THE PERFORMANCE OF MIXED EWMA-CUSUM CONTROL CHARTS BASED ON MEDIAN-BASED ESTIMATORS UNDER NON-NORMALITY

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Graphical Abstract

Abstract

Exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) charts have been regularly used to monitor small process mean shifts. More recently, a mixture of EWMA and CUSUM charts known as mixed EWMA-CUSUM (MEC) control chart has been introduced for better small shift detection. However, like its predecessor, the MEC chart requires the normality assumption to ensure optimal performances. In the presence of outliers, which is the cause of non-normality, the parameters of the chart may be overestimated, leading to an unreliable monitoring process. To mitigate this problem, this paper employed median-based estimators namely, the median and modified one-step \textit{M} estimator (MOM), to control the location parameter via the MEC control chart. In this study, the performance of robust MEC charts for Phase II monitoring of location was compared with the standard MEC chart that is based on the sample mean. The performance of the robust MEC charts in terms of the average run length (ARL) on various \textit{g} and \textit{h} distributions clearly shows that a robust MEC chart based on the MOM estimator performs well regardless of the distributional shapes.

Keywords: Non-normal, EWMA chart, CUSUM chart, Mixed EWMA-CUSUM chart, robust

Abstrak

Carta purata bergerak berpemberat eksponen (EWMA) dan carta hasil tambah longgokan (CUSUM) sering digunakan untuk memantau anjakan kecil min proses. Terkini, campuran carta EWMA dan CUSUM yang dikenali sebagai carta kawalan campuran EWMA-CUSUM (MEC) telah diperkenalkan bagi meningkatkan lagi prestasi carta. Namun, seperti pendahulunya, carta MEC memerlukan andaian normal untuk memastikan prestasi yang optimum. Apabila terdapat titik terpencil, yang merupakan punca ketidaknormalan, parameter carta mungkin dianggarkan lebih tinggi yang boleh menyebabkan ketidakbolehpercayaa pada proses pemantauan. Bagi mengurangkan masalah ini, kertas kerja ini menggunakan penganggar berasaskan median iaitu median dan penganggar-M satu langkah terubah suai (MOM) untuk mengawal parameter lokasi melalui carta kawalan MEC. Dalam kajian ini, prestasi carta-carta kawalan MEC teguh pada pemantauan Fasa II bagi lokasi dibandingkan dengan carta MEC piawai.
1.0 INTRODUCTION

Generally, a control chart is used to observe and conclude whether a process is in statistical control or otherwise. Ideally, the control chart shall signal as soon as possible when the process shifts into an out-of-control state. Shewhart [1] developed a control chart called the Shewhart control chart, which can be effectively used to monitor processes with large shifts. This chart is based on the present information of the process and thus, it is categorized as a memoryless control chart. Due to this feature, the Shewhart chart is less sensitive in monitoring small process shifts [2, 3, 4, 5, 6].

To improve Shewhart’s performance in monitoring small to moderate shifts, the synthetic chart was introduced [7]. Yet, it fails to outperform the CUSUM and EWMA charts when the change in the process is relatively small [8]. This is because the synthetic chart also neglects past information, just like the Shewhart chart.

In contrast, the CUSUM and EWMA charts, which are categorized as memory-type control charts, utilize past and current information of the data in the process. This feature makes the CUSUM and EWMA charts superior to the memoryless charts, i.e., the Shewhart and synthetic charts, in detecting small and moderate shifts [9, 10, 11, 12]. Recently, to improve the performance of the memory-type control charts, Abbas et al. [13] proposed to combine the EWMA and CUSUM structures into one new chart. This new chart is known as the MEC control chart.

It is well known that the standard estimators, i.e., the sample mean and standard deviation, are sensitive to outliers and yet, they define the construction of the abovementioned memoryless and memory-type control charts. As such, the performance of the charts may deteriorate due to the existence of outliers. Several researchers have claimed that control charts constructed using robust estimators have performed well under non-normality [14, 15, 16, 17]. Castagliola [18], for example, introduced the median chart via the EWMA control structure. Meanwhile, the study by Abdul Rahman et al. [19] explored the effects of utilizing robust location estimators, specifically the median and Hodges-Lehmann, on the performance of the CUSUM chart. In another robust study by Abdul-Rahman et al. [20], the researchers employed robust location and scale estimators namely, an automatic trimmed mean and median absolute deviation about the median (MADn), in Phase I of the EWMA chart. These robust control charts showed an improvement in the CUSUM and EWMA charts’ performances, particularly in detecting out-of-control status when the data set is contaminated.

More recently, Abdul-Rahman [21] proposed to control the location parameter via the median based estimators, i.e., MOM and its winsorized version, via the synthetic charting structure under non-normality, specifically using the $g$-and-$h$ distributions. The finding indicates good in-control robustness when the median based synthetic charts were designed for moderate and large shifts. Yet, due to the characteristic of the memoryless chart which only focuses on the present process information, the study claimed a lack of in-control robustness when the synthetic chart was designed for a small shift.

To mitigate this problem, this study employed median based estimators, i.e., the median and MOM, which possess the highest breakdown point, to replace the mean in constructing one of the memory-type charts, i.e., the MEC chart. The following sections explain in detail the structure of the MEC chart, together with the CUSUM and EWMA charts.

2.0 METHODOLOGY

2.1 Description of Cumulative Sum Control Chart

Introduced by Page [22], the CUSUM chart displays the cumulative data points of the present and previous samples in the process. Measurements of the samples are taken at a specified time and used to compute the CUSUM statistics, $C_i$ and $C_{i-1}$, which are defined as follows:

$$C_i^+ = \max \left[ 0, \left( \theta_i - \theta_0 \right) - K_0 + C_{i-1}^+ \right],$$

for $i = 1, 2, ..., m;$. \hfill (1)

and
where \( \hat{\theta} \) is the location estimator and \( K \) is the reference value which can be adjusted to reflect the desired sensitivity in detecting shift in the process.

The initial values of the statistics, \( C_i^+ \) and \( C_i^- \), are usually set equal to the target value, \( \theta_0 \). The standardized CUSUM control chart parameters are \( K_{\hat{\theta}} = k \times \sigma_{\hat{\theta}} \) and \( H_{\hat{\theta}} = h \times \sigma_{\hat{\theta}} \) where \( k \) and \( h \) are constants that correspond to a specific in-control average run length (ARL). Henceforth, the in-control ARL is denoted as \( ARL_0 \). Typically, \( k \) is set to \( \frac{z_{\alpha}}{\sqrt{2}} \). This approach makes the CUSUM more sensitive to the process shifts when the shift is relatively small [23].

### 2.2 Description of Exponentially Weighted Moving Average Control Chart

The idea of an EWMA control chart was originated by Roberts [24]. Each charting statistic in the EWMA chart signifies the weighted average of the present and all past subgroup values, giving more weight to current process data and less weight as the data get older. The EWMA charting statistic is defined as:

\[
Z_i = \lambda \hat{\theta}_i + (1 - \lambda)Z_{i-1}, \quad \text{for } i = 1, 2, ..., m, \tag{3}
\]

where \( \lambda \) is a smoothing constant that can take any value between 0 and 1. The starting value, \( Z_1 \), is typically set equal to the target mean value, \( \theta_0 \). The EWMA statistic, \( Z_i \), is plotted against the upper control limit (UCL) and lower control limit (LCL) which are defined as follows:

\[
UCL_i = \theta_0 + L_{\hat{\theta}} \sqrt{Var(\hat{\theta}) \frac{1}{2-\lambda} (1 - (1 - \lambda)^{2i})}, \tag{4}
\]

and

\[
LCL_i = \theta_0 - L_{\hat{\theta}} \sqrt{Var(\hat{\theta}) \frac{1}{2-\lambda} (1 - (1 - \lambda)^{2i})}, \tag{5}
\]

where \( L_{\hat{\theta}} \) is a positive coefficient value. This coefficient is usually set at a value that yields the pre-determined \( ARL_0 \).

Abbas et al. [13] made one of the most recent contributions where the researchers combined the EWMA and CUSUM charts in creating the mixed EWMA-CUSUM (MEC) chart.

### 2.3 Description of Mixed EWMA-CUSUM Control Chart

The salient features of CUSUM and EWMA are combined in the MEC chart to enhance small shift detection [13]. The MEC two plotting statistics are given as:

\[
MEC_i^+ = \max \left[ 0, (Z_i - \theta_0) - K_{\hat{\theta}} + MEC_{i-1}^+ \right], \quad \text{for } i = 1, 2, ..., m, \tag{6}
\]

and

\[
MEC_i^- = \min \left[ 0, (Z_i - \theta_0) + K_{\hat{\theta}} + MEC_{i-1}^- \right], \quad \text{for } i = 1, 2, ..., m, \tag{7}
\]

where \( Z_i \) denotes the EWMA statistic in Equation (3), \( i \) represents the sample number until \( m \) subgroups, and \( MEC_i^+ \) and \( MEC_i^- \) are the upper and lower MEC charting statistics, respectively. Both statistics are initially set to 0. Meanwhile, \( K_{\hat{\theta}} \) is the time-varying reference value in the MEC chart. In Equation (8), the value of \( \lambda \in (0, 1) \) and the initial value of \( Z_i \) is usually equal to the target mean value (\( Z_0 = \theta_0 \)). The variance of \( Z_i \), which was used in the computation of the parameters of the chart, is given as follows:

\[
Var(Z_i) = \sigma_{\hat{\theta}}^2 \left[ \frac{1}{2-\lambda} \frac{1}{1 - (1 - \lambda)^{2i}} \right]. \tag{8}
\]

There are two standardized parameters, \( K_{\hat{\theta}} = k \times \sqrt{Var(Z_i)} \) and \( H_{\hat{\theta}} = h \times \sqrt{Var(Z_i)} \), where the notation \( H_{\hat{\theta}} \) represents the control limit, \( k \) and \( h \) are the constants comparable to the one utilized in the standard CUSUM chart, which is the reference value and decision limit, respectively. The values of \( k \) and \( h \) are set to achieve the pre-determined \( ARL_0 \). When \( i \) in the Equation (8) approaching infinity (\( i \to \infty \)), \( Var(Z_i) = \sigma_{\hat{\theta}}^2 \frac{1}{2-\lambda} \), the two quantities become \( K_{\hat{\theta}} = k \times \sigma_{\hat{\theta}} \sqrt{\frac{1}{2-\lambda}} \), and \( H_{\hat{\theta}} = h \times \sigma_{\hat{\theta}} \sqrt{\frac{1}{2-\lambda}} \).

In utilizing the MEC chart for detecting an out-of-control process, both of the charting statistics, i.e., \( MEC_i^+ \) and \( MEC_i^- \) are plotted against the control limit, \( H_{\hat{\theta}} \). The process is said to be in statistical control if the two plotting statistics are randomly scattered between 0 and \( H_{\hat{\theta}} \). If either of the charting statistics exceeds \( H_{\hat{\theta}} \), the process is said to be out-of-control. Here, choices of \( h \), the amount of shift, \( \delta \), and \( \lambda \) paired with a fixed value of \( k \) would give practitioners the pre-determined \( ARL_0 \) [13].

#### 2.3.1 Robust Location Estimators

In this study, \( \hat{\theta}_i \) in Equation (3) which later be used in computing the MEC charting statistics as demonstrated in Equation (6) and (7) were computed using two robust estimators namely, the median and MOM. The sample median is computed as follows:

\[
\hat{\theta}_i = \begin{cases} 
\frac{1}{n} \sum_{i=1}^{n} X_i, & \text{if } n \text{ is odd} \\
\frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^{n} X_i + \frac{1}{n} \sum_{i=1}^{n} X_{n+1} \right), & \text{if } n \text{ is even},
\end{cases} \tag{9}
\]

where \( n \) is the sample size. Meanwhile, the MOM is computed as follows:

\[
\hat{\theta}_i = \frac{\sum_{i=1}^{n} X_i}{n - i + 1}, \tag{10}
\]

where \( X(i) \) is the ordered observation;
\( i_1 = \) number of observations \( X_i \) such that \( (X_i - M) < -K(MAD_n) \);
\( i_2 = \) number of observations \( X_i \) such that \( (X_i - M) > K(MAD_n) \);
with \( M = \) median, \( MAD_n = 1.4826 \text{med}_i |x_i - \text{med}_i x_j| \) and \( K = 2.24 \).

### 3.0 RESULTS AND DISCUSSION

#### 3.1 Case I: MEC Control Charts with Known Parameters

A Monte Carlo simulation analysis was carried out to assess the performance of the robust MEC charts based on the ARL. The ARLO assessed the in-control robustness of the MEC charts. Meanwhile, the true out-of-control condition was assessed via the out-of-control ARL (henceforth denoted as ARLi). A good control chart would have a reasonably large ARLO when the process is in-control. Meanwhile, the ARLi is expected to be as small as possible when the process is out-of-control [5].

In this study, 10,000 datasets were generated using SAS 9.4 to compute the ARL where several variables were manipulated to mimic the most frequently encountered conditions in real life.

The \( g \) and \( h \) distribution was employed to manipulate the shapes of the distributions. Each distribution was subsequently paired with two different sample sizes, \( (n = 5, 9) \) and various amounts of shift, \( (\phi = 0, 0.25, 0.5, 0.75, 1, 1.5, 2, 3) \) to test the strengths and weaknesses of the proposed robust MEC charts against the standard MEC chart.

To generate data using the \( g \) and \( h \) distributions, it is necessary to follow the subsequent steps:

i. Generate random variables following standard normal distribution, \( Z_{ij} \sim N(0,1) \).

ii. Transform the standard normal variables into random variables using an equation as follows:

\[
X_{ij} = \begin{cases} 
\left( \frac{\exp(g Z_{ij}) - 1}{g} \right)^{\frac{\exp\left(h Z_{ij}^2/2\right)}{2 \exp\left(h Z_{ij}^2/2\right)}} & , \quad g \neq 0 \\
Z_{ij} \exp\left(h Z_{ij}^2/2\right) & , \quad g = 0.
\end{cases}
\]

The parameters \( g \) and \( h \) are responsible for controlling the skewness and kurtosis, respectively. When \( g = 0 \) and \( h = 0 \), \( X_{ij} = Z \) represents a standard normal distribution. As \( h \) gets larger, the tails of the distribution become heavier. The same goes for \( g \), which controls the skewness. In this study, four different \( g \) and \( h \) distributions were generated as displayed in Table 1.

#### 3.1.1 Simulation Outcomes

In this study, the charting constants for the MEC charts, were derived for \( k = 0.5 \) and \( \lambda = 0.13 \) with the pre-determined ARL0 set at 370 under \( g = 0 \), \( h = 0 \) distribution, which is the standard normal distribution. The charting constants for different \( n \) are listed in Table 2.

<table>
<thead>
<tr>
<th>( n )</th>
<th>MEC-( \bar{X} )</th>
<th>MEC-( \bar{X} )</th>
<th>MEC-MOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>28.02</td>
<td>28.30</td>
<td>28.15</td>
</tr>
<tr>
<td>9</td>
<td>27.85</td>
<td>28.13</td>
<td>28.08</td>
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### Table 1

<table>
<thead>
<tr>
<th>( (g,h) )</th>
<th>Description</th>
</tr>
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<tr>
<td>(0.0)</td>
<td>Normal</td>
</tr>
<tr>
<td>(0.0, 0.5)</td>
<td>Symmetric heavy tail</td>
</tr>
<tr>
<td>(0.5, 0.0)</td>
<td>Skewed normal tail</td>
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<tr>
<td>(0.5, 0.5)</td>
<td>Skewed heavy tail</td>
</tr>
</tbody>
</table>

In this study, two median based MEC charts were constructed via the usage of the median and MOM estimators. Respectively, these robust charts are denoted as the MEC-\( \bar{X} \) and MEC-MOM charts whereby their performances were compared to the standard chart, henceforth denoted as the MEC-\( \bar{X} \) chart, based on the ARL. The results are displayed in Table 3.

Focusing on the normal distribution, when \( g = 0 \) and \( h = 0 \), all charts yield ARLO = 370 as they were initially designed in this study. In terms of the shift detection, all charts perform equally as shown by the ARLi in Table 3. Moreover, as \( n \) increases, the value of ARLi decreases for all charts, suggesting improvement in detecting shifts.

When \( g = 0 \) and \( h = 0.5 \), where the distribution is symmetric heavy-tailed, the in-control robustness of the MEC-\( \bar{X} \) and MEC-MOM charts remain unaffected since the values of the ARLO are not much different than the pre-determined value of 370. On the other hand, the standard MEC-\( \bar{X} \) chart produces ARLO values that are significantly greater than 370, regardless of the sample sizes examined, suggesting a lack of in-control robustness under this particular data scenario. In terms of the shift detection, the ARLi values indicate that all charts have similar performances especially when the shift sizes are getting larger.

It is noted that the in-control performances of the MEC-\( \bar{X} \) chart and MEC-\( \bar{X} \) chart are unaffected when the underlying process data follow skewed normal-tailed distribution, i.e., \( g = 0.5 \) and \( h = 0 \). Table 3 shows that all charts produced ARLO = 370 when \( n = 5 \) and \( n = 9 \). The MEC-MOM chart is observed with an improved ARLO as \( n \) increases. In terms of the shift detection capability, all three MEC charts perform similarly to the normality data scenario.

Finally, the performances of all charts are observed under extreme data conditions, which is \( g = 0.5, h = 0.5 \), i.e., skewed with the heavy tail distribution. Under this data scenario, the ARLO of the MEC-\( \bar{X} \) chart is highly affected when compared to the robust MEC-\( \bar{X} \) and MEC-MOM charts. The values in bold in Table 3 are much larger than the pre-determined value of 370. When \( n \) increases, the robust MEC-MOM chart...
remains unaffected, that is, the values of the ARL₀ are not much different than the pre-determined value of 370. The ARL₀ values for the robust MEC-\( \bar{X} \) deviate far from the pre-determined 370 when \( n = 5 \). As such, this standard MEC chart is no longer reliable for monitoring changes in the process despite showing comparable out-of-control performances with the median based MEC charts. Conversely, both median based MEC charts show good in-control robustness and fast detection when \( \delta > 0 \), judging by the ARL₀ that is close to 370 and relatively small ARL₁ values as the size of shifts increases, respectively. The finding generally indicates that the in-control performance of standard MEC-\( \bar{X} \) chart is highly affected under heavy-tailed distribution but relatively unaffected otherwise. In contrast, both robust MEC-\( \bar{X} \) and MEC-MOM charts perform consistently well in terms of the in-control performance regardless of distributions. Furthermore, the robust MEC-MOM chart performs better under heavy tail distribution regarding the in-control performance while still maintaining its shift detection capability across the normal and non-normal data scenarios.

<table>
<thead>
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<th>(g,h)</th>
<th>n</th>
<th>Methods</th>
<th>0</th>
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<th>0.75</th>
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<td>6.976</td>
<td>5.240</td>
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<td>3.064</td>
</tr>
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</table>

3.2 Case II: MEC Control Charts with Unknown Parameters

When the in-control parameters of the process are unknown, they have to be estimated based on the historical data in Phase I. Similar to the Case I, the performance of the MEC-MOM, MEC-\( \bar{X} \) and MEC-\( \bar{X} \) charts are assessed and compared using the ARL. The simulation process in Case II follows similar procedures as in Case I in computing the ARL (Phase II). An extra step was included prior to that to estimate parameters of the process based on a total subgroup of \( m = 50 \) with \( n = 5 \), \( g = 0 \) and \( h = 0 \) (Phase I).

Specifically, Phase I involved two series of simulation procedures. The first series was for determining the standard deviation of the sampling distribution of the location estimator, \( \sigma_0 \), based on \( 10^6 \) iterations. The second series involves \( 10,000 \) trials of 50 in-control Phase I with sample size \( n \) to estimate the process mean (\( \bar{X} \)).
Let the Phase I data be represented by $X_{ij} = \{X_{i1}, ..., X_{in}\}$ where $j = 1, 2, ..., m$. We assume that $X_{ij}$ to be independent and identically distributed (i.i.d) following an unknown distribution $W$ which has mean $\theta_0$ and standard deviation $\sigma_0$. $X_{ij} \sim W(\theta_0, \sigma_0)$. The $\theta_0$ was estimated using the mean of $\tilde{\theta}$, given by:

$$\theta_0 = \frac{\sum_{i=1}^{m} \tilde{\theta}_i}{m}$$  \hfill (12)

The charting constants for different sample sizes ($n$) are listed in Table 4.

### Table 4: Charting constants for different sample sizes ($n$) when process parameters are unknown

<table>
<thead>
<tr>
<th>$n$</th>
<th>MEC-$\bar{X}$</th>
<th>MEC-$\bar{X}$</th>
<th>MEC-MOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>36.61</td>
<td>37.00</td>
<td>36.74</td>
</tr>
<tr>
<td>9</td>
<td>37.00</td>
<td>37.78</td>
<td>36.87</td>
</tr>
</tbody>
</table>

#### 3.2.1 Simulation Outcomes

The ARL results are displayed in Table 4. Focusing on the normal distribution, when $g = 0$, $h = 0$, all control charts yield $ARL_0 = 370$ as they were initially designed in this study. In terms of the shift detection, all charts perform equally. Moreover, with an increase in the sample size, the value of the $ARL_1$ decreases for all charts, suggesting an improvement in detecting shifts.

Similar to Case I, even with deviation from the normality assumption, for example, when $g = 0$, $h = 0.5$, the in-control performance of the robust MEC charts remains unaffected. This is supported by the values of the $ARL_0$ in Table 5 which are very close to 370. On the other hand, the $ARL_0$ for the MEC-$\bar{X}$ chart is significantly greater than 370 regardless of the sample sizes examined. In terms of the shift detection, the $ARL_1$ value indicates that all charts have similar performances in out-of-control status.

It is noted that the in-control performances of both median based control charts, i.e., the MEC-$\bar{X}$ and MEC-MOM charts are not affected when the underlying process data follow skewed normal-tailed distribution, i.e., $g = 0.5$, $h = 0$, just like in Case I. Table 5 shows that both charts produced $ARL_0 = 370$ when $n = 5$ and $n = 9$. In addition, the out-of-control performance of charts under this data condition are comparable to the standard MEC chart.

Finally, the performances of MEC charts are observed under an extreme data condition, which is $g = 0.5$, $h = 0.5$, i.e., skewed with heavy tail distribution. The finding indicates that the $ARL_0$ of the MEC-$\bar{X}$ chart is highly affected when compared to the robust MEC charts. The bold values in Table 5 exceed the predetermined value of 370. In addition, the $ARL_0$ of MEC-$\bar{X}$ chart is larger for small shifts ($0.25 \leq \delta \leq 0.75$) when compared to the robust MEC charts. This implies delayed detection by the standard MEC chart when the change in the process is very small. When $n$ increases, the robust MEC-MOM chart remains unaffected, where the values $ARL_0$ are not much different than 370 suggesting good in-control robustness.

The finding indicates that the in-control robustness of the MEC-$\bar{X}$ chart is highly affected under the heavy-tailed distribution despite being able to perform well under skewed data scenario. In contrast, both robust MEC charts perform consistently in terms of the in-control robustness regardless of underlying distributions. Between the two median based robust charts investigated in this study, i.e., the MEC-$\bar{X}$ and the MEC-MOM chart, the latter performs better under heavy tail distribution in terms of the in-control robustness. With good in-control robustness, the performances of these median based MEC charts are highly reliable in detecting shifts in the process across all distributions considered in this study, unlike the MEC-$\bar{X}$ chart.

### Table 5: ARL values for the MEC charts with $k = 0.5$ at $ARL_0 = 370$ when process parameters are unknown

<table>
<thead>
<tr>
<th>($g,h$)</th>
<th>$n$</th>
<th>Methods</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.0]</td>
<td>5</td>
<td>MEC-$\bar{X}$</td>
<td>369.610</td>
<td>37.287</td>
<td>17.397</td>
<td>12.578</td>
<td>10.161</td>
<td>7.629</td>
<td>6.257</td>
<td>4.839</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>MEC-$\bar{X}$</td>
<td>369.824</td>
<td>37.498</td>
<td>17.520</td>
<td>12.662</td>
<td>10.224</td>
<td>7.686</td>
<td>6.294</td>
<td>4.878</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MEC-MOM</td>
<td>370.790</td>
<td>37.156</td>
<td>17.463</td>
<td>12.609</td>
<td>10.188</td>
<td>7.647</td>
<td>6.264</td>
<td>4.848</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>MEC-$\bar{X}$</td>
<td>369.610</td>
<td>37.287</td>
<td>17.397</td>
<td>12.578</td>
<td>10.161</td>
<td>7.629</td>
<td>6.257</td>
<td>4.839</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>MEC-$\bar{X}$</td>
<td>370.348</td>
<td>25.808</td>
<td>13.754</td>
<td>10.141</td>
<td>8.279</td>
<td>6.240</td>
<td>5.106</td>
<td>3.994</td>
</tr>
<tr>
<td>[0.5]</td>
<td>5</td>
<td>MEC-$\bar{X}$</td>
<td>371.070</td>
<td>37.663</td>
<td>17.499</td>
<td>12.665</td>
<td>10.204</td>
<td>7.680</td>
<td>6.282</td>
<td>4.891</td>
</tr>
<tr>
<td>[0.5,0]</td>
<td>5</td>
<td>MEC-$\bar{X}$</td>
<td>368.318</td>
<td>37.674</td>
<td>17.409</td>
<td>12.619</td>
<td>10.147</td>
<td>7.634</td>
<td>6.256</td>
<td>4.836</td>
</tr>
</tbody>
</table>
### 4.0 REAL APPLICATION

To demonstrate the application of the MEC chart on real data, all three charts in this study were applied on a projected rainfall (in milliliter, mm) in Kedah, a state in northwest Malaysia. The data were attained from 2019 until 2020 consisting of 104 samples of size 7 as presented in Figure 1.

The first half of the data was used to construct the control limits (Phase I) and the latter half was used to monitor out-of-control samples (Phase II). For \( n = 7 \), when the values of \( \lambda \) and \( k \) are fixed at 0.13 and 0.5, respectively (as in the simulated studies), \( h \) becomes 27.86. The outputs of the proposed charts are given in

<table>
<thead>
<tr>
<th>(g,h)</th>
<th>n</th>
<th>Methods</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MEC-( \bar{X} )</td>
<td>370.313</td>
<td>38.376</td>
<td>17.504</td>
<td>12.659</td>
<td>10.230</td>
<td>7.684</td>
<td>6.286</td>
<td>4.864</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>MEC-( \bar{X} )</td>
<td>379.081</td>
<td>25.938</td>
<td>13.789</td>
<td>10.200</td>
<td>8.309</td>
<td>6.263</td>
<td>5.106</td>
<td>3.990</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>5</td>
<td>MEC-( \bar{X} )</td>
<td>1667.373</td>
<td>42.473</td>
<td>18.615</td>
<td>13.053</td>
<td>10.302</td>
<td>7.754</td>
<td>6.041</td>
<td>4.969</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>MEC-( \bar{X} )</td>
<td>2845.329</td>
<td>27.619</td>
<td>13.923</td>
<td>10.349</td>
<td>8.112</td>
<td>6.024</td>
<td>4.999</td>
<td>3.998</td>
</tr>
</tbody>
</table>

Figures 2-4 where both statistics, \( MEC^+ \) and \( MEC^- \), are plotted against the control limits, \( h \) and \( h^- \), respectively.

Figure 2 represents the output for the MEC-\( \bar{X} \) chart. The chart shows 29 out-of-control samples (samples 14 – 41). Meanwhile, Figure 3 indicates 31 out-of-control samples (samples 11 – 41) MEC-\( \bar{X} \) was applied on the data. Figure 4 shows that the output for the robust MEC-MOM chart with 48 out-of-control samples (samples 5 – 52). This implies that the robust MEC-\( \bar{X} \) and MEC-MOM charts are more sensitive to the small changes in the data as indicated by this rainfall data scenario.

![Figure 1 The Scatter Plot of Weekly Rainfalls](image)
Figure 2 The Standard MEC-\bar{x} Chart

Figure 3 The Robust MEC Chart based on the median

Figure 4 The Robust MEC Chart based on MOM
5.0 CONCLUSION

The MEC control chart is targeted to enhance the performance of the EWMA and CUSUM charts under non-normality assumptions. This study proposed to improve the performance of the MEC chart under non-normality via the usage of median based estimators. The comparison of the MEC-$\bar{X}$ and MEC-MOM charts with MEC-$\bar{X}$ chart based on the ARL shows that the in-control robustness of the MEC-$\bar{X}$ and MEC-MOM charts are not affected under various g-and-h distributions unlike the MEC-$\bar{X}$ chart. The standard MEC chart, namely, the MEC-$\bar{X}$ chart, is easily perturbed when the distribution is heavy tail in nature. In general, this study observes that MEC-$\bar{X}$ and MEC-MOM charts can withstand various tested conditions and perform well on actual data. The findings indicate that robust MEC-MOM is not easily perturbed regardless of distributional shapes, shifts, and sizes. Thus, they are reliable to be used across various conditions that may be encountered in real life, unlike the MEC-$\bar{X}$ chart.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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