39916800} + \dots = \sin(\tau)$$

which converges to the exact solution, $w(\tau) = \sin(\tau)$ as well. It can be seen that the approximate solution obtained with HPSTM have good agreement with the other method. The absolute value of LVIM and HPSTM are compared in Table 3 while Figure 3 shows the behavior of the error between these two methods in Example 3.3.

Table 3 Comparison of the absolute errors for Example 3.3

t	Absolute Error	
	Laplace VIM [21]	**HPSTM**
0.0	0	0
0.1	0.098088117	0.098088117
0.2	0.195178731	0.195178731
0.3	0.290284207	0.290284207
0.4	0.382437042	0.382437042
0.5	0.470699039	0.470699039
0.6	0.554170473	0.554170473
0.7	0.632000687	0.632000687

	Absolute Error	
t	Laplace VIM [21]	HPSTM
0.8	0.703394091	0.703394091
0.9	0.76761991	0.76761991
1.0	0.824018986	0.824018985

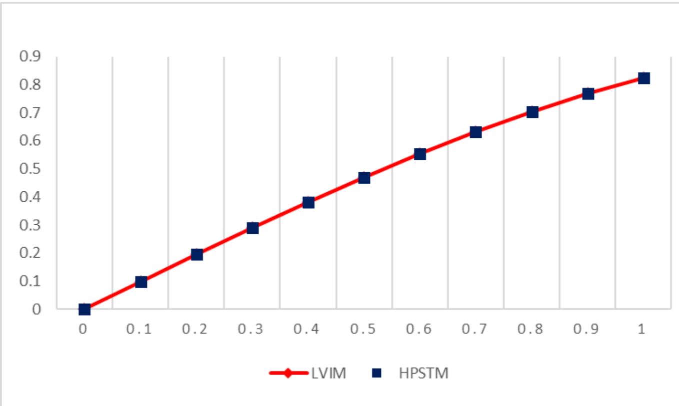

Figure 3 Graphs of the absolute errors for some values of t with LVIM and SVIM for Example 3.3.

4.0 CONCLUSION

The HPSTM is a stem of Sumudu transform in addition to incorporating techniques of HPM to acquire approximate resolutions of DDEs which turned out to be the exact solutions. The disadvantage for HPM which is determining the small parameter could be solved easily with this altered methodology. A fresh variation of HPM was obtained. This commended formula could be employed to decipher issues devoid of falling back to the frequency domain. The indicated property enables the operation of the clarification to be effortless and develops the accuracy as well resulting in quick convergence with physical problems. In this study, HPSTM was effectively employed in deciphering linear and nonlinear DDEs. It's estimable expressing suchlike methodology is successful in minimalizing quantity of calculations opposing to customary techniques in the face of preserving substantial precision of the numerical end product.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

Acknowledgement

This research is fully supported by research grants FRGS/2/2013/SG04/UKM/02/3 and PTS-2013-160. The authors fully acknowledged Ministry of Higher Education (MOHE) and Universiti Kebangsaan Malaysia for the approved fund which makes this important research viable and effective.

References

[1] Alomari, A. K., Noorani, M. S. M., & Nazar, R. 2009. Solution of Delay Differential Equation by Means of Homotopy Analysis Method. *Acta Applicandae Mathematicae*. 108(2): 395-412. https://doi.org/10.1007/s10440-008-9318-z.

[2] Dehghan, M., & Shakeri, F. 2007. Solution of a Partial Differential Equation Subject to Temperature Overspecification by He's Homotopy Perturbation Method. *Physica Scripta*. 75(6): 778. https://doi.org/10.1088/0031-8949/75/6/007.

[3] Dehghan, M., & Shakeri, F. 2008. Use of He's Homotopy Perturbation Method for Solving a Partial Differential Equation Arising in Modeling of Flow in Porous Media. *Journal of Porous Media*. 11(8). https://doi.org/10.1615/JPorMedia.v11.i8.50.

[4] Dehghan, M., & Shakeri, F. 2008. Solution of an Integro-differential Equation Arising in Oscillating Magnetic Fields using He's homotopy Perturbation Method. *Progress in Electromagnetics Research*. 78: 361-376. https://doi.org/10.2528/PIER07090403.

[5] Ghorbani, A., & Saberi-Nadjafi, J. 2007. He's Homotopy Perturbation Method for Calculating Adomian Polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation*. 8(2): 229-232. https://doi.org/10.1515/IJNSNS.2007.8.2.229.

[6] He, J. H. 1999. Homotopy Perturbation Technique. *Computer Methods in Applied Mechanics and Engineering*. 178(3-4): 257-262. https://doi.org/10.1016/S0045-7825(99)00018-3.

[7] He, J. H. 2003. Homotopy Perturbation Method: A New Nonlinear Analytical Technique. *Applied Mathematics and Computation*. 135(1): 73-79. https://doi.org/10.1016/S0096-3003(01)00312-5.

[8] He, J. H. 2006. Homotopy Perturbation Method for Solving Boundary Value Problems. *Physics letters A*. 350(1-2): 87-88. https://doi.org/10.1016/j.physleta.2005.10.005.

[9] Lakshmanan, S., Rihan, F. A., Rakkiyappan, R., & Park, J. H. 2014. Stability Analysis of the Differential Genetic Regulatory Networks Model with Time-varying Delays and Markovian Jumping Parameters. *Nonlinear analysis: Hybrid systems*. 14: 1-15. https://doi.org/10.1016/j.nahs.2014.04.003.

[10] Rakkiyappan, R., Velmurugan, G., Rihan, F. A., & Lakshmanan, S. 2016. Stability Analysis of Memristor-based Complex-valued Recurrent Neural Networks with Time Delays. *Complexity*. 21(4): 14-39. https://doi.org/10.1002/cplx.21618.

[11] Ratib Anakira, N., Alomari, A. K., & Hashim, I. 2013. Optimal Homotopy Asymptotic Method for Solving Delay Differential Equations. *Mathematical Problems in Engineering*. 2013. https://doi.org/10.1063/1.4858786.

[12] Rihan, F. A., Abdelrahman, D. H., Al-Maskari, F., Ibrahim, F., & Abdeen, M. A. 2014. Delay Differential Model for Tumour-immune Response with Chemoimmunotherapy and Optimal Control. *Computational and Mathematical Methods in Medicine*. 2014. https://doi.org/10.1155/2014/982978.

[13] Shakeri, F., & Dehghan, M. 2007. Inverse Problem of Diffusion Equation by He's Homotopy Perturbation Method. *Physica Scripta*. 75(4): 551. https://doi.org/10.1088/0031-8949/75/4/031.

[14] Watugala, G. 1993. Sumudu Transform: A New Integral Transform to Solve Differential Equations and Control Engineering Problems. *Integrated Education*. 24(1): 35-43. https://doi.org/10.1080/0020739930240105.

[15] Ganji, D. D., & Sadighi, A. 2006. Application of He's Homotopy-perturbation Method to Nonlinear Coupled Systems of Reaction-diffusion Equations. *International*

Journal of Nonlinear Sciences and Numerical Simulation. 7(4): 411-418. https://doi.org/10.1515/IJNSNS.2006.7.4.411.

[16] Singh, J., Gupta, P. K., & Rai, K. N. 2011. Homotopy Perturbation Method to Space-time Fractional Solidification in a Finite Slab. *Applied Mathematical Modelling*. 35(4): 1937-1945. https://doi.org/10.1016/j.apm.2010.11.005.

[17] Liu, Y., Li, Z., & Zhang, Y. 2011. Homotopy Perturbation Method to Fractional Biological Population Equation. *Fractional Differential Calculus*. 1(1): 117-124. https://doi.org/10.7153/fdc-01-07.

[18] Rekha, S., Balaganesan, P., & Renuka, J. 2021. Homotopy Perturbation Method for Mathematical Modelling of Dengue Fever. *Journal of Physics: Conference Series*. 1724(1): 012056. IOP Publishing. https://doi.org/10.1088/1742-6596/1724/1/012056.

[19] Domairry, G., & Aziz, A. 2009. Approximate Analysis of MHD Squeeze Flow between Two Parallel Disks with Suction or Injection by Homotopy Perturbation Method. *Mathematical Problems in Engineering*. 2009. https://doi.org/10.1155/2009/603916.

[20] T. A. Biala, O. O. Asim, and Y. O. Afolabi. 2014. A Combination of the Laplace Transform and the Variational Iteration Method for the Analytical Treatment of Delay Differential Equations. *International Journal of Differential Equations and Applications.* 13(3).

[21] Senu, N., Lee, K. C., Ahmadian, A., & Ibrahim, S. N. I. 2022. Numerical Solution of Delay Differential Equation using Two-Derivative Runge-Kutta Type Method with Newton Interpolation. *Alexandria Engineering Journal*. 61(8): 5819-5835. https://doi.org/10.1016/j.aej.2021.11.009.

[22] Anakira, N., Jameel, A., Hijazi, M., Alomari, A. K., & Man, N. 2022. A New Approach for Solving Multi-pantograph Type Delay Differential Equations. *International Journal of Electrical & Computer Engineering*. 12(2): 2088-8708. https://doi.org/10.11591/ijece.v12i2.pp1859-1868.

[23] Jameel, A. F., Anakira, N. R., Alomari, A. K., Al-Mahameed, M., & Saaban, A. 2019. A New Approximate Solution of the Fuzzy Delay Differential Equations. *International Journal of Mathematical Modelling and Numerical Optimisation*. 9(3): 221-240. https://doi.org/10.1504/IJMMNO.2019.100476.

[24] Izadi, M., & Srivastava, H. M. 2021. An Efficient Approximation Technique Applied to a Non-linear Lane-emden Pantograph Delay Differential Model. *Applied Mathematics and Computation*. 401: 126123. https://doi.org/10.1016/j.amc.2021.126123.

[25] Anakira, N. R., Abdelkarim, H., & Abu-Dawas, M. 2020. Homotopy Sumudu Transformation Method for Solving Fractional Delay Differential Equations. *Gen. Lett. Math.* 9(1): 33-41. https://doi.org/10.31559/GLM2020.9.1.4.