Jurnal Teknologi

Full paper

Conditions on the Edges and Vertices of Non-commuting Graph

M. Jahandideh[a*], M. R. Darafsheh[b], N. H. Sarmin[c], S. M. S. Omer[d]

[a]Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran
[b]School of Mathematics College of Science University of Tehran, Tehran, Iran
[c]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
[d]Department of Mathematics, Faculty of Science, University of Benghazi. Benghazi, Libya

*Corresponding author: m.jahandideh.kh@gmail.com

Article history

Received :9 August 2013
Received in revised form :
29 December 2014
Accepted :15 March 2015

Graphical abstract

If $\Gamma_G \cong \Gamma_H$ then $|G|=|H|$.

Abstract

Let G be a non-abelian finite group. The non-commuting graph of Γ_G is defined as a graph with a vertex set $G-Z(G)$ in which two vertices x and y are joined if and only if $xy \neq yx$. We define $\Gamma_G = (V(\Gamma_G), E(\Gamma_G))$ such that $V(\Gamma_G)$ is the vertices set and $E(\Gamma_G)$ is the edges set. In this paper, we invest some results on $|E(\Gamma_G)|$, the degree of a vertex of non-commuting graph and the number of conjugacy classes of a finite group. We found that that if $\Gamma_G \cong \Gamma_H$ is a finite group, then $|G|=|H|$.

Keywords: Finite group; non-commuting graph

Abstrak

Katalah G adalah suatu kumpulan terhingga yang bukan abelan. Graf tidak kalis tukar tertib Γ_G ditakrif sebagai graf yang mempunyai set bucu $G-Z(G)$ di mana dua bucu x dan y adalah berkait jika dan hanya jika $xy \neq yx$. Kita takrifkan $\Gamma_G = (V(\Gamma_G), E(\Gamma_G))$ yang mana $V(\Gamma_G)$ adalah set bucu dan $E(\Gamma_G)$ adalah set sisi. Dalam kertas kerja ini, kita hasilkan beberapa keputusan berkaitan $|E(\Gamma_G)|$, iaitu darjah kepada bucu graf tidak kalis tukar tertib dan bilangan kelas konjugat bagi kumpulan terhingga. Kita temui bahawa jika $\Gamma_G \cong \Gamma_H$, dengan H ialah kumpulan terhingga, maka $|G|=|H|$.

Kata kunci: Kumpulan terhingga, graf tidak kalis tukar tertib

1.0 INTRODUCTION

Let G be a non- abelian finite group. Various graphs could be attributed to G, one of which is the non-commuting graph, denoted by Γ_G. The set of vertices and edges of Γ_G are $V(\Gamma_G)$ and $E(\Gamma_G)$ espectively so that $(V(\Gamma_G) = G - Z(G))$ in which $Z(G)$ is the center of G and for every $x, y \in V(\Gamma_G)$ we have:

$$\{x, y\} \in E(\Gamma_G) \Leftrightarrow xy \neq yx.$$

It is apparent that if G is an abelian group, Γ_G would turn to a null graph. For this, G is assumed to be a non-abelian group. The centralizer of x within G, which is denoted by $C_G(x)$, is a subset of G which is defined as $\{g \in G | gx = xg\}$.

Assume that $\Gamma = (V, E)$ is a graph in which V is the set of vertices and E is the set of edges. This graph is assumed to be a finite graph whenever $|V|$, $|E|$ are finite. The degree of the vertex x which is shown by $deg(x)$ equals to the number of edges through x. According to[4], the non–commuting graph of a finite group G was first introduced by Paul Erdos in connection with the following problem: Let G be a group whose non–commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? By[4] the answer to this question is positive and this was the origin of many similar questions and research. In[1], the relations between some graph properties of Γ_G and the group theory properties of the group G are studied. In particular the following conjecture is raised:

Conjecture 1 Let G be a finite non-abelian group. If there is a group H such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

The main purpose of this paper is to put some conditions on $|E(\Gamma_G)|$ of the non–commuting graph so that if $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$. Our notation for graphs is standard and [2] is used as a general reference.

2.0 SOME RESULTS ON CONJUGACY CLASSES

Let G be a finite non-abelian group. The number of conjugacy classes of G is denoted by $k(G)$.

Lemma 2.1[1] Let G be a finite group and $k(G)$ be the number of conjugacy classes of G. Then

$$|E(\Gamma_G)| = 1/2\ |G|\ (|G| - k(G)).$$

Theorem 2.2 Let G and H be finite groups. If $\Gamma_G \cong \Gamma_H$, $(|G|, |H| - k(H)) = 1$ and $(|H|, |G| - k(G)) = 1$, then $|G| = |H|$.

Proof. We have $\Gamma_G \cong \Gamma_H$, thus $|E(\Gamma_G)| = |E(\Gamma_H)|$ and according to assumptions, we can obtain $|G|$ divides $|H|$. Using the same way, $|H|$ divides $|G|$. Therefore, $|G| = |H|$. ■

Theorem 2.3 Let G and H be finite groups. If $\Gamma_G \cong \Gamma_H$ and $k(G) = k(H)$, then $|G| = |H|$.

Proof. We use a contradiction proof. According to the assumptions, it can be written as $|G|^2 - |H|^2 = k(G)(|G| - |H|)$, since $|G| \neq |H|$, thus $k(G) = (|G| + |H|)$. According to the probability of commuting two randomly chosen elements of a finite group G which is equal to $(k(G))/|G|$ .Thus:

$$(k(G))/|G| = (|G| + |H|)/|G| = 1 + |H|/|G| > 5/8 .$$

Based on[3], G is an abelian group and this is a contradiction. Therefore $|G| = |H|$.

3.0 SOME RESULTS ON THE NUMBER OF EDGES

Lemma 3.1 Let G be a finite group. If $|E(\Gamma_G)| = p^n$, where p is a prime number $(p \neq 2)$, then

(i) If n is an even number, then $|G| = 2p^{\frac{n}{2}}$.

(ii) If n is an odd number, then $|G| = p^{\frac{n+1}{2}}$ where $p = 3, 5$.

Proof. Using a contradiction proof, it is shown that $n \neq 1$. There are two cases for $|G|$:

Case 1. If $|G| = 2p$ and $k(G) = 2p - 1$. According to $\frac{k(G)}{|G|} \leq \frac{5}{8}$, the result obtained is $3p \leq 4$ which is a contradiction.

Case 2. If $|G| = p$, then G is abelian and it is a contradiction.

Therefore $n \neq 1$. Now, is is proven that (i) is true, if n is an even number. In this case, there are three forms for $|G|$ which is stated as follows:

Case 1. $|G| = 2p^n$ and $k(G) = 2p^n - 1$. According to [3], $(k(G))/(|G|) \leq 5/8$ and $3p^n \leq 4$. Hence it is impossible for all odd prime number p and all even number n.

Case 2. $|G| = 2p^{n_1}$ and $k(G) = 2p^{n_1} - p^{n_2}$ $(n_1 \geq n_2)$. According to $(k(G))/(|G|) \leq 5/8$, we have $3p^{n_1-n_2} \leq 4$. If $n_1 \neq n_2$, then $3p^{n_1-n_2} > 4$. Thus it is concluded that $n_1 = n_2$. $n_1 + n_2 = n$ so $n_2 = n_1 = n/2$ and $|G| = 2p^{\frac{n}{2}}$.

Case 3. $|G| = p^{n_1}$ and $k(G) = p^{n_1} - 2p^{n_2}$, $(n_1 \geq n_2)$. In this case $3p^{n_1-n_2} \leq 16$. If $n_1 = n_2 = n/2$, then $|G| = p^{\frac{n}{2}}$ and $k(G) = -p^{\frac{n}{2}}$ as it is not possible. Using $3p^{n_1-n_2} \leq 16$, we conclude that $n_1 - n_2 = 1$, $p = 3, 5$. Therefore $n_1 = (n+1)/2$ and n_1 cannot be natural number. Hence we have $|G| = p^{\frac{n}{2}}$.

ii) If n is an odd number, then there exist three cases for $|G|$:

Case 1. $|G| = 2p^n$ and $k(G) = 2p^n - 1$. It is not possible for all odd prime numbers p and all odd numbers n.

Case 2. $|G| = 2p^{n_1}$ and $k(G) = 2p^{n_1} - p^{n_2}$, $(n_1 \geq n_2)$. We have $(k(G))/(|G|) \leq 5/8$, therefore $3p^{n_1-n_2} \leq 4$. If $n_1 \neq n_2$, then $3p^{n_1-n_2} > 4$. It follows that$n_1 = n_2$. Hence $n_1 = n_2 = n/2$. Since n is an odd number, n_1 can not be natural number. Therefore, this case is impossible.

Case 3. $|G| = p^{n_1}$ and $k(G) = p^{n_1} - 2p^{n_2}, (n_1 \geq n_2)$. We will gain $n_1 - n_2 = 1$, $p = 3,5$. In this case, $n_1 = (n+1)/2$, $n_2 = (n-1)/2$ and $|G| = 3^{\frac{n+1}{2}}$ or $5^{\frac{n+1}{2}}$. ■

Theorem 3.2 Let G and H be finite non–abelian groups. If $\Gamma_G \cong \Gamma_H$ and $|E(\Gamma_G)| = p^n$ (p is an odd prime number) then $|G|=|H|$.

Proof. This result can be proven easily by Lemma 3.1.

Lemma 3.3 Let G be a finite non-abelian group. If $|E(\Gamma_G)| = 2^n$ and n is an even number, then $|G| = 2^{\frac{n}{2}+1}$.

Proof. We have $|E(\Gamma_G)| = 2^n$ then $|G| = 2^{n_1}$ and $k(G) = 2^{n_1} - 2^{n_2}$ as $n_1 + n_2 = n + 1$ and $n_1 \geq n_2$. Using [3] we will have $3.2^{n_1} \leq 2^{n_2+3}$. Therefore, $n_1 = n_2 + 1$ *or* $n_1 = n_2 + 2$. If $n_1 = n_2 + 2$, then $3.2^{n_2+2} \leq 2^{n_2+3}$. Therefore $3 \leq 2$ and it is a contradiction. Thus $n_1 = n_2 + 1$ and on the other hand $n_1 + n_2 = n + 1$ and it is concluded that $n_2 = \frac{n}{2}, n_1 = \frac{n}{2} + 1$. As a result $|G| = 2^{\frac{n}{2}+1}$. ■

Theorem 3.4 Let G be a finite group. If H is a group, $\Gamma_G \cong \Gamma_H$ and $|E(\Gamma_G)| = 2^n$ (n is an even number), then $|G| = |H|$.

Proof. It follows from Lemma 3.3.

Lemma 3.5 Let G be a finite group. If $|E(\Gamma_G)| = p^2q$ (p, q are prime numbers and $p > q$), then $|G| = 3p$ or $5p$.

Proof. $2p^2q = |G|(|G| - k(G))$ is resulted by $|E(\Gamma_G)| = \frac{1}{2}|G|(|G| - k(G))$ and $|G| = 2p^2, 2q, p^2q, 2pq, pq$ or $2p$. Now we investigate all cases:

Case 1. If $|G| = 2p^2$, then $k(G) = 2p^2 - q$. According to $\frac{k(G)}{|G|} \leq \frac{5}{8}$, we have $3p^2 \leq 4q$ hence $|G| \neq 2p^2$.

Case 2. If $|G| = 2q$, then $k(G) = 2q - p^2 < 0$. Hence $|G| \neq 2q$. *Case* 3. If $|G| = p^2q$, then $k(G) = p^2q - 2$. This resulted as $3p^2q \leq 16$. There are not any two prime numbers that satisfy this inequality, thus $|G| \neq p^2q$.

Case 4. If $|G| = 2pq$ then $k(G) = 2pq - p$. We obtain $3q \leq 4$, and this is impossible.

Case 5. If $|G| = pq$, then $k(G) = pq - 2p$ and $3q \leq 16$. q can be 2, 3 or 5. If $q = 2$ then $k(G) = 0$ so $|G| = 3p$ or $5p$.

Case 6. If $|G| = 2p$ then $k(G) = 2p - pq \leq 0$. That is not possible. So $|G| \neq 2p$.

Using results in Lemma 3.5, we provide the following theorem:

Theorem 3.6 Let G and H be finite groups. If $\Gamma_G \cong \Gamma_H$ and $|E(\Gamma_G)| = p^2q$ (where p and q are prime numbers, $p > q$) then $|G| = |H|$.

Proof. Using recent lemma, we have $|G| = 3p$ or $5p$. Without loss of generality, suppose that $|G| = 3p$, so prove that $|H| = 3p$.
Suppose that $|H| = 5p$. We know that $\Gamma_G \cong \Gamma_H$ then $|V(\Gamma_G)| = |V(\Gamma_H)|$. It means $|G| - |Z(G)| = |H| - |Z(H)|$, there are three cases for $|Z(G)|$:

Case 1. If $|Z(G)| = 1$, then $|Z(H)| = 2p - 1$ and $2p - 1 \mid |Z(H)| = 5p$, this occurs when $p = 3$. Therefore, G is an abelian group and $G = Z(G)$. That is impossible. In this case $|G| = |H| = 3p$.

Case 2. If $|Z(G)| = 3$, then $|Z(H)| = 2p - 3$. It occurs when $p = 3$. Thus $|G| = |H| = 3p$.

Case 3. If $|Z(G)| = p$, then $|Z(H)| = 3p$ and $|Z(H)| \nmid 5p$, hence $|G| = |H| = 3p$.

Respectively, we can show that if $|G| = 5p$, then $|H| = 5p$. ■

Theorem 3.7 There is no finite group that the number of edges of its non–commuting graph be $2p$, where p is an odd prime.

Proof. Suppose that G is a finite group and $|E(\Gamma_G)| = 2p$. We have $4p = |G|(|G| - k(G))$, then $|G| = 4p$ or $|G| = 2p$.

If $|G| = 4p$, then $k(G) = 4p - 1$. Using $\frac{k(G)}{|G|} \leq \frac{5}{8}$, it is obtained that $3p \leq 2$. This not true for all odd prime numbers. Now, if $|G| = 2p$ then $k(G) = 2p - 2$. Using $\frac{k(G)}{|G|} \leq \frac{5}{8}$, we will have $3p \leq 8$. Again, this not true for all odd prime numbers. We conclude that, there is no such group.

■4.0 DEGREE OF A VERTEX OF NON–COMMUTING GRAPH

Lemma 4.11 Let G be a finite group. If x is one of the vertices of Γ_G, then

$$deg(x) = |G| - |C_G(x)|.$$

Theorem 4.2 Let G be a finite group such that there is an element $g \in G - Z(G)$ with $deg(g) = p^2q$, where p and q are prime numbers. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

Proof. From $|C_G(g)|\left(\frac{|G|}{|C_G(g)|} - 1\right) = p^2q$ we deduced that $|C_G(g)| = p, p^2, q, pq$ and p^2q, hence
$|G| = p(pq+1), p^2(q+1), q(p^2+1), pq(p+1)$ and $2p^2q$
Since the corresponding element $g' \in H - Z(H)$ has also degree p^2q we will obtain
$|H| = p(pq+1), p^2(q+1), q(p^2+1), pq(p+1)$ and $2p^2q$. We use contradiction to show $|G| = |H|$.
Since $|G| = p(pq+1)$ and $|G| \neq |H|$, then there exists four forms for $|H|$:

1. From $|G| = p(pq+1)$ we obtain $|C_G(g)| = p$, hence $|Z(G)| = 1$. If $|H| = p^2(q+1)$ and since $\Gamma_G \cong \Gamma_H$, we have $|Z(H)| = p^2 - p + 1$. Therefore $|C_G(g')| = p^2$ and $|Z(H)| \nmid |C_G(g')|$. This case is impossible.

2. If $|H| = q(p^2+1)$, where $|C_G(g')| = q$. Using this equality $|G| - |Z(G)| = |H| - |Z(H)|$ thus, $|Z(H)| = q - p + 1$. The order of $Z(H)$ must divide $|C_G(g')|$. It means $(q - p + 1) \mid q$. This is impossible.

3. If $|H| = pq(p+1)$, we must have $|C_G(g')| = pq$. In this case $|Z(H)| = pq - p + 1$ and since the $|Z(H)| \mid |C_G(g')|$, there is three cases for $|Z(H)|$:

Case 1. If $|Z(H)| = p = pq - p + 1$, then $p(q-1) = p - 1$. It is not possible.

Case 2. If $|Z(H)| = q = pq - p + 1$, then $p = 1$. It is contradiction.
Case 3. $|Z(H)| \neq 1$. It is clear.
If $|G| = p(pq+1)$, then $|H| = p(pq+1)$. It means $|G| = |H|$.

Simply, we can consider different scenarios to reach the desired result. ■

Theorem 4.3 Let G be a finite group such that there is an element $g \in G - Z(G)$ with $deg(g) = p^2q^2$, where p and q are prime numbers. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

Proof. From $|C_G(g)|\left(\frac{|G|}{|C_G(g)|} - 1\right) = p^2q^2$ we have
$|C_G(g)| = p, p^2, q, q^2, pq, p^2q, pq^2$ and p^2q^2. Respectively
$|G| = p(pq^2+1), p^2(q^2+1), q(p^2q+1), q^2(p^2+1), pq(pq+1), p^2q(q+1), pq^2(p+1)$ and $2p^2q^2$.
Since the corresponding element $g' \in H - Z(H)$ has also degree p^2q^2, we will obtain

$|H| = p(pq^2+1), p^2(q^2+1), q(p^2q+1), q^2(p^2+1), pq(pq+1), p^2q(q+1), pq^2(p+1)$ and $2p^2q^2$. Without loss of generality, assume that $|G| = 2p^2q^2$, from $|G|$ we obtain $|C_G(g)| = p^2q^2$ and since $|G| \neq |H|$, there exists seven cases for $|H|$ stated as follows:

1. If $|H| = p(pq^2+1)$, we gain $|Z(H)| = 1$. Using of this equality $|G| - |Z(G)| = |H| - |Z(H)|$, thus $|Z(G)| = p^2q^2 - p + 1$. It is impossible, since $(p^2q^2 - p + 1) \nmid p^2q^2$.

2. If $|H| = p^2(q^2+1)$, then $|Z(H)| = 1$ or p. If $|Z(H)| = 1$, then $|Z(G)| = p^2q^2 - p^2 + 1$. This is not true since $(p^2q^2 - p^2 + 1) \nmid p^2q^2$. If $|Z(H)| = p$, then $|Z(G)| =$

$p^2q^2 - p^2 + p$, but we have $(p^2q^2 - p^2 + p) \nmid p^2q^2$. Therefore $|H| \neq p^2(q^2+1)$.

3. If $|H| = q(p^2q+1)$, we have $|Z(H)| = 1$. Using the equality $|G| - |Z(G)| = |H| - |Z(H)|$, $|Z(G)| = (p^2q^2 - q + 1)$. It is impossible, because $(p^2q^2 - q + 1) \nmid p^2q^2$.

4. If $|H| = q^2(p^2+1)$, then $|Z(H)| = 1, q$. If $|Z(H)| = 1$, we have $|Z(G)| = (p^2q^2 - q^2 + 1)$ and p^2q^2 is not divisible by $(p^2q^2 - q^2 + 1)$. Now, assume that $|Z(H)| = q$, in this case $|Z(G)| = (p^2q^2 - q^2 + q)$. Again it is not true.

5. If $|H| = pq(pq+1)$, then $|Z(H)| = 1, q$ or p. Clearly, this is not true.

6. If $|H| = p^2q\ (q+1)$, then $|Z(H)| = 1, p, p^2, q$ or pq.
If $|Z(H)| = 1$, then $|Z(G)| = (p^2q^2 - p^2q + 1)$.
If $|Z(H)| = p$, then $|Z(G)| = (p^2q^2 - p^2q + p)$.
If $|Z(H)| = q$, then $|Z(G)| = (p^2q^2 - p^2q + q)$.
If $|Z(H)| = p^2$, then $|Z(G)| = (p^2q^2 - p^2q + p^2)$.
If $|Z(H)| = pq$, then $|Z(G)| = (p^2q^2 - p^2q + pq)$.

All of the above are impossible, because $|Z(G)| \nmid p^2q^2$ for all mentioned cases.

7. If $|H| = pq^2(p+1)$, then $|Z(H)| = 1, p, q^2, q$ or pq. As in 6 it is not true.

Therefore, $|G| = |H|$.

5.0 CONCLUSION

One of the important graphs that could be attributed to G is non-commuting graph. It defines as a graph with a vertex set $G - Z(G)$ in which two vertices x and y are joined if and only if $xy \neq yx$. In introduction, we mentioned two conjectures. In this research, we put some conditions on the number of edges set and degree vertices so that the conjectures become true.

Acknowledgement

The first author would like to acknowledge Universiti Teknologi Malaysia for invitation of her research attachment from 16th July to 15th September 2013. The third author would also like to acknowledge Universiti Teknologi Malaysia for the Research University Grant (GUP) Vote No. 08H07.

References

[1] Abdollahi, A., S. Akbari and H. R. Maimani. 2006. Non-Commuting Graph of a Group. *J. Algebra*. 298: 468–492.
[2] Bondy, J. A and J. S. Murty. 1977. *Graph Theory with Applications*. American Elsevier Publishing Co, Inc.
[3] Gustafson, W. H. 1973. What is the Probability That Two Group Elements Commute? *The American Mathematical Monthly*. 80(9): 1031–1034.
[4] Neuman, B. H. 1976. Problem of Paul EdrÄos on Groups. *J. Austral. Math.Soc*. 21: 467–472.