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Non-Homogeneous Hidden Markov Model for Daily Rainfall Amount in Peninsular Malaysia

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Graphical abstract

Abstract

The non-homogeneous hidden Markov model (NHMM) generates the rainfall observation depends on few weather states which serve as a link between the large scale atmospheric measures. The daily rainfall at 20 stations from Peninsular Malaysia for 33 years sequences is analyzed using NHMM during the northeast monsoon season. A NHMM with six hidden states are identified. The atmospheric variable was obtained from NCEP Reanalysis Data as predictor. The gridded atmospheric fields are summarized through the principle component analysis (PCA) technique. PCA is applied to sea level pressure (SLP) to identify their principal spatial patterns co-varying with rainfall. The NHMM can accurately simulate the observed daily mean rainfall, correlations between stations for daily rainfall amounts and the quantile-quantile plots. It can be concluded that the NHMM is a useful method to simulate the daily rainfall amounts that may be used to prepare strategies and planning for the unpredicted disaster such as flood and drought.

Keywords: Hidden Markov model; daily rainfall; PCA

Abstrak

Model Markov tersembunyi bersifat tak homogen (NHMM) *membangkitkan data* taburan *hujan* bergantung kepada beberapa keadaan cuaca yang berperanan sebagai penghubung dengan atmosfera skala besar. Taburan hujan harian untuk 20 stesen di Semenanjung Malaysia selama 33 tahun dianalisis menggunakan NHMM pada musim monsun timur laut. NHMM dengan enam keadaan tersembunyi dikenal pasti. Pembolehubah atmosfera telah diperolehi daripada Reanalysis Data NCEP sebagai peramal. Grid sistem atmosfera diringkaskan melalui teknik Analisis Komponen Utama (PCA). PCA menganalisis tekanan aras laut (SLP) untuk mengenal pasti corak utama ruang berbeza-beza dengan taburan hujan. NHMM mensimulasikan min pengamatan taburan hujan harian, korelasi antara stesen-stesen untuk jumlah taburan hujan harian dan plot-plot quantile. Kesimpulannya, NHMM adalah kaedah yang berguna untuk simulasi jumlah hujan harian yang boleh digunakan untuk menyediakan strategi dan perancangan untuk bencana tidak disangka seperti banjir dan kemarau.

Kata kunci: Model Markov tersembunyi; taburan hujan harian; PCA

■1.0 INTRODUCTION

Rainfall modeling in space and time has important impacts on human and physical environments. Once the rainfall behavior is known, it is easy to predict the trends of future rainfall. In addition, it will help us to be aware to the unpredicted disaster such as flood. However, existing stochastic models of rainfall typically do not incorporate atmospheric information. Stochastic rainfall modeling which do not incorporate atmospheric information can only be used to simulate rainfall under climatic conditions which are stochastically similar to those used to fit the model. A model that has succeeded in incorporating atmospheric information will be useful in studies of climate variability or climate change. Hence this study will emphasis on modeling the rainfall process using hidden Markov model by linking synoptic atmospheric patterns to rainfall occurrence and amount. In order to incorporate the time-varying covariates, the non-homogeneous Markov chain model will be applied as well.

The main goal of this study is to assess the performance of NHMM in tropical climate**.** The study on the daily occurrence using hidden Markov model in rainfall occurrence was pioneered by Zucchini and Guttorp [1]. They introduced unobserved climate states which had different distribution of rainfall. Hughes and Guttorp [2] extended the HMM which was used by Zucchini and Guttorp [1] and described a non-homogeneous hidden Markov model (NHMM) to link the synoptic atmospheric information to local precipitation. Hughes and Guttorp [3] extended the NHMM to the case of spatial dependence by using autologistic model for the transition probability of rainfall given the weather state. Hughes *et al*. [4] modeled a 15 year sequence of winter data for 30 rainfall station in south-western Australia. The results showed that the model accurately reproduces the observed rainfall statistics and provides some useful insights into the rainfall process in south-western Australia.

Charles *et al*. [5] presented the nonhomogeneous hidden Markov model (NHMM) for climate change condition in southwestern Australia. They found that the NHMM can be placed against the criteria of Wilby *et al*. [6] for a useful downscaling model. Charles *et al*. [7] extended the NHMM by incorporating rainfall amount. The results showed that the extended NHMM accurately simulates the survival curves for dry (wet) spell lengths, wet day probabilities, daily rainfall amount distribution, and intersite correlations for daily rainfall amounts. Bellone *et al*. [8] used NHMM to model the rainfall amount independently at each rainfall station as gamma deviates with gauge-specific parameters in Washington. They verified that the model responded to shifts in atmospheric circulation from a reserved data set.

Robertson *et al*. [9] applied the HMM to tropical rainfall occurrence over Northeast Brazil. They found that interannual variability in the frequency of occurrence of dry spells could be simulated well. Charles *et al*. [10] investigated the ability of the extended NHMM to reproduce observed interannual and interdecadal rainfall variability when driven by observed and modeled atmospheric field. Robertson *et al*. [11] applied hidden Markov model to 11 stations over North Queensland. The results showed that the hidden Markov model able to simulate accurate station level simulations of the interannual variability of daily rainfall amount and occurrence. Robertson *et al*. [12] used NHMM in conjunction with a crop model to investigate spatial and temporal disaggregation of seasonal rainfall for simulating maize yield at 10 stations over the southeastern United States. Greene *et al*. [13] applied 4-state homogenous hidden Markov model to a network of 13 stations in central western India. Their results have shown enough evidence to the HMM representation of monsoon spatio-temporal variability. Robertson *et al*. [14] applied NHMM to a network of 17 station over Indramayu district, Indonesia. They obtained results with accurate levels of interannual variance for more skillful quantities (onset date, seasonal total and rainfall frequency).

In this study, we fitted the non-homogeneous hidden Markov model for daily rainfall amounts in peninsular Malaysia during northeast monsoon season. The performance of NHMM is evaluated on its ability to reproduce the rainfall simulations. The rainfall simulations data are compared to the observations data on the daily mean, correlation and quantile-quantile plot.

2.0 MATERIALS AND METHODS

2.1 Data

Daily rainfall data from 20 stations over Peninsular Malaysia from 1975 to 2008 was obtained from the Malaysia Meteorological Service (MMS). These stations were selected from each region based on the completeness of the data and length of records.The station locations are given in Figure 1. The atmospheric variables were taken from the analysis product of the National Centers for Environmental Prediction (NCEP) [15]. The rainfall data is obtained from the Malaysia Meteorological Service (MMS). The rainfall daily data are categorized into 2 categories namely "wet day" which is defined by a day with a rainfall amount exceeds or equals a fixed threshold of 0.3 mm and labeled as "1". On the other hand, a day with rainfall amount less than 0.3 mm is categorized "dry day" and labeled as "0". The threshold of 0.3 mm represents measurable rainfall is defined by World Meteorological Organisation [16]. The 120-day period beginning November 1 (November - December - January - February, NDJF) were selected, corresponding to the Northeast monsoon over Peninsular Malaysia. The 29 of February for leap year was withdrawn. The descriptive statistics for each rainfall stations during the Northeast monsoon is shown in Table 1.

No	Station	longitude	latitude
1	Arau	100.27	6.43
2	Kodiang	100.30	6.37
3	Pendang	100.48	5.99
4	SIK	100.73	5.81
5	Senai	103.67	1.63
6	Kluang	103.32	2.02
7	Tangkak	102.57	2.25
8	Malacca	102.25	2.27
9	Mersing	103.83	2.45
10	Endau	103.67	2.59
11	Subang	101.55	3.12
12	Gombak	101.73	3.27
13	Kuantan	103.22	3.78
14	Sitiawan	100.70	4.22
15	Ipoh	101.10	4.57
16	Gua Musang	101.97	4.88
17	Selama	100.70	5.14
18	Bkt Berapit	100.48	5.38
19	Kota Bharu	102.28	6.17
20	Alor Star	100.40	6.20

Figure 1 The selected rainfall stations in Peninsular Malaysia

Table 1 Descriptive statistics for each rainfall stations

	Arau	Kodiang	Pendang	SIK	Senai	Kluang	Tangkak	Malacca	Mersing	Endau
Mean	3.1	2.8	3.2	4.1	7.2	6.4	4.7	4.5	11.8	14.9
Variance	105.5	91.7	92.5	107.4	316.5	265.0	127.6	123.1	890.2	1115.4
Kurtosis	72.9	63.5	32.9	18.1	77.0	45.9	26.2	23.5	37.9	21.1
Skewness	6.8	6.6	5.0	3.8	6.5	5.4	4.3	4.2	5.2	4.0
Sum	12093	11246	12506	16390	28414	25489	18655	17749	46844	59150

	Subang	Gombak	Kuantan	Sitiawan	Ipoh	Gua Musang	Selama	Bkt Berapit	Kota Bharu	Alor Star
Mean	7.6	5.6	11.7	5.6	6.9	5.8	8.5	4.9	11.0	3.1
Variance	221.5	161.1	948.5	161.3	201.6	162.7	246.4	146.5	1069.6	91.4
Kurtosis	14.6	17.1	51.5	20.8	13.7	25.0	9.4	23.7	73.1	33.2
Skewness	3.3	3.7	5.8	3.9	3.3	4.2	2.7	4.2	6.9	5.1
Sum	30009	22113	46485	22045	27482	22962	33720	19544	43394	12101

2.2 Method

The non-homogeneous hidden Markov model (NHMM) generates the rainfall observation that is dependents on few weather states which serve as a link between the large scale atmospheric measures [2]. The assumption for the hidden process can be summarized as:

$$P(S_t|S_1^{t-1}, \mathbf{X}_1^T) = P(S_t|S_{t-1}, \mathbf{X}_t) \quad (1)$$

where S_t is the weather state at time t and $\mathbf{X}_t$ is a vector of atmospheric variables at time t, $1 \le t \le T$. The notation $\mathbf{X}_1^T$ indicates all the values of $\mathbf{X}_t$ from time 1 to T and similarly S_1^{t-1} denotes all the values of S_t between time 1 and time t-1[2].

The parameterization we adopt for $P(S_t|S_{t-1} = 1, \mathbf{X}_t)$ is

$$P(S_t = j|S_{t-1}, \mathbf{X}_t) \propto \gamma_{ij}\exp[-\frac{1}{2}(\mathbf{X}_t - \mu_{ij})\Sigma^{-1}(\mathbf{X}_t - \mu_{ij})'] \quad (2)$$

where Σ is the variance-covariancematrix for the atmospheric data. The μ_{ij} parameters represent the mean vectors of the atmospheric variables when the state of the weather at the previous time point was state i and the current state of the weather is j, while the γ_{ij} parameters can be interpreted as baseline transition probabilities [2]. The constraints $\sum_j \gamma_{ij} = 1$ and $\sum_j \mu_{ij} = 0$ are imposed to ensure identifiability of the parameters

The hidden Markov model assumptions for the observed process can be summarized by

$$f_{R_t|S_1^T, R_1^{t-1}, \mathbf{X}_1^T}(r) = f_{R_t|S_t}(r) \quad (3)$$

where f denotes a probability density function, $\mathbf{R}_t$ is the vector of precipitation amounts at a network of stations at time t and $\mathbf{R}_1^{t-1}$ indicate all the precipitation data from time 1 to t-1 [8].

Amounts are introduced by modelling rainfall at each station, given the weather state, as a mixture of a point mass at zero and a gamma distribution. The resulting parameterization is

$$f_{R_t|S_t}(r) = \prod_{i=1}^{N}[p_{si}G(r_t^i - c; \alpha_{si}, \beta_{si})]^{1[r_t^i > c]}(1 - p_{si})^{1[r_t^i \le c]} \quad (4)$$

where N is the number of rain stations, p_{si} is the rainfall probability at Station i and time t. With $G(r_t^i; \alpha_{si}, \beta_{si})$, the density at r_t^i of a gamma distribution is indicated with parameters α_{si} and β_{si}which depend on the state s and the Station i:

$$G(\boldsymbol{r}; \alpha_{si}, \beta_{si}) = \frac{\beta_{is}^{\alpha_{is}}}{\Gamma(\alpha_{is})}\boldsymbol{r}^{\alpha_{is}-1}\mathrm{e}^{-\mathrm{r}\beta_{\mathrm{is}}} \quad (5)$$

where $\Gamma(\alpha_{is})$ is the gamma function with argument α_{is}. The indicator function $^1[r_t^i > c]$ takes on a values of 1 if the rainfall amount at time t and station i is above the prescribed cutoff c; it takes on a value of 0 if the precipitation amount is below c [8]. Thus amounts below c are treated as no rainfall.

In this study, the sea level pressure is used as the classifier to the hidden state in the NHMM. Before the sea level pressure is used as classifier, principle component analysis (PCA) is applied to reduce the dimension of the raw sea level pressure data. The atmospheric variables typically are available on several grid nodes. Therefore, principle component analysis is preformed to summarize the atmospheric data from several grid nodes into few values. The k-th PCA at time t is

$$P_t(k) = \sum_j \mathbf{e}_j^k \left[\frac{x_t(j) - \bar{x}(j)}{s(j)}\right] \quad (6)$$

where $x_t(j)$ is the sea level pressure at node j at time t, $\bar{x}_t(j)$ and $s(j)$ are the mean and standard deviation of the seal level pressure time series at the j-th node. The $\mathbf{e}^k$ denotes the k-th eigen vector of the covariance matrix of the sea level pressure.

The Spearman Rank Correlation is used to measures the strength of the relationship between historical and simulated rainfall amounts at each station pair.

$$\rho = 1 - \frac{6\sum D_i^2}{n(n^2-1)}$$

where D_i is the difference in ranks between the i-th station pair rainfall amounts.

2.3 Parameter Estimation

Parameter estimates are obtained by numerically maximizing the likelihood. The likelihood of the observed data:

$$\begin{aligned} \boldsymbol{L}(\theta) &= f_{R_1^T|X_1^T=x_1^T,\theta}(r_1^T) \\ &= \sum_{s_1,s_2,\dots,s_T=1}^{m} f_{(R_1^T,\boldsymbol{S}_1^T|X_1^T=x_1^T,\theta}(r_1^T,\boldsymbol{s}_1^T) \\ &= \sum_{s_1,s_2,\dots,s_T=1}^{m} P(S_1|\mathbf{X}_t)\, f_{R_1,S_1=s_1(r_1)} \prod_{t=2}^{T} P(S_t|S_{t-1},\mathbf{X}_t)\, f_R \end{aligned} \quad (7)$$

where θ is the vector of the model parameters. The likelihood is computationally intractable. In order to make the calculation possible, the forward-backward procedure, a recursive algorithm is developed to solve the HMM [17]. Therefore, the likelihood function can be simplified as

$$L(\theta) = \delta\mathbf{P}(r_1)\Gamma\mathbf{P}(r_2)\Gamma\mathbf{P}(r_3)\dots\Gamma\mathbf{P}(r_T)\mathbf{1}' \quad (8)$$

$$\mathbf{P}(r_t) = \mathrm{diag}(p_1(r_t),\dots,p_m(r_t)) \quad (9)$$

where δ is the initial distribution, Γ is the transition probability matrix for the Markov chain, and $\mathbf{P}(r)$ is the $m\times m$ diagonal matrix with *i*th diagonal element of the state dependent probability $p_i(r)$ as shown in expression (9). The parameter of the likelihood function is then maximized by the Baum-Welch method which is also known as the EM algorithm [18].

Fitting an NHMM to precipitation data involves the choice of a model order and of the atmospheric variable to be included. The number of the NHMM is determined before including the atmospheric variables. The standard likelihood-based method which is Bayesian information criterion (BIC) is used in this study. The BIC is defined as

$$\mathrm{BIC} = 2l - k\log(T) \quad (10)$$

where l is the log-likelihood, k is the number of model parameters and T is the number of days of data.

3.0 RESULTS AND DISCUSSION

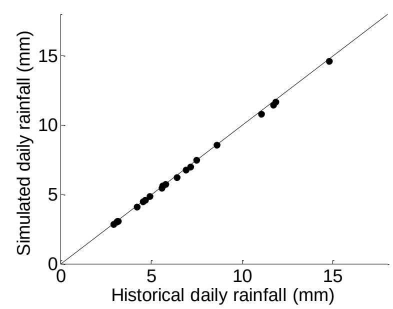
The NHMM model was fitted to 20 Malaysian rainfall stations. This model generated simulations of rainfall amounts that incorporated the sea level pressure. The standard hidden Markov model (HMM) was first fitted with different number of weather state from 2 to 8 states. It was noticed that the score of BIC reached a turning point that consists of six hidden states. Therefore, the hidden state of the model was set to six and the PCAs were then fitted to the model. It is noticed that the best model consists of six hidden states and includes PCA (1) with the minimum value of BIC. The BIC scores are given as follows:

Table 2 Comparison between the various hidden states of the model by BIC

Model	Number of parameter	BIC
S2	243	336516
S3	368	334897
S4	495	334971
S5	624	332313
S6	755	331951
S7	888	332166
S8	1023	332827
S6 P(1)	760	328195*
S6 P(1) P(2)	765	328235
S6 P(1) P(2) P(3)	770	328276
S7 P(1)	895	329384
S7P(1) P(2)	900	329433
S7P(1) P(2) P(3)	905	329482

Figure 2 (a) compares the historical daily mean rainfall amount with the simulated daily mean rainfall amount. All station show points that are much closer to the *x=y* line. This shows that the NHMM does well in reproducing the daily mean rainfall. Figure 2 (b) compares the historical versus simulated Spearman rank inter-site correlation. There is a negative bias in the simulated correlations across the historical correlation range. When the correlation between rain gauge stations is strong, the simulated Spearman rank correlation is underestimated.

(a)



(b)

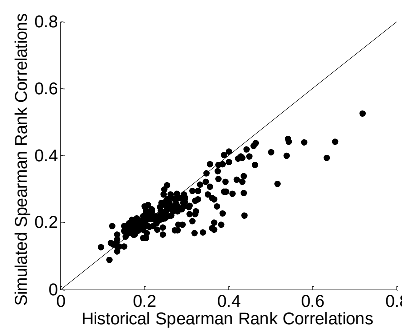


Figure 2 Historical versus simulated based (a) Daily mean rainfall amount. (b) Spearman rank correlations between all station pairs

The rainfall patterns of daily rainfall associated with each of the states which were identified by the NHMM including the first summary variable for sea level pressure are plotted in Figure 3 (a) and (b). The six weather states describe daily rainfall conditions varying from wet to dry in terms of rainfall probability at each station in Figure 3 (a) and the rainfall distribution on wet days which in terms of mean rainfall amounts in Figure 3 (b). State 3 has a high probability of rainfall over the entire region with a mean of rainfall amounts over 10 mm for all the station. On the contrary, state 5 corresponds to the dry days with a mean of rainfall amounts lower than 10 mm for majority of the stations. State 2 has a high probability of rainfall all over the region except the northwestern region and the mean of rainfall amounts are extremely high in the southwestern region. The state 1 has a probability of almost the same with state 2 but the mean of rainfall amounts are quite low in the southwestern region. State 4 represents a moderate dry condition over the entire region and state 6 represents a moderately wet condition over the entire region except the northwestern region.

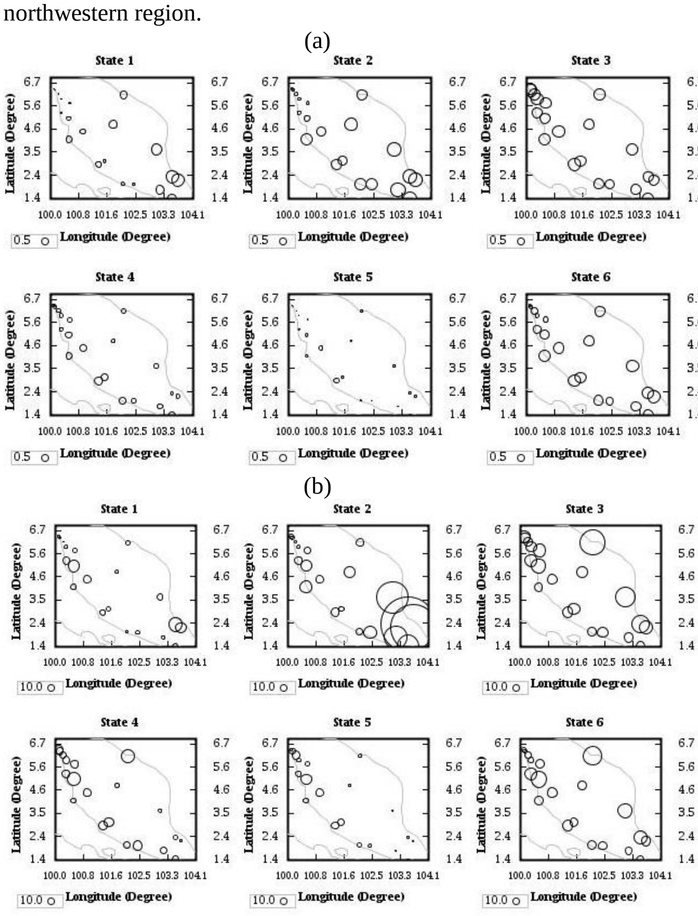


Figure 3 (a) Rainfall Probabilities and (b) Rainfall amounts corresponding to the six weather states identified by the NHMM including the first summary variable for sea level pressure

Figure 4 shows the quantile-quantile plots to compare the fits of the historical rainfall and the simulated rainfall from 6-hidden states of NHMM. The NHMM does well in reproducing the daily rainfall distributions at all selected stations, since the quantile function evaluated at the simulated rainfall amount is quite close to the historical rainfall amounts, yielding points that are very closed to the $x=y$ line. For stations Kota Bharu and Malacca, the simulated rainfall amounts are lower than the historical especially at the maximum amount, thus stations Kota Bahru and Malacca are underestimated. However, for the stations Arau and Kluang, the simulated rainfall amounts are higher than the historical, thus it is overestimated especially at the maximum amount.

4.0 CONCLUSION

The non-homogeneous hidden Markov model (NHMM) can be used to generate simulations of rainfall amounts that are linked to the atmospheric information. In this study, NHMM was applied to daily rainfall data from 20 stations over Peninsular Malaysia and was linked to the sea level pressure. The 6-hidden state NHMM was determined by Bayesian information criterion (BIC). The performance of the model was evaluated by comparing the correlation and the quantile-quantile plot between observation and simulation rainfall amounts. The NHMM seems to reproduce the overall distribution of simulated rainfall amounts reasonably well although the fit varies from station to station. It can be concluded that the NHMM is a useful method to simulate the daily rainfall amounts that may be used to prepare strategies and planning for the unpredicted disaster such as flood and drought.

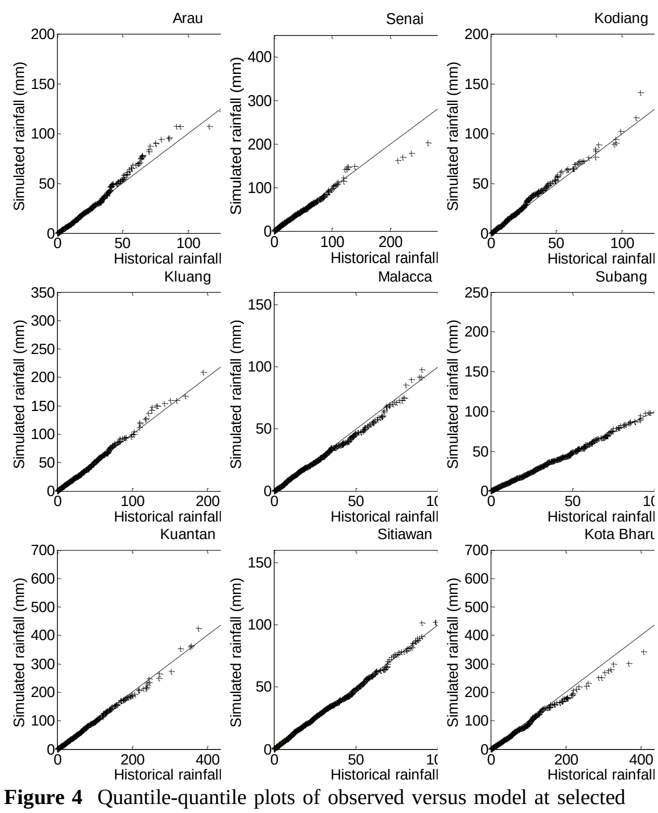


Figure 4 Quantile-quantile plots of observed versus model at selected stations

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