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Parameter Estimation of Lotka-Volterra Model : A Two-Step Model

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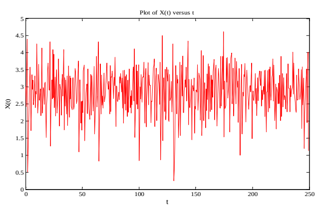
Graphical abstract



Abstract

The deterministic power law logistic model is used to describe density-dependent population growth for cases when ordinary logistic model is found to be insufficient. This paper estimates the parameters of stochastic power law logistic model specifically the Lotka-Volterra model by employing the two-step approach. The Bayesian approach is implemented in the first step of estimating the regression spline parameters. Combining the existing and proposed nonparametric criterion, the structural parameters of SDE are estimated in the second step. Results indicate high percentage of accuracy of the estimated diffusion parameter of Lotka-Volterra model supporting the adequacy of the proposed criterion as an alternative to the classical methods.

Keywords: Stochastic differential equation; regression spline; Bayesian approach; truncated power series basis; Lotka-Volterra model

Abstrak

Model hukum kuasa berketentuan logistik digunakan bagi menerangkan pertumbuhan penduduk bersandar-kepadatan untuk kes-kes apabila model hukum kuasa logistik biasa didapati tidak mencukupi. Kertas ini menganggarkan parameter bagi model hukum kuasa stokastik logistik khususnya untuk model Lotka-Volterra dengan menggunakan pendekatan dua-langkah. Pendekatan Bayesian dilaksanakan dalam langkah pertama bagi menganggarkan parameter regresi splin. Dengan menggabungkan kriteria bukan parametrik yang sedia ada dengan yang dicadangkan, parameter struktur SDE dapat dianggarkan dalam langkah kedua. Keputusan menunjukkan peratusan yang tinggi dalam ketepatan anggaran parameter model Lotka-Volterra menyokong kecukupan kriteria yang dicadangkan sebagai alternatif kepada kaedah klasik.

Kata kunci: Persamaan pembezaan stokastik; regresi splin; pendekatan Bayesian; singkatan asas siri kuasa, model Lotka-Volterra

1.0 INTRODUCTION

Population dynamics studies the changes in population size and age composition, the biological and environmental processes influencing those changes. It also deals with the affect of birth and death rates, immigration and emigration on population. The Malthusian growth model or simple exponential model is amongst the pioneer model used to model population dynamics. This model is refined and adjusted with a more general formulations was later proposed and expanded. One of such model is Power Law Logistic Model.

1.1 Power Law Logistic Differential Equation

Power law logistic differential equation used to describe population dynamic has the following form,

$$\frac{dN}{dt} = aN^{\xi} - bN^{\eta} \tag{1}$$

where N is population density, a and b are growth and crowding coefficients respectively, and a, b, ξ, and η are constants.

Let $\xi = 1$, $N = x(t)$, $\eta = s+1$, Eq.(1) becomes

$$\frac{dx(t)}{dt} = ax(t) - bx^{s+1}(t) \tag{2}$$

For a more general equation, let $C = -b$ thus

$$\frac{dx(t)}{dt} = ax(t) + Cx^{s+1}(t) \tag{3}$$

The above model considers population interactions with a solution

$$x(t)=e^{at}u(t)$$

where $u(t)=\dfrac{a^{1/s}x_0}{\left[a+Cx_0^s\left(1-e^{ats}\right)\right]^{1/s}}$.

2.2 Lotka-Volterra Model

Extending Eq. (1),

$$dx(t)=diag\left(x_1(t),\ldots,x_n(t)\right)\left[(a+Cx^s(t))dt\right] \quad (4)$$

where $x=\left(x_1,\ldots,x_n\right)^T$, $a=\left(a_1,\ldots,a_n\right)^T$, $C=\left(c_{ij}\right)_{nxn}$, $x^s=\left(x_1^s,\ldots,x_n^s\right)^T$.

A general equation of Lotka-Volterra model for interacting n species is described by the

$$dx(t)=diag\left(x_1(t),\ldots,x_n(t)\right)\left[(a+Cx(t))dt\right] \quad (5)$$

Population dynamics are usually affected by the noise either intrinsicly or extrinsicly by perturbing the deterministic Lotka-Volterra model into Itô stochastic equation. In this work, only extrinsic perturbation is applied in the model. Considering a linear growth condition, every element of growth coefficients will be perturbed as $a_i \to a_i+\sigma dwt$. The stochastic differential equation for this model is

$$dx(t)=diag\left(x_1(t),\ldots,x_n(t)\right)\left[(a+Cx(t))dt+\sigma dw(t)\right] \quad (6)$$

where $dw(t)=\left(dw_1(t),\ldots,dw_n(t)\right)^T$, n dimensional Brownian motions, and $\sigma=\left(\sigma_{ij}\right)_{n\times n}$ is a matrix representing the intensity of noise with the assumptions, $\sigma_{ij}>0$ if $1\le i\le n$ whilst $\sigma_{ij}\ge 0$ if $i\ne j$. Pang *et al*. (2008) studies the asymptotic properties of this model when the noise is relatively small. Many estimates in their paper indicate strong existence of a stationary distribution. Mao (2011) had shown theoretically this model has stationary distribution. The pdf $f_{X(t_i)}$ does not depend on t_i, therefore a single sample path can be utilized to estimate the parameter of SDE. Consider a scalar or one dimensional case:

$$dx(t)=x(t)\left[(b-ax(t))dt+\sigma dw(t)\right] \quad (7)$$

Considering only two parameters of SDE replacing $b=\theta$ and $a=\dfrac{\theta}{c}$ where c is a constant. Thus the general form of the scalar model becomes

$$dx(t)=\theta x(t)\left[(1-\frac{x(t)}{c})dt+\sigma dw(t)\right]. \quad (8)$$

2.0 MATERIALS AND METHODS

Parameter estimation of stochastic differential equation (SDE) is largely based on classical methods such as non-linear least squares, maximum likelihood, methods of moment and filtering such as the extended Kalman filter. Nonparametric approach in estimating the parameters of SDE has recently been introduced by Varziri *et al*. (2008) who developed Approximate Maximum Likelihood Estimation (AMLE).

A new version of a two-step method is proposed via the minimization of the negative of natural logarithm of approximate probability density function to estimate the drift and the spline parameters of the SDE. The estimated disturbance intensity was then repeatedly improved by a noise estimator. This approach however causes computational burden since it involves the approximation of transitional probability. Furthermore, Bayesian approach with spline implementations have not been considered in parameter estimation of SDE.

Wide literatures may be found in the implementation of regression spline. For example: Budiantara (2001), Lee (2002), Molinari *et al*. (2004), Leathwick *et al*. (2005), Hunt and Li (2006), Calderon *et al*. (2010) but few have involved Bayesian approach. Works employing Bayesian regression spline include Li and Yu (2006) who estimated the term structure with Bayesian regression splines based on nonlinear least absolute deviation. The method was found to be robust to outliers in a chosen case study. Lang and Brezger (2004) proposed a Bayesian version for P-splines for generalized additive models. The approach has the advantages of allowing simultaneous estimation of smooth function and smoothing parameter, and had been extended to more complex formulations. Wallstrom *et al*. (2008) implemented BARS (Bayesian Adaptive Regression Splines) in C by manipulating B-splines for normal and Poison cases. This has improved the original implementation of BARS in S.

The objective of this paper is to estimate SDE parameters, with Bayesian regression spline in the first step for estimating the spline parameters. For the second step, a criterion introduced by Varah (1982) and a our proposed criterion with a non-likelihood based with a spline approach are used to estimate the SDE parameters.

2.1 Proposed Methods

Consider a one dimensional Itô SDE given by

$$\frac{dx(t)}{dt}=f\left(x,t,\theta\right)+g\left(x,t,\sigma\right)\frac{dW(t)}{dt}, \quad (9)$$

where $f(x,t,\theta)$ is the average drift term, $g(x,t,\sigma)$ is the diffusion term, and $dW(t)$ is the Brownian noise. A two-step method with a non-likelihood based approach will be used to estimate the structural parameter of SDE by firstly estimating the parameters of regression spline with Bayesian approach in the first step and next estimate the parameters of the drift and diffusion term in SDE in the second step.

2.2 Two-Step Method. The First Step

The general equation of regression splines with truncated power series basis is :

$$s(t)=\sum_{j=1}^{m}\alpha_j t^{j-1}+\sum_{j=1}^{k}\delta_j\left(t-\xi_j\right)_+^{m-1}, \quad (10)$$

where s is known as a spline of order m with knots $\xi_1,\ldots,\xi_k$, t is the independent variable, $\alpha_1,\ldots,\alpha_m$ and $\delta_1,\ldots,\delta_k$ are some sets of coefficients. Given a choice of

$\boldsymbol{\lambda} = (\xi_1, \xi_2, ..., \xi_k)$, let $x_2(t) = t, ..., x_m(t) = t^{m-1}$, $x_{m+1}(t) = (t-\xi_1)_+^{m-1}$, ..., $x_{m+k}(t) = (t-\xi_k)_+^{m-1}$ and set $\boldsymbol{\beta} = (\alpha_1, ..., \alpha_m, \delta_1, ..., \delta_k)'$. The least squares spline estimator can be rewritten as $s(t) = \sum_{j=1}^{m+k} \beta_{\lambda j} x_j(t_i)$. From Eubank (1988), adhoc rules for locating knots are as the followings:

1.0 For $m = 2$, linear splines, place knots at points where the data shows a change in slope.

2.0 For $m = 3$, quadratic splines, the knots are located near the local minima, maxima or inflection points of the data

3.0 For $m = 4$, cubic splines, the knots are arranged near the inflexion points in the data

In the first step of the proposed procedure, the values of α and δ are estimated using the Bayesian approach with the assumption of normal error and diffuse prior for the parameters. The estimation is carried out in Winbugs with 10^6 MCMC simulations after selecting suitable number and the location of knots. The best number and location of knots of the fitted spline are determined by calculating the Generalized Cross Validation (GCV):

$$GCV(\lambda) = \frac{\sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m+k} \beta_{\lambda j} x_j(t_i) \right)^2}{\left(1 - \frac{(m+k)}{n} \right)^2}, \tag{11}$$

where y_i is the observed data, $\sum_{j=1}^{m+k} \beta_{\lambda j} x_j(t_i) = s(t)$ is the spline equation, k is the number of knots, m is the degree of spline, n is the number of the observations. The least value of GCV indicates the best fit of $s(t)$.

2.3 Two-Step Method. The Second Step

In the second step, we first estimate the parameter of the average drift equation by minimizing

$$\sum_{i=1}^{n} \left[\dot{\hat{x}}_i - f(\hat{x}, t, \theta) \right]^2, \tag{12}$$

where $\dot{\hat{x}}_i$ is the derivative of the spline approximation of the true solution of ordinary differential equation (ODE) which is used to represent the average drift term, $\hat{x}$ is the consistent estimator of the true solution, t is the independent variable and θ represents the parameter of ordinary differential equation. $f(\hat{x}, t, \theta)$ is the ODE used to represent the average drift term in SDE. Minimizing Eq. (12) was primarily introduced by Varah (1982) and used by many authors including Brunel (2008). By minimizing Eq. (12), it is expected to minimize the deviation between the differential of spline and the estimated ordinary differential equation.

A new criterion is proposed in estimating the diffusion term. Consider a one dimensional Itô SDE given by

$$\frac{dx(t)}{dt} = f(x, t, \theta) + g(x, t, \sigma) \frac{dW(t)}{dt}, \tag{13}$$

where $f(x, t, \theta)$ is the average drift term, $g(x, t, \sigma)$ is the diffusion term, and $dW(t)$ is the Brownian noise. Rearranging equation (13), we have

$$\frac{dx(t)}{dt} - f(x, t, \theta) = g(x, t, \sigma) \frac{dW(t)}{dt}. \tag{14}$$

The discrete approximation of true solution of SDE using numerical descretization such as Euler or Milstein can be considered here. Here, Milstein numerical approximation is used to estimate the solution of (13) given as,

$$x_{i+1} = x_i + h_i f(x_i) + g(x_i)\Delta W_i + \frac{1}{2} g(x_i) g'(x_i)((\Delta W_i) - h_i).$$

SDE can be rewritten in a form of difference quotient such as in Taylan *et al*. (2008) where

$$\dot{x}_i \approx f(x_i) + g(x_i) \frac{\Delta W_i}{h_i} + \frac{1}{2} g(x_i) g'(x_i) \left(\frac{(\Delta W_i)^2}{h_i} - 1 \right) \tag{15}$$

(RHS is the Milstein scheme). In order to estimate θ and σ they minimize

$$\sum_{i=1}^{N} \left(\dot{x}_i - f(x_i) - g(x_i) \frac{\Delta W_i}{h_i} - \frac{1}{2} g(x_i) g'(x_i) \left(\frac{(\Delta W_i)^2}{h_i} - 1 \right) \right)^2 \tag{16}$$

where x_i' s are the observed data, $i = 1, ..., N$, $\Delta W_t = W_{t_{i+1}} - W_{t_i}$, and $h_i = t_{i+1} - t_i$. Therefore, there is an argument to support equation (13) which can be approximated by a difference quotient $\dot{x}_i = \frac{x_{i+1} - x_i}{h_i}$. The average drift term in Eq. (13) is of ODE form and approximated by regression spline. (See Varah (1982) and Brunel (2008)). Thus, $f(x, t, \theta)$ can be approximated by $f(\hat{x}, t, \theta)$. Therefore, the approximation of equation (14) is obtained, that is $\dot{x}_i - \dot{\hat{x}} \approx g(x, t, \sigma) \frac{dW(t)}{dt}$.

In order for the values of σ to produce the least variation between both of RHS and LHS of Eq. (14), the squared difference of both quantities is minimized. In order to estimate ϕ in the second step, a new criterion is introduced by rearranging the terms as follows,

$$\text{minimizing} \sum_{i=1}^{n} \left[\dot{x}_i - \dot{\hat{x}}_i - g(x, t, \sigma) \frac{dW(t)}{dt} \right]^2. \tag{17}$$

The Brownian motion $dW(t)$ is approximated by $\Delta W_t \sim N(0, h_i)$ where $\Delta W_t = Z_i \sqrt{\Delta t_i} = Z_i \sqrt{h_i}$ and $\frac{dW(t)}{dt} \approx \frac{Z_i}{\sqrt{h_i}}$. It can be seen that Z_i has a standard normal distribution. The approximated values of $\frac{dW(t)}{dt}$ can be generated by standard normal random numbers generator in MATLAB.

2.4 Application

A sample path is simulated by fixing the parameters with $\theta = 3$ and $\sigma = 0.2$,

thus $dx(t) = 3x(t)\left[(1 - \frac{x(t)}{3})dt + 0.2dw(t)\right]$

Simulated sample path with EM scheme with $x(0) = 3$ and step size $h = 0.5$ is shown in Figure 1.

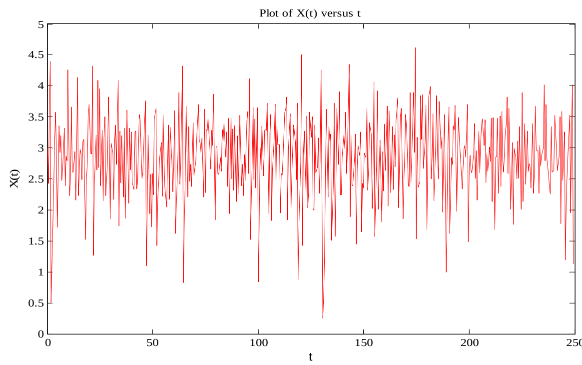


Figure 1 A simulated sample path of Lotka Volterra model

A Two-step method will be used to estimate the parameters of Lotka-Volterra model with a simulated data generated from the sample path shown 1. The optimal knot selection is shown in Table 1, a linear spline is chosen since it has the smallest GCV value and the single optimal knot is 242.5.

Table 1 Optimal Knot selection for simulated data

Order of spline	Optimal Single Knot	GCV
Linear Quadratic	242.5	245.645
Cubic	119	245.971
	144.5	246.821

Using winbugs software with 10^6 MCMC simulations, the estimated spline parameters are listed in Table 2, followed by the plot the regression spline of the simulated data in Figure 2.

Table 2 Estimated Bayesian spline parameters for simulated data

m	Location of knot	Interval	Estimated Bayesian spline parameters
1	242.5	$t < 242.5$	2.841,-0.0002247
		$t \geq 242.5$	-1.239,0.01524,0.08174

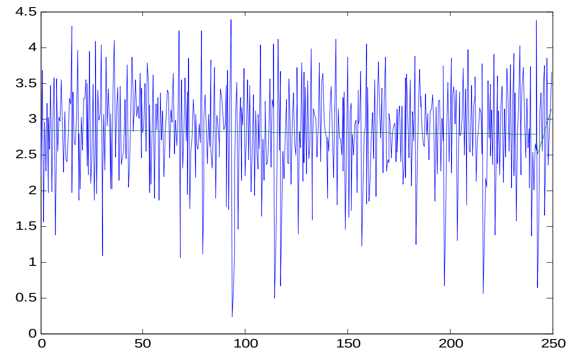


Figure 2 Linear regression spline of simulated data

From Table 3, the estimated drift and diffusion parameter are 0.0123 and 0.19225 respectively with percentage of accuracy being 0.41% and 96.12%. High percentage of accuracy is obtained for the diffusion parameters which employ the proposed non-parametric criterion as in Eq. (17).

Table 3 Estimated parameters of Lotka-Volterra SDE model

Estimated drift parameter	0.0123
% of Accuracy	0.41
Estimated diffusion parameter	0.19225
Minimum of HANPIC	4.3610
% of Accuracy	96.12
Standard Deviation	0.10026
Lower Bound 95% CI	0.12674
Upper Bound 95% CI	0.25776
Mean	0.07648

3.0 CONCLUSION

The estimation of diffusion parameter of Lotka-Volterra model resulted in satisfactory estimates with high accuracy for simulated data. This indicates the proposed criterion produced good estimate of the diffusion parameter of SDE. However the drift parameter estimation is unsatisfactory with low accuracy using the criterion introduced by Varah which may be due to the solution component of the system is oscillatory in nature and out of phase with each other (Varah, 1982). As a solution, a spline fit with many knots such as B-spline is possibly more suitable which is not considered in this case study.

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