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Learning Functions of Two Variables Based on Mathematical Thinking Approach

Hamidreza Kashefi[a*], Zaleha Ismail[b], Yudariah Mohammad Yusof[c]

[a]*Centre for Engineering Education, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia*
[b]*Department of Science and Mathematics Education, Faculty of Education, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia*
[c]*Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia*

*Corresponding author: khamidreza4@live.utm.my

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Abstract

Multivariable function is one of the most important concepts in the learning of advanced mathematics. We had implemented a teaching approach to support students in the learning of two-variable functions by promoting mathematical thinking in face-to-face Multivariable Calculus classroom. This study investigates the obstacles and difficulties faced by students in the learning of two-variable functions based on the mathematical thinking approach. The findings indicated that students displayed various difficulties in finding the range and sketching the graph of two-variable functions. The students' difficulties and obstacles such as poor mastery of algebraic manipulation, poor grasp of prior knowledge or lack of it, idiosyncrasy attributed from previous mathematical experience, and restricted mental images of two-variable functions could be classified as difficulties with techniques, concepts, and studying mathematics. Based on students' responses, the difficulties were considered mainly conceptual in nature and few were related to techniques and studying mathematics.

Keywords: Algebraic manipulation; mathematical thinking; multivariable calculus; prior knowledge; students' difficulties; two-variable functions

Abstrak

Dalam pembelajaran matematik di peringkat tinggi, fungsi banyak pemboleh ubah merupakan satu konsep yang amat penting. Kami telah melaksanakan pendekatan pengajaran yang menyokong pembelajaran fungsi banyak pemboleh ubah dengan mempromosikan pemikiran matematik dalam keadaan bersemuka di bilik darjah. Kajian bertujuan menyiasat halangan dan kepayahan yang dihadapi pelajar dalam suasana pendekatan baru ini. Dapatan kajian menggambarkan kepayahan ketika pelajar mencari julat dan melakar graf Fungsi Dua Pemboleh ubah. Disamping itu kepayahan dan halangan dalam bentuk kelemahan manipulasi algebra, kekurangan pengetahuan sedia ada, kesilapan yang mencirikan pengalaman matematik terdahulu dan imej mental yang terbatas boleh diklasifikasikan sebagai kepayahan teknik, konsep dan pembelajaran matematik. Berdasarkan maklum balas dari pelajar, kebanyakan kepayahan berunsur konsepsi berbanding berkaitan teknik dan pembelajaran matematik.

Kata kunci: Manipulasi algebra; pemikiran matematik fungsi banyak pemboleh ubah; pengetahuan sedia ada; kepayahan pelajar; Fungsi Dua Pemboleh ubah

1.0 INTRODUCTION

The concept of multivariable function is considered fundamental in advanced mathematics and its applications (Trigueros and Martínez-Planell, 2010). It means that more advanced topics in the engineering undergraduate curriculum cannot be grasped without understanding of multivariable functions. Although its understanding is essential for students in many fields of study, little is known about students' conceptions and obstacles (Martinez-Planell and Trigueros, 2009). There are very few research based studies that probe how students construct the concept of multivariable functions and the obstacles that they encounter.

Dubinsky (1991) used Action – Process – Object – Schema theory, better known as APOS theory, to describe certain mental construction for learning mathematical concepts. In this theory, the Actions are routinized as Processes, encapsulated as Objects and embedded in a Schema of knowledge. Breidenbach *et al*. (1992) based on APOS theory described what it means to understand a concept such as function and how students can make that construction. Dubinsky and Yiparaki (1996) and other researches (see for example Asiala *et al*., 1996) suggested several specific pedagogical strategies for helping students to make the mathematical knowledge constructions. The main strategies used were ACE (Activities, Class discussion, and Exercises) teaching cycle, cooperative learning groups to engage in problem solving activities and the use of an interactive mathematical programming language. This theory also can be used to describe the construction of two-variable functions and the development of them by students (Trigueros and Martínez-Planell, 2010).

In an earlier research, Gray and Tall (see Gray and Tall, 1994, Grey *et al*., 1999) had introduced a similar cycle of mental construction as in APOS theory, called "procept" which is the amalgam of three components namely a process which produces a mathematical object, and a symbol which is used to represent either process or object. Reflecting on the theoretical development on the construction of mathematical knowledge in elementary and advanced mathematics, Gray and Tall (2001) then proposed three distinct types of mathematics worlds to describe certain mental construction for learning mathematical concepts. They suggested that there are three different ways of constructing mathematical concepts from *perception* of objects (as occurs in geometry), *actions* on objects (as in arithmetic and algebra) and *properties* of objects which lead to formal axiomatic theories.

In a further study, Tall (2004) point out that there are not only three distinct types of mathematics worlds; there are in fact three significantly different worlds of mathematical thinking: *conceptual-embodied, proceptual-symbolic, axiomatic-formal*. This theory underlies the creation of computer software which Tall called *generic organizer* and used it in his researches (Tall, 1986, 1989, 1993, 2000, 2003) to support students' mathematical construction and build embodied approach to mathematical concepts. In designing the generic organiser, it requires the selection of an important foundational idea to focus on. Tall used the notion of *cognitive root* as a cognitive unit containing the seeds of cognitive expansion to formal definitions and later theoretical development. Tall showed how *the notion of local straightness* (for rate of change/differentiation) and *area under the graph* (for cumulative growth/integration) can be cognitive roots in building an embodied understanding of the calculus. However, the generic organiser does not guarantee the understanding of the concept and Tall (1993, 1997) reported some cognitive obstacles faced by students when using this organiser. Tall believed that the learner requires an external *organising agent* in the shape of guidance from a teacher, textbook, or some other agency. In this way, Tall suggested that the combination of a human teacher and a computer environment can support students' mathematical knowledge construction and prevent misleading factors. In the case of real function, this theory insists on a flexible blend of embodiment and symbolism. As for the transition from one variable to two, two variables form one vector variable and the idea of local straightness becomes local flatness and the locally straight approach that Tall advocate was based on a blend of embodiment and formalism (Tall, 2010).

Tall and Dubinsky and their colleagues **(**Tall, 1997, 2010; Dubinsky, 1991; Dubinsky *et al*., 2005) endeavored to explain the construction of mathematical concepts in Basic Calculus. They focused on students' difficulties and used computers as a way of supporting students' mathematical thinking to overcome these difficulties **(**Tall, 1992, 2000, 2003; Dubinsky and Yiparaki, 1996; Asiala *et al*., 1996). However, there are very few researches that investigate the support on students' thinking powers in mathematical construction and the obstacles they faced in Multivariable Calculus concepts. In a study done by Roselainy and her colleagues (Roselainy, 2009; Roselainy, Yudariah, and Mason, 2007; Roselainy, Yudariah, and Sabariah, 2007) on students learning Multivariable Calculus, they presented a model of active learning that was based on invoking students' mathematical thinking powers, supporting mathematical knowledge construction, and promoting generic skills that students need to be aware of. Here, we further extended the study where we had adopted Roselainy *et al*.'s model and Tall's theory on three worlds of mathematical thinking in our approach. Based on this mathematical thinking approach, we first attempt to demonstrate the ways and means of supporting students in the learning of Multivariable Calculus specifically in the learning of two-variable functions. We then seek to uncover what instigate students' difficulties and obstacles when they encounter non-routine problems involving functions of two variables.

■2.0 MULTIVARIABLE CALCULUS THROUGH MATHEMATICAL THINKING APPROACH

Depending on scholars' perspectives who define the term of mathematical thinking in different ways, there is no consensus on the definition of mathematical thinking (Sternberg, 1996). According to Selden and Selden (2005), there are three different perspectives on the nature of advanced mathematical thinking. In the first perspective, advanced mathematical thinking is defined as thinking that required deductive and rigorous reasoning about mathematical ideas that were not entirely accessible to the five senses (Edwards *et al*., 2000). Whilst, in the second perspective advanced mathematical thinking is considered as involving overcoming the epistemological obstacles together with ways of thinking that are helpful (Harel and Sowder (2005). Finally, in the third perspective Rasmussen and his colleagues (2000, 2005) discussed advanced mathematical thinking in terms of practice which they called "*advancing mathematical activity*" to emphasise the progression and growth of students' reasoning in relation to their previous activity. Despite the different definition and perspectives, there is an acceptance that mathematical thinking is the main goal of mathematics education (Kardage, 2008) which in turn plays an important role in the learning and teaching of mathematics to address students' mathematical learning difficulties. There is quite an extensive study on mathematical thinking such as works by Mason, Burton, and Stacey (1982), Dubinsky (1991), Schoenfeld (1992), Yudariah and Tall (1999), Gray and Tall (2001), Tall (1995, 2004), and Roselainy (2009) to name a few.

In the study of Multivariable Calculus, Roselainy and her colleagues (Roselainy, 2009; Yudariah and Roselainy, 2004; Roselainy, Yudariah, and Mason, 2005, 2007; Roselainy, Yudariah, and Sabariah, 2007) adopted the theoretical foundation of Tall (1995) and Gray *et al*. (1999) and used frameworks from Mason, Burton, and Stacey (1982) and Watson and Mason (1998) to develop the mathematical pedagogy for classroom practice. They highlighted some strategies to support students to empower themselves with their own mathematical thinking powers and help them in constructing new mathematical knowledge and generic skills, particularly, communication, team work, and self-directed learning (Yudariah and Roselainy, 2004). Roselainy and her colleagues had tried to connect explicitly the processes of mathematical thinking such as specializing and generalizing, imagining and expressing, conjecturing and convincing, organizing and characterizing with the different types of mathematical structures such as definitions, facts, theorems, properties, examples, techniques, and proofs, and to the generic skills (Roselainy, Yudariah, and Sabariah, 2007). See Figure 1.

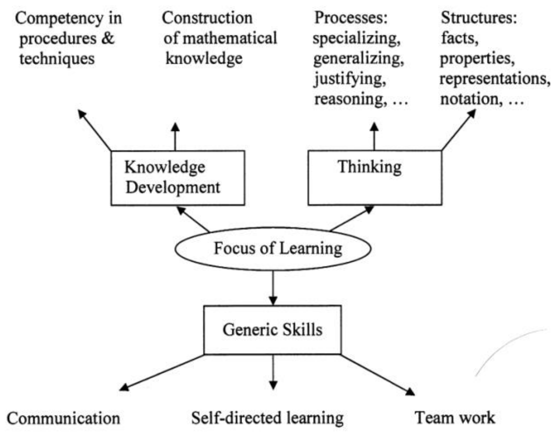

Figure 1 Focus of mathematical learning

Roselainy and her colleagues (Yudariah and Roselainy, 2004; Roselainy, Yudariah, and Mason, 2005) used mathematical themes through specially designed prompts and questions to provide linkages between mathematical ideas, to expose the structures of the mathematics, and to support students' generic skills. Some of the themes that used by them were, *invariance amidst change*, which form the basis for many mathematical theorems and technique, and *doing and undoing*, which can help students identify features or structures that should be the focus of attention. They endeavoured to design '*prompts and questions'* based on Watson and Mason (1998) to draw students' attention to the mathematical processes and structures involved in facilitating their understanding of concepts learnt. Some examples of prompts that they used were: *Give me one or more examples*, *Find a counter-example*, and *Compare examples* (Roselainy, 2009). The questions such as *What is the same*?, *What is different*?, *What can change and what stays the same*?, *What connects the different examples*? and *What happens in general*? were some common questions that they usually used. In this way, students' attention was focused and directed to the prompts and questions in the beginning until students were aware of the questions asked in the class and became increasingly directed over time as they gradually use the prompts and questions themselves (Sabariah, Yudariah, and Roselainy, 2008).

Roselainy, Yudariah, and Sabariah (2007) to achieve the focus of learning, they had chosen active learning, as it would give students the opportunities to be interactive with the subject matter. They considered the following aspects in the implementation of active learning in Multivariable Calculus classroom.

- classroom tasks– by categorizing book as *Illustrations* (using examples with complete solution and explanation) with prompts and questions, *Structured Examples* (using typical examples and then generic examples to lead students towards a generality) with prompts and questions, *Reflection* (asking important ideas and concepts), *Review exercise*, and *Further Exercises*.
- classroom activities– by utilizing quick feedback, small group, working in pairs, students' own examples, assignments, discuss and share, reading and writing.
- encouraging communication– by designing prompts and questions to initiate both written and oral mathematical communication through discussion and sharing of ideas among the students.
- supporting self-directed learning– by creating structured questions to strengthen the students' understanding of mathematical concepts and techniques.
- identifying types of assessment– by incorporating both summative and formative types such as quizzes and tests, quick classroom feedback and written assignments. Figure 2 gives a summary of Roselainy *et al*.'s model for active learning (Roselainy, Yudariah, and Sabariah, 2007).

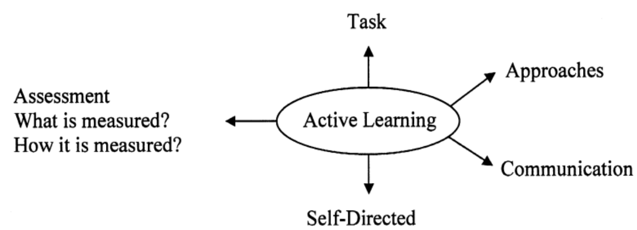

Figure 2 Model of active learning

3.0 METHOD

The study was carried out at Islamic Azad University of Kermanshah (IAUKSH) during the fall semester of 2011. The sample of the study involved a class of 59 first year engineering students enrolled in a Multivariable Calculus. The students comprised of 45 males and 14 females first year undergraduates aged 18 to 20 in Electrical Engineering, Computer Engineering, Civil Engineering, and Mechanical Engineering. The first-named author with more than 8 years experience of teaching Multivariable Calculus course taught this class.

The Multivariable Calculus offered by IAUKSH is a three credit undergraduate course and covers functions of several variables, partial derivatives, multiple integrals, vector functions and vector calculus. The book entitled "*Engineering Mathematics for Independent Learners"* which covers all the above topics written by the second-named author, Sabariah, and Roselainy (2009) was translated to Persian and introduced as a textbook. The instructional design of the book took into consideration students learning on their own and the the contents were organised in a specified manner. In most lessons, the mathematical tasks were designed so that students would experience the mathematical thinking processes themselves and eventually could identify the general class of problems they were working on (Yudariah and Roselainy, 2004).

In this study, we only focused on students' learning and difficulties in the topic of functions of several variables that covered the definition of two-variable functions, the domain and the range, sketching the graph of two-variable functions and also functions of three or more variables. The topic was taught in 3 hours meeting per week including lecture and tutorial sessions over a period of 3 weeks. The tutorial session was combined as part of the lectures, thus each week the meeting consist of two 1 hour and 30 minutes class with a mix of lectures and activities. In a typical class meeting, the instructor used the first 30 minutes of the session to introduce the topic on a particular mathematical concept through lecture and whole class discussion. This was then followed by students working on the structured examples individually and in groups for about 30 minutes. The class ended with "question and answer" session where time is spent discussing the examples, reviewing or addressing difficulties faced by students. Tutorial questions were taken from the textbook and students could discuss these questions in any of the class session. By encouraging the students to talk, to listen, to read, to write and to reflect on their mathematical thinking and problem solving, we sought to enhance students' awareness of their own thinking (Sabariah, Yudariah, and Roselainy, 2008).

For instance, in teaching the definitions of two-variable functions and the domain and range the instructor used two structured examples from the textbook to demonstrate the focus of attention students should be attending to. Table 1 showed an extract of one of the structured example.

Table 1 Example 1(a) - Finding domain, range and sketching a graph

Example 1 (a):	**Questions and Prompts:**
Given $z = 1 - x^2 - y^2$ i. Evaluate f(2,1), f(-4, 3), f(0,-5) and f(u,v). ii. Find the domain and range. iii. Sketch the domain of f	• Which pairs of variables are the input variables? • Which variable is the output variable? • Is there any restriction on the input variables for which the function is defined? • How do you represent the set of all inputs graphically?

For this problem, the following themes and powers (see Table 2) were identified for students to focus on.

Table 2 Themes, powers and mathematical activities of Example 1(a) (from Roselainy, 2009)

Theme: Invariance amidst Change Sub-theme: Range of Change Activities: Specialising and Generalising, Characterising, Expressing	
Problem: Finding the domain of a function	**Focus of Attention**: property of function, values of domain and range, graph of function
Example 1 (a): Given $z = 1 - x^2 - y^2$ i. Evaluate f (2,1), f (-4, 3), f (0,-5) and f (u,v). ii. Find the domain and range. iii. Sketch the domain of f	The *Questions and Prompts* were to direct students' attention to the roles of the independent and dependent variables as well as to the property of the function, z.

Example 1(a) was followed by two more examples, Example 1(b) and 1(c). In Example 1(b), the function in 1(a) was changed by only one aspect of the function to square root function whilst in Example 1(c), the function in 1(b) was inversed (see Table 3). These examples provided to help student in revising the procedure of finding the domain of a square root function and of an inverse function (Roselainy, 2009). The prompts and question were designed to direct students' attention in understanding the importance the various properties of functions in determining its domain.

Table 3 Invariance amidst change (from Roselainy, 2009)

Sub-theme: Range of Change Activities: Specialising and Generalising, Characterising, Expressing	
Problem: Finding the domain of a square root function	**Focus of Attention**: property of function, values of domain and range, graph of function
Example 1(b): Given $z = \sqrt{1 - x^2 - y^2}$ i. Describe and sketch the domain. ii. Determine the range. iii. Write down at least three possible values of $f(x, y)$.	**Questions and Prompts:** Compare 1(a) and 1(b). • What remains the same? • What has changed? • What condition is necessary for the function to be defined? • How do you represent the set of all inputs graphically?
Example 1(c): Given $z = \frac{1}{\sqrt{1-x^2-y^2}}$ i. Describe and sketch the domain. ii. Determine the range. iii. Write down at least three possible values of $f(x, y)$.	**Questions and Prompts:** • What condition is necessary for the function to be defined? • How does the condition affect the input variables? Output variable? • How do you determine the set of input variables? Output variable? • Compare (a), (b) and (c). ○ What is the same? ○ What is different?

The following example (Table 4) provided to help students for moving from a few instances to making conjecture about a wide class of cases (Mason, Burton, and Stacey, 1982). In fact, by using some specific examples and then the students' own examples are tried to help students see the "general in the particular" and also to see the "particular in the general" (Roselainy, 2009).

Table 4 Specialising and generalising (from Roselainy, 2009)

Sub-theme: Range of Change Activities: Specialising and Generalising, Characterising, Expressing	
Example 2: Let $f(x, y) = \sqrt{4 - x^2 - y^2}$ i. Find the domain and range of f. ii. Sketch the graph of the domain	**Questions and Prompts:** Compare Examples 1(a) and 2. • What remains the same? • What has changed? • What was the property of $f(x, y)$ which required the condition $4 - x^2 - y^2 \geq 0$? • What information in Example. 1(a) did you use to solve Example. 2?
(iii) Could you give one example that is like Examples. 1(a) or 2? (iv) Please give another example? (v) Can you give a general example?	

After demonstrating these two examples, the instructor then asked students to work on structured examples from the textbook. The students were asked to work in groups of 3 to 4 people, thus there were different groupings in the class, with some students working in groups of fours and a few in threes. The same teaching strategies were used to teach other subtopics such as sketching the graph of quadric surfaces and the concept of functions of three or more variables.

Data for this study was collected through written assessments such as quiz, test and midterm exam followed by semi-structure interviews with selected students. For the purpose of this study, we only highlighted the problems of the quiz, test and a problem in the midterm exam which were related to the domain, the range, and the graph of two-variable functions.

Students were given the quiz at the end of week 1. The most important goal of the quiz was to identify students' difficulties in finding the domain, range and sketching the domain of two-variable functions. The quiz problem was a part of a question from the structured examples of the textbook that students discussed about in their group during the class session. The quiz problem was as follows:

Find the domain and range of $f(x, y) = \sqrt{64 - 4x^2 - y^2}$. *Sketch the graph of domain.*

The test was conducted at the end of week 3 and used to identify how Roselainy *et al*.'s method can support students in solving non-routine problems. To achieve these goals the following example (Table 5) from the *Illustrations* of the textbook was selected (Yudariah, Sabariah, and Roselainy , 2009, p. 43).

Table 5 An example from the *Illustrations* of the textbook

Example 1.19:	Questions/Prompts:
Sketch the graph of the following functions: i. $f(x, y) = 9 - x^2 - y^2$ ii. $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$	• What are the traces in the coordinate planes? ○ Can you identify the curves? • What is different? How do these traces built up the surfaces?

By changing the variables and constants and adding some new questions, two problems in the test were prepared as follows:

1. *Suppose* $f(y, z) = 9 - y^2 - z^2$.
   a. *Find and sketch the domain.*
   b. *Determine the range.*
   c. *Sketch the graph of function.*
2. *Sketch the graph of* $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$. *Does the graph represent a function?*

The midterm exam was conducted at the end of week 7 and include a problem unfamiliar to the students: *Determine the domain and range of* $f(x, y) = ln\sqrt{1 - x^2 - y^2}$. *Sketch the graph of the domain.*

Several students were selected to answer the semi-structured interviews based on their responses to each written assessments. During the sessions, the reasons of their responses especially their capabilities and difficulties in solving the problems were uncovered. Some common questions in the semi-structured interviews were: what students understand about the domain and range of two-variable functions, what they do to find the domain and the range, and what were their difficulties to solve the problems. To ensure that the data was accurately captured, the researcher audio taped the semi-structured interviews and transcribed the responses immediately after each of the interviews was completed.

Many studies have been carried out about the nature of mathematical errors, their interpretation and the ways of overcoming them (see Radats, 1979; Orton, 1983a, b; Borasi, 1994; Hirst, 2002; Yee and Lam, 2008). We adopted Miles and Huberman's (1994) qualitative analysis method as the main framework in analyzing the data obtained from students' responses. According to Miles and Huberman, a qualitative data analysis consists of three stages: data reduction, data display, and conclusion drawing. In the "data reduction" stage, students' responses to written assessments undergo the process of selecting, focusing, simplifying, abstracting and transforming the data into a certain order. In the next stage, the "data display" stage, the researcher tries to organize and compress the assembly of available information that consents conclusion drawing. Finally, "conclusion drawing and verification" is involved with emerging, and inducing of meanings from the data and testing them for their credibility, their robustness and their validity.

The analysis of students' responses in the assessments given to them involving functions of two variables would provide detailed information concerning the degree of understanding attained and the common difficulties, errors and misconception. Peng and Luo (2009) introduced a framework to analyse students' mathematical errors. The framework includes two separate dimensions as the nature of mathematical error and the phrases of error analysis which are linked together in a complex way. The nature of mathematical error includes four keys as mathematical, logical, strategical, and psychological. There are four keys for the phrases of error analysis, namely, identify, interpret, evaluate, and remediate. We used this framework for students' error analysis; however, the nature of mathematical error and difficulty classification and the way of diagnosing them were changed to the scheme described by Mason (2002) based on mathematical thinking approach. According to Mason (2002), students' mathematical difficulties were divided into difficulties with "concepts", with "techniques", and with "studying mathematics". See Table 6.

Table 6 A framework for mathematical error analysis (adopted from Mason (2002) and Peng and Luo (2009))

Dimension	Analytical categorization	Description
Nature of mathematical errors	**Conceptual**	Technical terms, mathematical concepts, compound concepts and process becoming objects, seeing behind symbols and notation, definitions
	Technical	Algebraic manipulations, confusion of notation, insufficient facility or competence
	Studying Mathematics	Using examples and exercises as routs, not remembering, drawing and reading diagrams
Phrases of error analysis	**Identify**	Knowing the existence of mathematical error
	Interpret	Interpreting the underlying rationality of mathematical error
	Evaluate	Evaluating students' levels of performance according to mathematical error
	Remediate	Presenting teaching strategy to eliminate mathematical error

The data analysis started with the data reduction and data display stages of Miles and Huberman's method for written assessments to identify students' errors in Peng and Luo's framework. Semi-structured interviews of selected students helped us to prepare for interpreting students' errors in Peng and Luo's framework that is the last stage of Miles and Huberman's method. You can see the identifying students' errors based on students' responses to the assessments and interviews in the results section. The interpreting and addressing students' errors followed with the evaluation and diagnosing three types of students' mathematical difficulties based on Mason scheme. In the discussion section, interpret and evaluate stages of students' errors will be discussed. The students' difficulties informed us on their struggle in making sense of the new mathematical ideas and concepts encountered in their learning. Consequently, this information helps us to further modify and improve our strategies based on the mathematical thinking approach. The way of remediate these errors and difficulties will suggest in the conclusion section based on mathematical thinking approach.

■4.0 RESULT

Analysis of students' responses to the quiz problem showed that some students faced difficulty in solving the problem due to various reasons. Although the quiz problem was discussed by students in their groups, students displayed difficulties in solving it. Figure 3 represents a typical student's response, student A illustrating poor algebraic manipulation that had caused him difficulty in finding the domain of $f(x,y)=\sqrt{64-4x^2-y^2}$. By squaring both sides of the equation, student A obtained $64 \geq 4x^2+y^2$ and subsequently deduced $8 \geq 2x+y$ and thus wrote the domain of the function *f* as $D_f=\{(x,y)|\ 2x+y \leq 8\}$ incorrectly. This student not only could not sketch the graph of the domain but also could not find the range and wrote: "*I got stuck"*. Like others who could not find the range, student A said that his lack of understanding of the range of single variable functions was the reason of his difficulty.

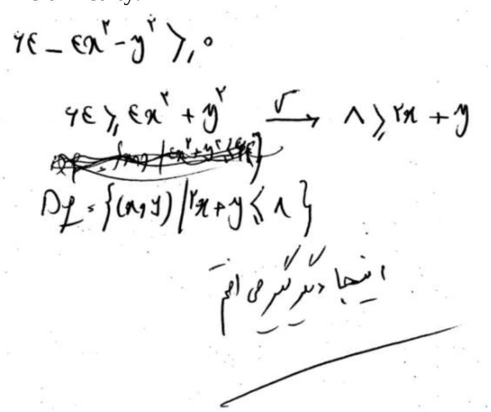

Figure 3 Student A difficulty in algebraic manipulation

Some students sketched the graph of *f* as a circular disc. Figure 4 shows a student B response typical of those who wrote the domain as $D_f=\{(x,y)|\ 4x^2+y^2 \leq 64\}$ and sketched the graph of the domain as a circular disc of radius 8. In the interview, student B explained the reason of his response as "*... because this inequality* [points to $4x^2+y^2 \leq 64$] *identifies a circle area for the domain ...*". However, some students in their interviews noted that solving many problems similar to $f(x,y)=\sqrt{a-x^2-y^2}$ was the reason of sketching the domain of $f(x,y)=\sqrt{64-4x^2-y^2}$ as a circular disc.

Figure 5 represents the similar difficulty faced by student C. The student wrote the domain as $D_f=\{(x,y)|\ 4x^2+y^2 \leq 64\}$ that is the set of points on or within the circle of radius 8 centred at the origin. Based on the graph of the domain, this student wrote the range as $R_f=[-8,8]$. The student had not really understood what this graph represented and had assumed that it was the graph of the function.

Below is an excerpt from student C responses during the interview:

Interviewer: *How did you find the domain as* [points to the student's response]*?*
Student C: ... *By finding the restrictions of the input variables ...*
Interviewer: *Why the graph of the domain is a circular disc?*
Student C: ... *Because the quantity under the square root is nonnegative provided* $64-4x^2-y^2 \geq 0$, *or* $4x^2+y^2 \leq 64$ *that represents a circle area ...*
Interviewer: ... *Well, here you wrote* $R_f=[-8,8]$, *can you explain how you found it?*
Student C: ... *Yeah, if we look at the graph* [points to the circle] *we can see the values on the vertical axis change between -8 and 8 ...*
Interviewer: *This problem was discussed in your group in the class. Why you could not solve it?*
Student C: ... *Sorry ... I forgot the answer ...*

In solving the first problem of the test, some students wrote the domain of $f(y,z)=9-y^2-z^2$ in terms of *x* and *y* as $D_f=\{(x,y)|\ x,y \in R\}$. Figure 6 shows a student D response in finding and sketching the domain of *f*. The student had not only incorrectly identified the domain as $\{(x,y)|\ x,y \in R\}$ but also sketched it in the wrong coordinate plane. During the interview, this student noted the reason of the difficulty as "*... sorry ... these* [points to *x* and *y* in the domain statement] *are y and z ... I just looked at the problem and solved it quickly ...*". The student was not aware of the different symbols used and their role in representing the function. The obstacle faced was due to their inflexibility in handling symbolic representation. Similar difficulties were reported in ... state related references). This student wrote the range as $R_f=(\infty,9]$ correctly; however, many students could not find the range because of their poor understanding of this concept in Basic Calculus.

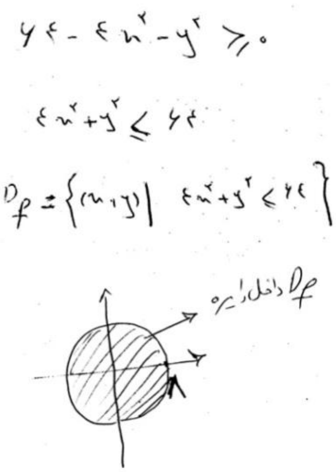

Figure 4 Student B attempt in finding and sketching the domain of $f(x,y)=\sqrt{64-4x^2-y^2}$

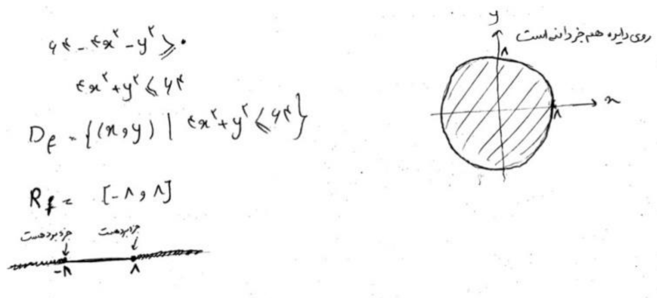

Figure 5 Student C attempt in finding the domain and range of $f(x,y) = \sqrt{64 - 4x^2 - y^2}$

Figure 6 Student D attempt in finding and sketching the domain of $f(y,z) = 9 - y^2 - z^2$

Also, there were few students who sketched the domain of *f* as a circle of radius 3. Figure 7 represents a student E response that wrote the domain as $D_f = \{(x,y) | \, y^2 + z^2 = 9\}$ and sketched it as a circle. This student, like other students who could not sketch the graph of *f* indicated that their difficulty was due to unfamiliarity of the problem. The student's struggle was due to lack of experience in making generalisation, that is, they were not aware of what is general among the specific examples.

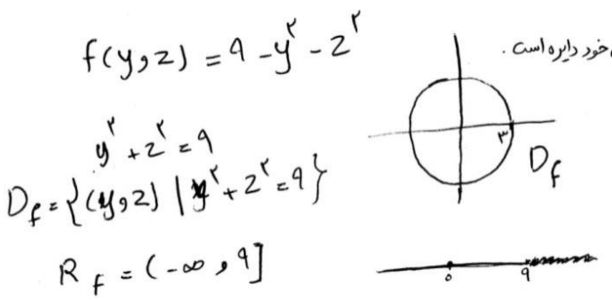

Figure 7 Student E attempt in finding and sketching the domain of $f(y,z) = 9 - y^2 - z^2$

Below is an excerpt of student E responses during the interview:

Interviewer: *Why did you sketch the graph of the domain as a circle?*
Student E: [reads her response] ... *Because this* [points to $y^2 + z^2 = 9$] *is a circle equation ...*
Interviewer: *Well, can you show the domain on the graph?*
Student E: ... [reads the problem] ... *Okay, it is including all points on the circle ...*
Interviewer: *Okay, can you evaluate* $f(0,0)$, *... for this function?*
Student E: ... *The equation is* $f(y,z) = 9 - y^2 - z^2$ *with* $y = 0$ *and* $z = 0$ ... $(0,0) = 9$ ...
Interviewer: ... *As you see we can evaluate the values of f for each* (y,z) *and there are no restrictions for y and z ... so f is defined for all ordered pairs (y, z) ... You didn't sketch the graph of f. What was your difficulty for sketching that?*
Student E: ... *We did not solve any problem like this* [points to $f(y,z) = 9 - y^2 - z^2$] *before ...*

Figure 8 shows the same difficulties for a student F. The student not only wrote the domain as $D_f = \{(y,z) | \, y,z \in R^2, \; y^2 + z^2 = 3^2\}$ and sketched it as a circle but also found the range of *f* based on the graph of domain as $R_f = [-8, 8]$.

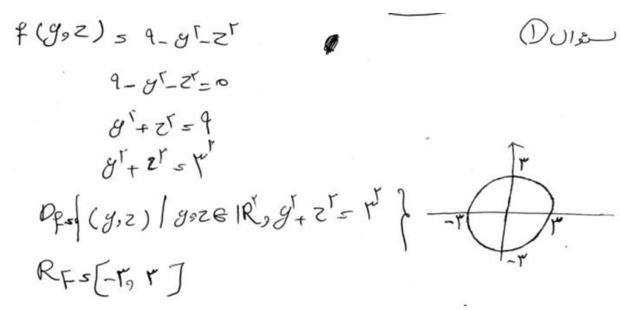

Figure 8 Student F attempt in finding the domain and range of $f(y,z) = 9 - y^2 - z^2$

The student's responses during the interview confirmed that student F was not aware of the properties of the function and had confused this problem with $f(y,z) = \sqrt{9 - y^2 - z^2}$.

Interviewer: *Why did you sketch the graph of the domain as a circle?*
Student F: ... *Oh, that's right ... sorry, I thought f is a square root function ...*
Interviewer: ... *Now, let me see here ... Okay, how did you find the range?*
Student F: ... *Well, by finding the values on the vertical axis…*

In solving the second problem in the test, majority of students sketched the graph of $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$ correctly. Most of these students in the interviews explained that they could sketch the graph of surface by memorizing the six common types of quadric surfaces. Figure 9 represents a student G response where the student had sketched the graph correctly without finding the traces or finding the intersecting points with the coordinate axes. Some students wrote the graph did not represent a function but no reasons were given; however, other students believed that the graph represents a function.

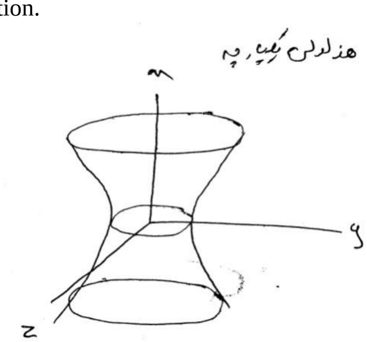

Figure 9 Student G attempt in sketching the graph of $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$

Below is an excerpt of student G responses during the interview:

Interviewer: *How did you sketch the graph of the surface?*
Student G: ... *Because this* [points to the equation] *is a hyperboloid of one sheet ... I knew the graph of it* ...
Interviewer: *Does the graph represent a function?*
Student G: ... *No, because hyperboloid of one sheet is not a function* ...

Few students did not sketch the graph $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$ because they thought it is a three-variable function. They wrote the domain as $D_f = \{(x, y, z) |\, x, y, z \in R\}$ and the range as $R_f = \{w | w \in R\}$; however, finding the domain and the range did not ask in the problem. Figure 10 shows a student H response.

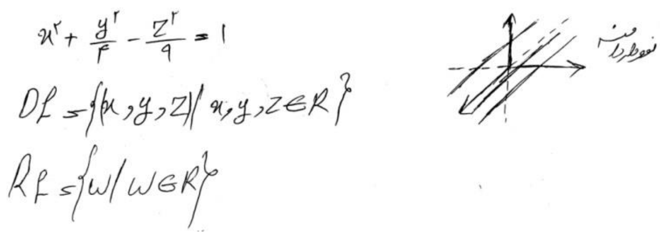

Figure 10 Student H error in finding the domain and range of $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$

Below is an excerpt of student H responses during the interview:

Interviewer: *Why did you find the domain and range?*
Student H: [reads the problem and his response] ... *I don't know ... I confused ...*
Interviewer: *What did you do to find the domain and the range?*
Student H: ... *Look at the input variables for finding the domain and the output variable for finding the range ...*
Interviewer: *Okay, can you say what input variables in this problem are?*
Student H: [reads his response] ... [points his response] ... *x, y and z ...*
Interviewer: *Well, what is the output variable?*
Student H: ... *I think w, because a three-variable function is defined as* $= f(x, y, z)$ ...
Interviewer: ... *So where is w?*
Student H: ... *Let me see here ... I don't know...*
Interviewer: ... *Why you did not sketch the graph?*
Student H: ... *Because we cannot sketch the graph of three-variable functions ...*

In the midterm exam, some students wrote the domain of $f(x, y) = ln\sqrt{1 - x^2 - y^2}$ as $D_f = \{(x, y) |\, x^2 + y^2 \leq 1\}$. Students' responses to the interviews clarified that most of them did not consider that the input variable for natural logarithm function cannot be 0. Figure 11 represents a student J response typical of those who wrote the domain as $D_f = \{(x, y) |\, x^2 + y^2 \leq 1\}$.

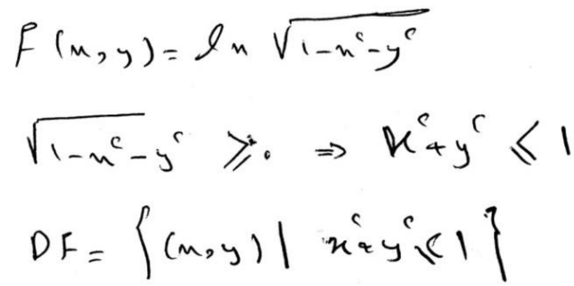

Figure 11 Student J attempt in finding the domain of $f(x, y) = ln\sqrt{1 - x^2 - y^2}$

Below is an excerpt of student J response to the interview:

Interviewer: *What did you do when you wanted to find the domain?*
Student J: ... *Look at the properties of the function and then by finding the restrictions of x and y determine the values that define the function ...*
Interviewer: *Okay, what are the properties of this function?*
Student J: … the *natural logarithm and square root ....*
Interviewer: *Are there any restrictions for x and y?*
Student J: *Yeah ... because of ln then* $\sqrt{1 - x^2 - y^2} \geq 0$ *, or* $x^2 + y^2 \leq 1$ ...
Interviewer: *Why you did not find the range?*
Student J: ... *Sorry, I always have difficulty in finding the range ... there is no routine way to find the range ...*

Few students wrote the range as $R_f = [0, 1]$ based on the graph of domain. Figure 12 shows a student K response that wrote the domain as $D_f = \{(x, y) |\, x^2 + y^2 \leq 1\}$ correctly and sketched it as a circular disc of radius 1. The student found the range as $R_f = [0, 1]$ based on the graph of the domain.

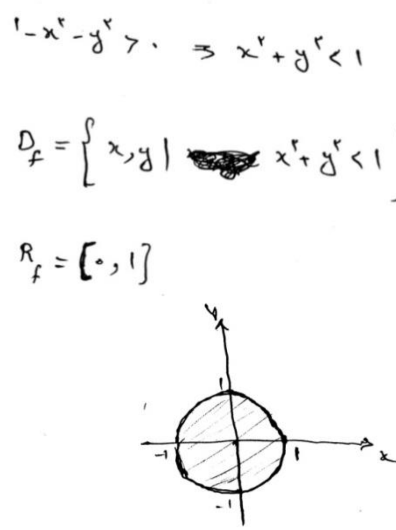

Figure 12 Student K attempt in finding the range of $f(x, y) = ln\sqrt{1 - x^2 - y^2}$

5.0 DISCUSSION

Results of the study showed that many students struggled as they encounter new mathematical ideas and concepts in the learning of two-variable functions through mathematical thinking approach. During the quiz session, three students were absent and two students could not give any answer at all. Solving problems done by particular students in the group and students' resistance during the discussion were some of the reasons cited by them for their difficulties in solving the quiz problem although they had earlier discussed similar problem in their group. Seventeen students were unable to obtain the domain correctly. Some students' difficulties were due to poor mastery of algebraic manipulation; and nine of seventeen students obtained the domain as $D_f = \{(x, y)|\ 4x^2 + y^2 \leq 64\}$. The errors made by students were either technical or conceptual. Eighteen students did not sketch the graph of domain. Nine of these students sketched the graph of the domain as a circular disc of radius 8. Based on the interviews, four students said that they did not know the inequality $64 - 4x^2 - y^2 \geq 0$ represents an ellipse; however, five students noted that solving many problems similar to $f(x, y) = \sqrt{a - x^2 - y^2}$ coerced them to over generalised. Overall, the students' main difficulties and obstacles were with related to the concept or with studying mathematics. Based on students' responses, finding the range was the difficult part of the problem and twenty-five students did not obtain the correct answer. During the interviews, most of these students believed that their difficulties were due to poor understanding of the range of single variable functions. Four students obtained the range of *f* based on the graph of the domain. The students' responses to the interviews revealed that their difficulty may be related to the negative effect of students' previous experience on finding the range of single variable function. The errors made by students on finding the range therefore appeared to be conceptual.

The quiz problem was part of a question from the structured examples in the textbook where students had solved together as a group in the class. The structured example in the textbook (Table 7) was given as follows (Yudariah, Sabariah, and Roselainy , 2009, p. 8):

Looking closely at the three parts of the above question in the textbook we can see that comparing part (b) and part (c) would help students to understand the differences between the polynomials functions and square root functions in finding the domain. May be the similarity and differences between the equations of the circle and ellipse could be made more explicit by changing the function in part (b) to $f(x, y) = \sqrt{64 - x^2 - y^2}$ first. Furthermore, students were not accustomed in using the prompts and questions in solving the different problems. They prefer to solve the problems according to their idiosyncrasies attributed from previous experience.

In the test, one student was absent and two students could not respond at all. In finding the domain of $f(y, z) = 9 - y^2 - z^2$, twelve students wrote the domain in terms of *x* and *y* as $D_f = \{(x, y)|\ x, y \in R\}$. Seventeen students could not sketch the domain correctly of which four of these students sketched the graph of domain as circular disc. Most of these students in the interviews noted that they had confused this problem with $f(y, z) = 9 - x^2 - y^2$. Using and solving many problems in terms of *x* and *y* may burden the students when encountered with a problem such as $f(y, z) = 9 - y^2 - z^2$, where they have the tendency to over generalised and found its domain in terms of *x* and *y*. Students' difficulties in finding and sketching the domain were found to be mainly conceptual or related with studying mathematics. In all, twenty-four students did not obtain the range due to the students' poor prior knowledge. Three students obtained the range of *f* based on the graph of the domain. The errors appeared were conceptual. In the remaining part of the problem there were more errors displayed by students. The most striking errors made by students concerned sketching the graph of *f*. Thirty-seven students could not sketch the graph of *f* correctly; although three of these students managed to sketch the graph correctly albeit in the wrong orientation. Students' responses during the interviews revealed that the difficulty were due to their inflexibility in handling different symbols. The errors made by students were considered conceptual.

Table 7 A structured example in the textbook

Question 1 (a) Given $f(x, y) = 1 - 4x - y$. i. Evaluate *f* (2,1), *f* (-4, 3), *f* (0,-5) and *f* (*u*,*v*). ii. Find the domain and the range. iii. Sketch the domain of *f*	**Questions/Prompts:** • Which pairs of variables are the input variables? • Which variable is the output variable? • Is there any restriction on the input variables? • How do you represent the set of all inputs graphically?
(b) Suppose $f(x, y) = 1 - 4x^2 - y^2$. i. Write down at least three possible values for *f* (*x*,*y*). ii. Determine the domain and the range. iii. Sketch the domain.	**Questions/Prompts:** • Which variables do you look at when you want to find the domain? The range? • Is there any restriction on the variables?
(c) Let $f(x, y) = \sqrt{64 - 4x^2 - y^2}$. i. Describe and sketch the domain. ii. Determine the range. Write down at least three possible values for *f* (*x*,*y*).	**Questions/ Prompts:** Compare 1(b) and 1(c). • What remains the same? • What has changed? • What condition is necessary for the function to be defined?

In solving the second problem of the test, thirty-eight students could sketch the graph of $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$. Majority of these students sketched the graph without using the traces. Most of these students in the interviews explained how they could sketch the graph by memorizing six common types of quadric surfaces. Fifteen students without noting any reasons wrote the graph did not represent a function; however, twenty-three students wrote the graph represents a function. Three students did not sketch the graph of surface because they thought $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$ is a three-variable function. These students did not know the two different representations of two-variable functions as $z = f(x, y)$ and $f(x, y, z) = 0$. They wrote the domain as $D_f = \{(x, y, z)|\ x, y, z \in R\}$ and the range as $R_f = \{w|\ w \in R\}$ but did not use the information further. These difficulties could arise and may be related to the example (see Table V) in the textbook where the test problems were selected from. In the textbook example, the authors asked "*Sketch the graph of the following functions*" and the use of the word function appeared problematic.

Overall, students' difficulties were with the concepts or with studying mathematics.

In finding the domain of $f(x,y) = ln\sqrt{1-x^2-y^2}$ from the midterm exam, twenty-three students obtained the domain of f is $\{(x,y)|\ x^2+y^2 \leq 1\}$ incorrectly. Most of these students because of their poor prior knowledge did not know the properties of the natural logarithm and they did not consider that the input variable for natural logarithm function cannot be 0. It is a cause of concern that thirty-eight students were unable to obtain the range. Because of the negative effect of previous mathematical experience two students found the range of f based on the graph of the domain. The difficulties were considered conceptual. The composite function that combined the square root function and the natural logarithm function as inner and outer functions made this problem more challenging to most of the students.

6.0 CONCLUSION

This study investigated students' difficulties and obstacles in the learning of two-variable functions through mathematical thinking approach. The findings indicated that though Roselainy *et al*.'s method help in making the mathematical thinking processes explicit learning it also highlighted students' struggle as they encounter new mathematical ideas and concepts. In particular, students displayed various difficulties and obstacles when they encounter unfamiliar problems. Using Mason's error classification (2002), the students' difficulties were mainly considered conceptual in nature and few were technical or related to studying mathematics

Students' poor algebraic manipulations and students' idiosyncrasy attributed from previous mathematical construction were difficulties that students displayed in finding the domain. Solving more problems and considering the different class of problems can help students in the learning of two-variable functions. The use of different prompts and questions had enhanced students' awareness of their own thinking in making sense of the concept and help them to recognise the cause of their difficulties.

The findings indicated that one of the greatest students' difficulties was in finding the range of two-variable functions. According to students' responses, the difficulty was due to poor prior knowledge. Providing different examples and questions about the range of single and two-variable functions and designing appropriate prompts and questions to guide students in making connections between them can help students in the learning of the range. Using these strategies also can reduce the negative effect of previous mathematical construction in finding of the range of two-variable functions observed among few students.

Many students sketched the graph of surface by memorizing the formulas and graphs of six common types of quadric surfaces and most of them had difficulties in sketching the graph of unfamiliar problems. Restricted mental images of two-variable functions could be the reasons that had caused difficulties when students are faced with examples slightly beyond their experiences. The errors made by students in sketching the graph were conceptual or related to studying mathematics. It seems that students need more support in sketching the graph of two-variable functions. The prompts and questions in Roselainy *et al*.'s method were more focused on invoking students' own mathematical powers in making sense of the mathematical ideas, the meaning of the concepts and the symbols that are from the symbolic world of mathematics. Although, there were some efforts in introducing sketching graphs in the embodied world of mathematics, it appears insufficient. Perhaps the use of other approaches such as using computer facilities and interactive software can help students in visualising the regions and surfaces and in the interpretation of their graphs.

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