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LINEAR MATRIX INEQUALITIES IN ROBUST UNIFIED SMOOTH SLIDING MODE CONTROLLER DESIGN

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Abstract. A wide range of problems encountered in system control theory can be reduced to a few standard convex or quasiconvex optimisation problems involving linear matrix inequalities (LMI). With recent developed of interior point methods, the optimisation problems can be solved numerically very efficiently. One of the applications of the LMI may be seen in solving the sliding mode control problems. The sliding mode control system is capable of total invariance to the matched uncertainties while remain in the sliding mode. But the system may still face the undesirable distractions cause by the mismatched uncertainties, and chattering problem. In this paper, the sliding surface is designed with integration of an optimal guaranteed cost H infinity criterion to attenuate the mismatched disturbances. The guaranteed cost surface is derived from a convex optimisation procedure formulated as an LMI problem. A unified smooth control law is applied to solve the chattering problem. The results showed that the controller may improve the performance with total chattering elimination and mismatched disturbances rejection.

Keywords: Linear matrix inequalities (LMI), sliding mode control, mismatched uncertainties, chattering free, optimal guaranteed cost H infinity criterion

Abstrak. Sebahagian besar masalah yang dihadapi dalam teori kawalan sistem boleh dikurangkan kepada beberapa masalah pengoptimuman cembung atau *kuasi*-cembung piawai yang melibatkan ketaksamaan matriks lelurus (LMI). Dengan perkembangan terbaru tentang cara titik dalaman, masalah pengoptimuman tersebut dapat diselesaikan secara efisien dengan kaedah berangka. Satu daripada aplikasi LMI boleh dilihat dalam penyelesaian masalah kawalan ragam gelincir. Sistem kawalan ragam gelincir berkemampuan supaya tidak terpengaruh secara keseluruhan oleh ketidakpastian padanan apabila berada dalam ragam gelincir. Akan tetapi, sistem masih menghadapi gangguan yang tidak diingini apabila diusik oleh ketidakpastian tidak terpadan, serta masalah gelugutan. Dalam kertas kerja ini, permukaan gelincir direka bentuk dengan integrasi suatu kriteria H infiniti terjamin kos optimum untuk mengurangkan gangguan tidak terpadan. Permukaan kos terjamin tersebut diterbitkan daripada prosedur pengoptimuman cembung yang diformulasikan sebagai masalah LMI. Satu kawalan licin seragam diaplikasikan untuk menyelesaikan masalah gelugutan. Keputusan menunjukkan bahawa pengawal tersebut dapat memperbaiki prestasi dari segi penyingkiran gelugutan secara keseluruhan dan penyisihan gangguan tidak terpadan.

Kata kunci: Ketaksamaan matriks lelurus (LMI), kawalan ragam gelincir, gangguan tidak terpadan, bebas gelugutan, kriteria H infiniti terjamin kos optimum

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1.0 INTRODUCTION

It is well known that sliding mode control is capable of rendering uncertain systems to be in total invariance to the matched uncertainties while remains in the sliding mode. However, the system may still face the undesirable distractions caused by the chattering problem due to the discontinuous fast switching action of the control input across the switching surface, and the controller may fail if the system suffers from mismatched uncertainties. Various strategies have been proposed by numerous researchers to solve the mismatched problem [1-7]. In [7], Takahashi and Peres proposed an H_∞-norm guaranteed cost design for sliding surfaces in variable structure controllers which is suitable for approximate rejection of mismatched uncertainties. A convex optimisation approach via linear matrix inequalities (LMI) is employed in the surface design. However, the study does not solve the chattering problem. Thus in this paper, a sliding mode controller with a unified smooth control (US-SMC) is proposed based on the idea of two-phase slinding mode controller by Wang and Lee [8] and Zhang and Panda [9] together with the LMI approach to completely eliminate the chattering problem as well as robust to mismatched uncertainties.

A wide range of problems encountered in system control theory can be reduced to a few standard convex or quasiconvex optimisation problems involving linear matrix inequalities (LMI). One of the applications of the LMI may be seen in solving the sliding mode control problems. A strict LMI has the form

$$F(\chi) \underline{\underline{\Delta}} F_0 + \sum_{i=1}^{f} \chi_i F_i > 0 \tag{1}$$

where $\chi \in \Re^f$, $F_i = F_i^T \in \Re^{n \times n}$, $i = 0, \ldots, f$. The LMI is equivalent to a set of f polynomial inequalities in χ, the leading principal minors of $F(\chi)$ must be positive [10].

The LMI (1) is a convex constrain on χ, which the set $\{\chi | F(\chi) > 0\}$ is convex, so that any robust control problem with constraints of this type and a convex performance objective can be reduced to a convex programming problem. The convex programming problems are particularly attractive since their optimal solutions are global and efficient algorithms such as interior-point algorithms exists for finding optimal solutions [11].

Multiple LMIs $F^{(1)}(\chi) > 0, \ldots, F^{(f)}(\chi) > 0$ can be expressed as single LMI as follows [12]:

$$\left[F^{(1)}(\chi) > 0, \ldots, F^{(f)}(\chi)\right] = diag\left[F^{(1)}(\chi) > 0, \ldots, F^{(f)}(\chi)\right] > 0 \tag{2}$$

Nonlinear (convex) inequalities are converted to LMI form using Schur complements [10]. The LMI

$$\begin{bmatrix} Q(\chi) & S(\chi) \\ S^T(\chi) & R(\chi) \end{bmatrix} > 0 \tag{3}$$

where $Q(\chi) = Q^T(\chi), R(\chi) = R^T(\chi)$, and $S(\chi)$ depend affinely on χ, is equivalent to

$$R(\chi) > 0, \qquad Q(\chi) - S(\chi)R^{-1}(\chi)S^T(\chi) > 0 \tag{4}$$

2.0 SLIDING MODE CONTROLLER DESIGN AND MISMATCHED UNCERTAINTIES PROBLEM

Consider the following system with uncertain parameters:

$$\dot{\mathbf{x}}(t) = (A_o + \Delta_A)\mathbf{x}(t) + g(\mathbf{x},t) + (B_o + \Delta_B)\mathbf{u}(t) + (E_o + \Delta_E)\mathbf{d}(t) \tag{5}$$

where, $\mathbf{x}(t) \in \Re^n$; $\mathbf{u}(t) \in \Re^m$; $\mathbf{d}(t) \in \Re^l$; $A_o \in \Re^{n \times n}$; $B_o \in \Re^{n \times m}$; $E_o \in \Re^{n \times l}$; $g : \Re^n \times \Re_+ \to \Re^n$. Δ_A, Δ_B, Δ_E are the uncertainties associated with each matrix, which assumed to belong to a polytopic uncertainty domain $\wp$, a set which is the convex hull of finitely many points called polytope [13] defined as follows:

$$\wp \triangleq \begin{cases} (\Delta_A, \Delta_B, \Delta_E)(\alpha) \middle| (\Delta_A, \Delta_B, \Delta_E)(\alpha) = \sum_{i=1}^{v} \alpha_i (\Delta_{Ai}, \Delta_{Bi}, \Delta_{Ei}), \\ \sum_{i=1}^{v} \alpha_i = 1, \qquad \alpha_i \geq 0, \forall\, i = 1, \ldots, v \end{cases} \tag{6}$$

where $(\Delta_{Ai}, \Delta_{Bi}, \Delta_{Ei})$ are the polytope vertices with $i = 1, \ldots, v, \alpha_i$ are the weights going from zero to one and v is the number of vertices.

The sliding surface is chosen to be a linear subspace determined by the intersection of switching surfaces S_i [14], given by

$$S_i : \left\{ \mathbf{x} \in \Re^n \middle| \psi_i(\mathbf{x}) = c_i^T \mathbf{x} = \mathbf{0} \right\} \qquad i = 1, 2, \ldots, n \tag{7}$$

$$\mathbf{S} : \left\{ \mathbf{x} \in \Re^n \middle| \boldsymbol{\psi}(\mathbf{x}) = C\mathbf{x} = \mathbf{0} \right\} = \bigcap_{i=1}^{n} S_i \tag{8}$$

Assumed that the matrix $|CB_o| \neq 0$, resulting in the equivalent control $\mathbf{u}_{eq}(\mathbf{x})$ as the solution to the sliding surface (8) with $\boldsymbol{\psi}(\mathbf{x}) = 0$, as below:

$$\mathbf{u}_{eq}(\mathbf{x}) = -(CB_o)^{-1} C\{(A_o + \Delta_A)\mathbf{x}(\mathbf{t}) + g(\mathbf{x},t) + \Delta_B u(t) + (E_o + \Delta_E)\mathbf{d}(t)\} \tag{9}$$

Suppose that all parameter uncertainties, disturbances and nonlinearities are in the range space of matrix B_o, that is,

$$R(B_o) \supset R(\Delta_A) + R(\Delta_B) + R(E_o) + R(\Delta_E) + R(g(\mathbf{x},t)) \tag{10}$$

Then, the resulting reduced order sliding motion reduced to

$$\dot{\mathbf{x}}(t) = \left(I - B_o(CB_o)^{-1}C\right)A_o\mathbf{x}(t) \tag{11}$$

Hence, for a simple system, the following equivalent control can be used.

$$\mathbf{u}_{eq}(\mathbf{x}) = -(CB_o)^{-1}CA_o\mathbf{x}(t) \tag{12}$$

The above sliding motion does not depend on the exogenous signal. The relation (10) yields the famous matching condition which leads to the powerful invariance property of matching uncertainties in sliding mode control. However, it would be easier to investigate the invariance property of sliding modes in system (5) by considering its behaviour in the space of new variables in regular form, after performing of canonical transformation.

By assumption, B_0 has full rank m, so that there exists an orthogonal matrix of elementary row operations $T \in \Re^{n\times n}$ such that

$$TT^T - \mathbf{I} \qquad TB_o = \left[\frac{0}{\overline{B}_2}\right] \qquad T\Delta_B = \left[\frac{0}{\Delta_{B2}}\right] \tag{13}$$

where $\overline{B}_2 \in \Re^{m\times m}$, $\left|\overline{B}_2\right| \neq 0$. Defined a transformed state variable, $\overline{\mathbf{x}} = T\mathbf{x}$, the system (5) and switching function (8) becomes

$$\dot{\overline{\mathbf{x}}} = T(A_o + \Delta_A)T^T\overline{\mathbf{x}} = Tg\left(T^T\overline{\mathbf{x}}\right) + T(B_o + \Delta_B)\mathbf{u} + T(E_o + \Delta_E)\mathbf{d} \tag{14}$$

$$\boldsymbol{\psi}(\overline{\mathbf{x}}) = CT^T\overline{\mathbf{x}} = 0 \tag{15}$$

Consider some relaxations in matching condition (10), such that only

$$R(\Delta_B) + R(g(\mathbf{x},t)) \subset R(B_o) \tag{16}$$

but

$$R(\Delta_A) + R(E_o) + R(\Delta_E) \not\subset R(B_o) \tag{17}$$

Then the equation of ideal sliding motion becomes [15] :

$$\dot{\overline{\mathbf{x}}}_1 = \left\{\left(\overline{A}_{11} + \overline{\Delta}_{A11}\right) - \left(\overline{A}_{12} + \overline{\Delta}_{A12}\right)\overline{C}_2^{-1}\overline{C}_1\right\}\overline{\mathbf{x}}_1 + \left(\overline{E}_1 + \overline{\Delta}_{E1}\right)\mathbf{d} \tag{18}$$

$$\overline{\mathbf{x}}_2 = -\overline{C}_2^{-1}\overline{C}_1\overline{\mathbf{x}}_1 \tag{19}$$

with
$$TA_oT^T = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}; \qquad T\Delta_A T^T = \begin{bmatrix} \overline{\Delta}_{A11} & \overline{\Delta}_{A12} \\ \overline{\Delta}_{A21} & \overline{\Delta}_{A22} \end{bmatrix};$$

$$Tg\left(T^T\overline{\mathbf{x}}\right) = \begin{bmatrix} \overline{g}_1\left(T^T\overline{\mathbf{x}}_1, T^T\overline{\mathbf{x}}_2\right) \\ \overline{g}_2\left(T^T\overline{\mathbf{x}}_1, T^T\overline{\mathbf{x}}_2\right) \end{bmatrix}; \quad TE_o = \begin{bmatrix} \overline{E}_1 \\ \overline{E}_2 \end{bmatrix}; \quad T\Delta_E = \begin{bmatrix} \overline{\Delta}_{E1} \\ \overline{\Delta}_{E2} \end{bmatrix}; \tag{20}$$

$$CT^T = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix}; \qquad \left|\overline{C}_2\right| \neq 0$$

It is clearly seen that the sliding motion equations (18-19) are not free from the effects of disturbance factors. Thus, combine system of control have to be implemented to compensate the effect of the mismatched elements.

3.0 LMI SOLUTION TO SWITCHING SURFACE DESIGN

A controlled output vector $\mathbf{z}(t) \in \Re^p$ is defined, to introduce the H_∞ guaranteed cost design, in which

$$\mathbf{z}(t) = L\mathbf{x}(t) \tag{21}$$

and
$$z(s) = H(s)d(s) \tag{22}$$

$H(s)$ is defined as the closed-loop transfer function from **d** to **z** for the system in sliding motion, in which valid for $t > 0$, as the reaching time is defined as $\tau = 0$. $z(s)$ and $d(s)$ are the Laplace transforms of $\mathbf{z}(t)$ and $\mathbf{d}(t)$, respectively.

An H_∞ optimal guaranteed cost surface, $\mathbf{S}_{op}$ is derived so that the closed-loop transfer function $H(s)$ has minimal H_∞ norm for the worst case uncertainty matrices Δ_A, Δ_E :

$$\mathbf{S}_{op} : \left\{C_{op}\mathbf{x} = \mathbf{0} \mid \mathbf{x} \in \Re^n\right\} \tag{23}$$

$$C_{op} = \arg\min_{C} \max_{\Delta_A, \Delta_E} \left\|H(s)\right\|_\infty \tag{24}$$

Due to the present lack of better synthesis methods which exactly solve the problem defined above, the resulting surfaces will actually be suboptimal. The available methods constrain to the solutions to be found inside the class of quadratically stabilising controllers [5], where

$$\mathbf{S}_{qop} : \left\{\mathbf{x} \in \Re^n \mid C_{qop}\mathbf{x} = \mathbf{0}\right\} \tag{25}$$

$$C_{qop} = \arg\min_{C \in C_q} \max_{\Delta_A, \Delta_E} \left\|H(s)\right\|_\infty \tag{26}$$

C_q is the set of quadratically stabilising sliding controllers. The solution will leads to an upper bound for the hypothetical optimal controller cost. If $\Delta_A = \Delta_E = 0$, the optimisation problem becomes the exact H_∞ optimal control [5].

The matrix L in (21) is assumed to be constrain to

$$\overline{L}_2^T \overline{L}_2 > 0 \tag{27}$$

which is relevant for the H_∞ guaranteed cost optimisation problem. Assumption (27) must be taken in order to make the H_∞ a non-singular problem. Hence, from equations (18) and (21) yield,

$$\dot{\overline{\mathbf{x}}}_1 = \left(\widetilde{A}_1 - \widetilde{A}_2 K\right)\overline{\mathbf{x}}_1 + \widetilde{E}\mathbf{d} \tag{28a}$$

$$\mathbf{z} = \left(\overline{L}_1 - \overline{L}_2 K\right)\overline{\mathbf{x}}_1 \tag{28b}$$

in which, $\widetilde{A}_1 = \overline{A}_{11} + \overline{\Delta}_{A11}$, $\widetilde{A}_2 = \overline{A}_{12} + \overline{\Delta}_{A12}$, $\widetilde{E} = \overline{E}_1 + \overline{\Delta}_{E1}$, and $K = \overline{C}_2^{-1}\overline{C}_1$.

From assumption (6), matrices $\widetilde{A}_1$, $\widetilde{A}_2$, and $\widetilde{E}$ also belong to polytope-type set with known vertices. This set is defined as

$$\widetilde{\wp} \triangleq \left\{ \begin{array}{l} \left(\widetilde{A}_1, \widetilde{A}_2, \widetilde{E}\right)(\alpha) = \sum_{i=1}^{v} \alpha_i \left(\widetilde{A}_{1i}, \widetilde{A}_{2i}, \widetilde{E}_i\right), \\ \sum_{i=1}^{v} \alpha_i = 1, \alpha_i \geq 0, \forall\, i = 1, \ldots, v \end{array} \right\} \tag{29}$$

The vertices of polytope $\widetilde{\wp}$ correspond to the vertices of $\wp$, defined in (6), transformed through the mapping T, with the same weighting α_i [6].

The design of a robust sliding mode control is possible if there exists a $C \in \Re^{m\times(n-m)}$ which guarantees robust stability of (28) and there exists a control law K which makes the sliding function asymptotically stable for a specified sliding surface [16].

Definition 1: [17]

Let the constant $\gamma > 0$ be given. The system (28) is said to be stabilisable with disturbance attenuation γ if there exists a state feedback matrix K such that the following conditions are satisfied.

(i) The matrix $\left(\widetilde{A}_1 - \widetilde{A}_2 K\right)$ is a stable matrix.
(ii) $\left\|H(s)\right\|_\infty < \gamma$ (from Small Gain Theorem).

The proof of Definition 1 can be found in [17].

Definition 2: [7]
The uncertain system defined in (28) with the pair $\left\langle \overline{L}_1, \widetilde{A}_1 \right\rangle$ *observable is said to be quadratically stabilisable with disturbance attenuation* γ *if there exists* $P \in \Re^{(n-m)\times(n-m)}$, $P = P^T > 0$ *and* $K \in \Re^{m\times n-m}$ *such that*

$$\left(\widetilde{A}_1 - \widetilde{A}_2 K\right)^T P + P\left(\widetilde{A}_1 - \widetilde{A}_2 K\right) + P\widetilde{E}\widetilde{E}^T P + \gamma^{-2}\left(\overline{L}_1 - \overline{L}_2 K\right)^T \left(\overline{L}_1 - \overline{L}_2 K\right) < 0, \quad (30)$$

$$\left(\widetilde{A}_1, \widetilde{A}_2, \widetilde{E}\right) \in \widetilde{\wp}$$

Define the set

$$K_q \underline{\underline{\Delta}} \left\{ K \in K_q \mid \left(\widetilde{A}_1 - \widetilde{A}_2 K\right) \textit{is quadratically stable} \right\} \quad (31)$$

An uncertain linear system is said to be robustly stabilisable if $K_q \neq 0$. The set K_q is composed by all matrices K which stabilise each pair of $\left(\widetilde{A}_1, \widetilde{A}_2, \widetilde{E}\right) \in \widetilde{\wp}$ [18].

The optimal H_∞ norm guaranteed cost, γ_{qop}, is defined as, and can be found from

$$\gamma_{qop} = \inf\left\{ \gamma \left\| H(s) \right\|_\infty < \gamma, K \in K_q \right\} \quad (32)$$

The technique of finding $\left\| H(s) \right\|_\infty$ involves a search over $\gamma > 0$. The famous bounded real lemma provides this characterisation [19]. The optimal guaranteed cost control problem in Definition 2 can be solved with linear matrix inequality (LMI) approach.

The inequality developed in (30) is equivalent to

$$\left(\widetilde{A}_{1i} - \widetilde{A}_{2i} K\right)^T P + P\left(\widetilde{A}_{1i} - \widetilde{A}_{2i} K\right) + P\widetilde{E}_i\widetilde{E}_i^T P + \gamma^{-2}\left(\overline{L}_1 - \overline{L}_2 K\right)^T \left(\overline{L}_1 - \overline{L}_2 K\right) < 0 \quad (33)$$

for each vertex $i = 1, \ldots, v$ of set $\widetilde{\wp} = \sum_{i=1}^{v} \alpha_i \left(\widetilde{A}_{1i}, \widetilde{A}_{2i}, \widetilde{E}_i\right)$.

The inequality (33) is not linear and is not jointly convex in (P, K, γ). However, with the bijective transformation

$$Y \underline{\underline{\Delta}} P^{-1}, \quad \text{and} \quad Z \underline{\underline{\Delta}} - KP^{-1} \quad (34)$$

equation (33) becomes

$$\begin{aligned} &\left(\widetilde{A}_{1i} + A_{2i} Z Y^{-1}\right)^T Y^{-1} + Y^{-1}\left(\widetilde{A}_{1i} + A_{2i} Z Y^{-1}\right) + Y^{-1}\widetilde{E}_i\widetilde{E}_i^T Y^{-1} \\ &+\gamma^{-2}\left(L_1 - L_2 Z Y^{-1}\right)^T \left(\overline{L}_1 - \overline{L}_2 Z Y^{-1}\right) < 1 \end{aligned} \quad (35)$$

Multiplied (35) on the left and right by Y, gives:

$$\begin{aligned} & Y\left(\tilde{A}_{1i}+A_{2i}ZY^{-1}\right)^T+\left(\tilde{A}_{1i}+A_{2i}ZY^{-1}\right)Y+\tilde{E}_i\tilde{E}_i^T \\ & +\gamma^{-2}Y\left(\overline{L}_1+\overline{L}_2ZY^{-1}\right)^T\left(\overline{L}_1+\overline{L}_2ZY^{-1}\right)Y<0, \\ & -\left(Y\tilde{A}_{1i}^T+Z^T\tilde{A}_{2i}^T+\tilde{A}_{1i}Y+\tilde{A}_{2i}Z\right)-\tilde{E}_i\tilde{E}_i^T \\ & -\gamma^{-2}\left(Y\overline{L}_1^T+Z^T\overline{L}_1^T\right)\left(\overline{L}_1Y+\overline{L}_2Z\right)>0 \end{aligned} \tag{36}$$

Refer to the Schur complements in (3) and (4), with

$$Q=-\left(Y\tilde{A}_{1i}^T+Z^T\tilde{A}_{2i}^T+\tilde{A}_{1i}^TY+\tilde{A}_{2i}Z\right);\quad S=\left[\tilde{E}_i\left(Y\overline{L}_1^T+Z^T\overline{L}_2^T\right)\right];\quad R=\begin{bmatrix}\mathbf{I} & 0\\ 0 & \delta\mathbf{I}\end{bmatrix},$$

and define

$$\delta \triangleq \gamma^2, \tag{37}$$

the inequality in (33) can be written in LMI form as

$$\Lambda_i(Z,Y,\delta)=\begin{bmatrix} -\left(\tilde{A}_{1i}^TY+Z^T\tilde{A}_{2i}^T+\tilde{A}_{1i}Y+\tilde{A}_{2i}Y\right) & \overline{E}_i & \left(Y\overline{L}_1^T+Z^T\overline{L}_2^T\right) \\ \tilde{E}_i^T & I & 0 \\ \left(\overline{L}_iY+\overline{L}_2Z\right) & 0 & \delta I \end{bmatrix}>0 \tag{38}$$

Associated to each LMI in (38), there is a set of solution V_i, defined as

$$V_i \triangleq \left\{ \begin{aligned} & (Z,Y,\delta)\in\Re^{m\times n-m}\times\Re^{n\times n}\times\Re \\ & \mid Y\geq\varepsilon I,\ \varepsilon>0\in\Re;\ \delta>0;\ \Lambda_i(Z,Y,\delta)>0 \end{aligned} \right\},\quad \forall i=1,\ldots,v \tag{39}$$

The parameter ε above is a small positive number, and has been introduced to make V_i a closed set, in which a search algorithm is guaranteed to converge [20]. Each vertex of the uncertainty polytope $\tilde{\wp}=\sum_{i=1}^{v}\alpha_i\left(\tilde{A}_{1i},\tilde{A}_{2i},\tilde{E}_i\right)$ is verified by the set V_i mentioned accordingly. The simultaneous satisfaction of the inequality for all vertices, i.e. the intersection of such vertices yields the overall solution for the uncertain system control problem.

$$V \triangleq \bigcap_{i=1}^{v} V_i \tag{40}$$

Theorem 1[7]: *The set V is convex.*
Proof: Proofs of the theorem can be found in [7] and Lemma 6.1 of [18].

Theorem 2 [7]: *$V \neq \varnothing$ if and only if the uncertain system in (28) and is quadratically stabilisable with disturbance attenuation γ in which the stabilizing gain to assure such attenuation is given by $K \triangleq -ZY^{-1}$.*

Proof: The proof of Theorem 2 is quite straight forward from Definition 2, bijective transformation of (33) and Theorem 1, since V yields the solution for $\Lambda(Z,Y,\delta)>0$. When (Z,Y,δ) varies in V then all stabilising state feedback gains are generated from $K=-ZY^{-1}$. If such a matrix does not exist, it is clear that $V=\varnothing$.

Remark 1 [7]: It should be noted that the constraints $\Lambda_i(Z,Y,\delta)>0$ are defined only for the uncertainty polytope vertices. However, any triple (Z,Y,δ) satisfying such constraints also satisfied any constraint $\Lambda(Z,Y,\delta)>0$, defined with the uncertain parameter vertices $\left(\widetilde{A}_{1i}, \widetilde{A}_{2i}, \widetilde{E}_i\right)$ replaced by any parameters $\left(\widetilde{A}_1, \widetilde{A}_2, \widetilde{E}\right)$ chosen inside the convex polytope. This occurs due to the fact that matrices Λ_i are affine in these uncertain parameters. The consequences of the validity of the constraints inside the whole polytope are that both the quadratic stability and the disturbance attenuation derived in Theorem 2 are also valid in the whole polytope.

From equations (39) and (40), the optimal H_∞ norm guaranteed cost problem in (32) is equivalent to

$$\delta_{qop} = \inf\{\delta \mid (Z,Y,\delta) \in V\} \tag{41}$$

The problem can be solved using LMI Control Toolbox in Matlab® [21]. The objective function, δ is linear to $\Lambda(Z,Y,\delta)>0$, and therefore also convex. As a by-product of the optimisation on δ, the set (Z,Y,δ) is also sub-optimal, as

$$\left(Z_{qop}, Y_{qop}, \delta_{qop}\right) \in V \tag{42}$$

The controller matrix is given by

$$K_{qop} = -Z_{qop}Y_{qop}^{-1} \in K_q \tag{43}$$

The corresponding sliding mode function matrix is obtain, after canonical transformation of (18 and 19), is therefore equal to

$$C_{qop} = \left[\overline{C}_2 K_{qop} \quad \overline{C}_2 \right] T \tag{44}$$

There is some freedom in the final choice of matrix C_{qop}. Matrix $\overline{C}_2$ can be of any values, for instance, provided it remains non-singular. However, the simplest method of determining C_{qop} from K_{qop} is that employed by Utkin and Young [22]; namely, letting $\overline{C}_2 = \mathbf{I} \in \Re^{m \times m}$. The approach has the merit of minimising the amount of calculation in proceeding from K_{qop} from C_{qop} and hence reduces the possibility of numerical errors [15].

4.0 SLIDING MODE CONTROLLER WITH A UNIFIED SMOOTH CONTROL (US-SMC) CONTROLLER DESIGN

In this section, the proposed US-SMC controller which is void of chattering phenomenon in the control input is presented. The structure of controller is based on the idea of the two-phase sliding mode controller by Wang and Lee [8] and Zhang and Panda [9]. The US-SMC controller is proposed as:

$$\mathbf{u}(t) = \mathbf{u}(t)_{eq} + \kappa_s \boldsymbol{\psi} = -(CB_o)^1 CA_o \mathbf{x} + \kappa_s C\mathbf{x} = V_{inSM} \tag{45}$$

with reachability condition $\psi\dot{\psi} = C_{qop} B_o \kappa_s \psi^2 < 0$, where $\kappa_s > 0$ as $C_{qop} B_o < 0$. It can be seen from equation (45) that the US-SMC does not contain any switching (including saturation) component, therefore the control input signal is expected to be completely free from chattering phenomenon.

Theorem 3: *The system in (5) with* $\mathbf{u}(t) = \mathbf{u}(t)_{eq} + \kappa_s \boldsymbol{\psi}$ *is quadratically stable, if and only if there exists* $P \in \Re^{n \times n}$, $P = P^T > 0$, *such that* $A_{eq}^T P + PA_{eq} < 0$ *with*

$$A_{eq} = \left\{ \left(I - B_o \left(C_{qop} B_o \right)^{-1}{}_{qop} \right) A_o + \kappa_s B_o C_{qop} \right\} \tag{46}$$

Proof:

$$\mathbf{u}(t) = \mathbf{u}(t)_{eq} + \kappa_s \boldsymbol{\psi}$$

$$\dot{\mathbf{x}}(t) = \left\{ \left(I - B_o \left(C_{qop} B_o \right)^{-1}{}_{qop} \right) A_o + \kappa_s B_o C_{qop} \right\} \mathbf{x}(t) = A_{eq} \mathbf{x}(t) \tag{47}$$

Let the Lyapunov function candidate for the system is chosen as

$$V_q(t) = \mathbf{x}^T(t) P \mathbf{x}(t) \tag{48}$$

where $\mathbf{x}(t)$ represent the solution of the system and $P > 0$ is the solution of the matrix Lyapunov equation $A^T P + PA = -Q$, for a given positive definite symmetric matrix $Q > 0$. Differentiating $V_q(t)$ with respect of time,

$$\dot{V}_q(t) = \dot{\mathbf{x}}^T(t)P\mathbf{x}(t) + \mathbf{x}^T(t)P\dot{\mathbf{x}}(t) = \mathbf{x}^T(t)\left\lfloor A_{eq}{}^T P + PA_{eq} \right\rfloor \mathbf{x}(t) = -\mathbf{x}^T(t)Q\mathbf{x}(t) \tag{49}$$

From the Rayleigh principle,

$$\mathbf{x}^T(t)Q\mathbf{x}(t) \le \lambda_{\min}(Q)\|\mathbf{x}(t)\|^2 \tag{50}$$

gives
$$\dot{V}_q(t) = -\mathbf{x}^T(t)Q\mathbf{x}(t) \le \lambda_{\min}(Q)\|\mathbf{x}(t)\|^2 < 0 \tag{51}$$

since $\lambda_{\min}(Q)\|x(t)\|^2 > 0$, with $Q = -A_{eq}{}^T P - PA_{eq} > 0$.

Theorem 3 explains that a system with the controller $\mathbf{u}(t) = \mathbf{u}(t)_{eq} + \kappa_s \boldsymbol{\psi}$ is guaranteed to be quadratically stable if the condition stated is fulfilled.

5.0 SIMULATIONS AND RESULTS

The designed robust sliding mode controller will be applied on the Rotary Inverted Pendulum (RIP) model as shown in Figure 1. With the reference coordinates of the system as shown, the mathematical model of the RIP model can be obtained as equation (52). It can be seen that the system equation contains uncertainties, nonlinearities and mismatched disturbances.

$$\begin{bmatrix} \dot{\theta}_n \\ \dot{\theta}_p \\ \ddot{\theta}_a \\ \ddot{\theta}_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-25.4059(\cos\theta_p)}{1-0.37565(\cos^2\theta_p)}\dfrac{(\sin\theta_p)}{\theta_p} & \dfrac{-12.75736E_{ff}}{1-0.37565(\cos^2\theta_p)} & 0 \\ 0 & \dfrac{24.11471}{1-0.37565(\cos^2\theta_p)}\dfrac{(\sin\theta_p)}{\theta_p} & \dfrac{4.54873(\cos\theta_p)E_{ff}}{1-0.37565(\cos^2\theta_p)} & 0 \end{bmatrix} \begin{bmatrix} \theta_a - \theta_{aREF} \\ \theta_p \\ \dot{\theta}_a \\ \dot{\theta}_p \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ 0 \\ \dfrac{1.05354(\sin\theta_p)}{1-0.37565(\cos^2\theta_p)} \\ \dfrac{-0.37565(\cos\theta_p)(\sin\theta_p)}{1-0.37565(\cos^2\theta_p)} \end{bmatrix} \dot{\theta}_p^2 + \begin{bmatrix} 0 \\ 0 \\ \dfrac{23.59264E_{ff}}{1-0.37565(\cos^2\theta_p)} \\ \dfrac{-8.41213(\cos\theta_p)E_{ff}}{1-0.37565(\cos^2\theta_p)} \end{bmatrix} V_{in} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \zeta \tag{52}$$

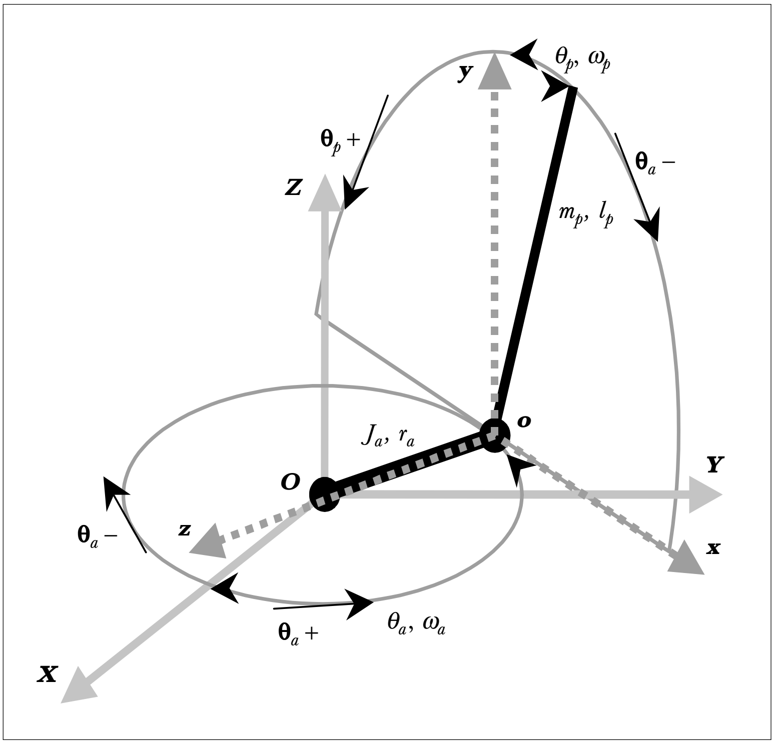


Figure 1 Reference coordinates of the RIP model

To solve the gain matrix of the sliding mode controller proposed, a MATLAB® program file is developed to calculate the quadratic H_∞ guaranteed cost gain matrix, solved by the LMI solver of the MATLAB® LMI Control Toolbox. The H_∞ gain is then be utilised to obtain the sliding mode gain matrix. The final results obtained are as below:

$$\gamma_{qop} = 9.4003 \times 10^{-1} \tag{53}$$

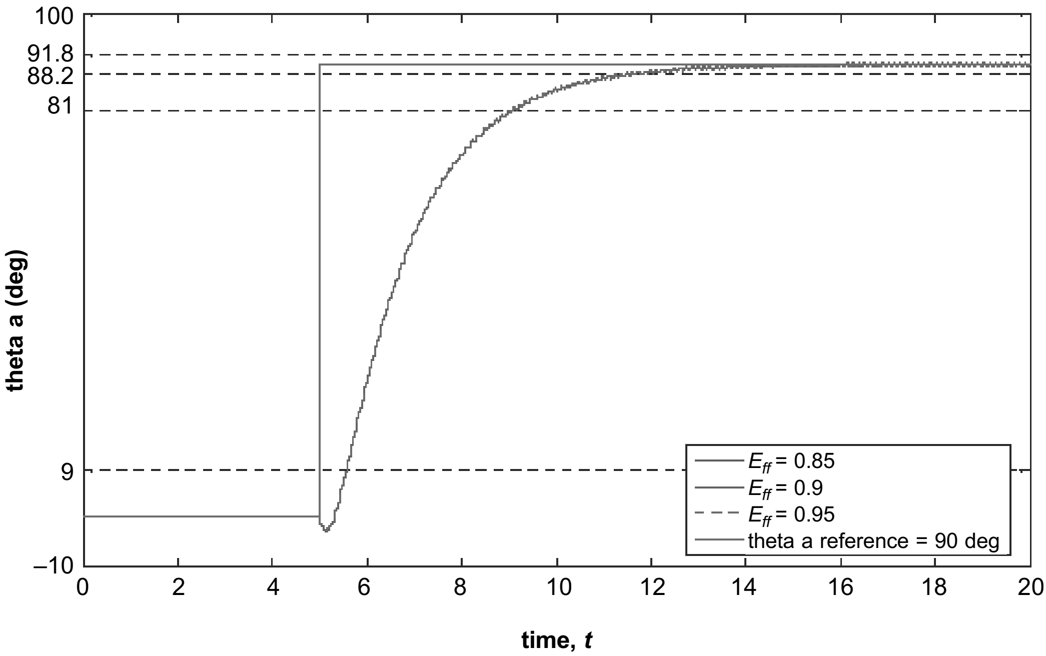
$$C_{qop} = \begin{bmatrix} 0.2207 & 5.7878 & 0.3952 & 1.1114 \end{bmatrix} \tag{54}$$

The actual output of the optimisation procedure is the above C_{qop} matrix times 10^3. Since the sliding surface is the null space of C_{qop}, it does not vary under such scaling [7]. The sub-optimal quadratically stabilising sliding controller set, C_{qop} obtained will be utilised in the SIMULINK® simulations.

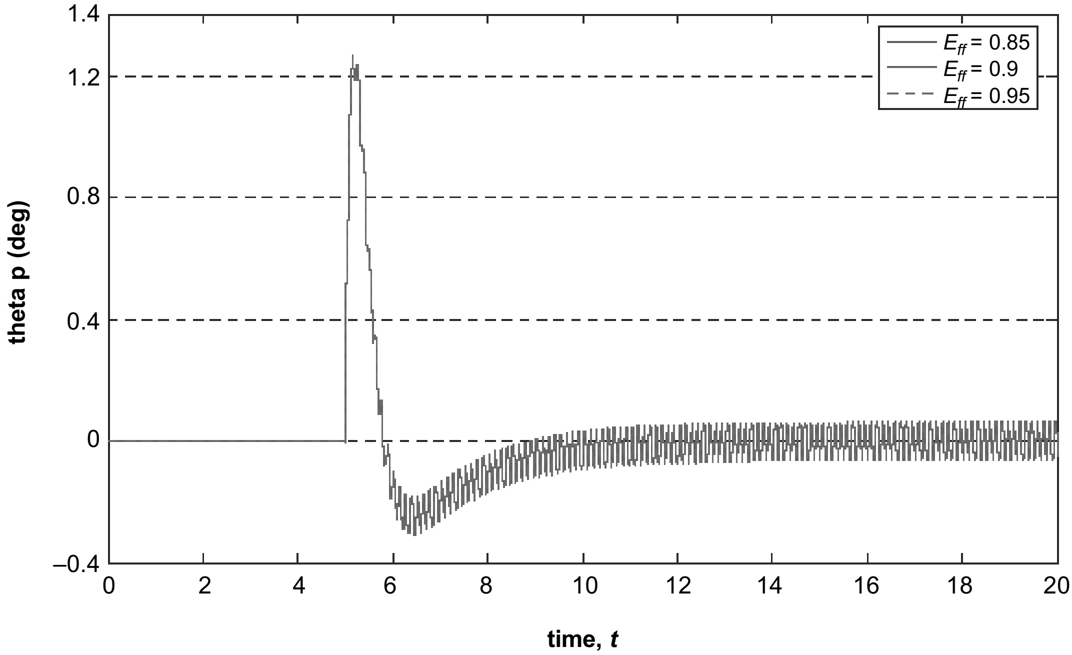
The θ_{aREF} used to test the performance is a step input which reaches 90°, five seconds after the simulation starts. The sliding structure constant κ_s is set to 200. Matched and mismatched uncertainties are used in simulations to test the performance of the controller, with E_{ff} equals to *0.85*, *0.9*, and *0.95*, and the external uncertainty

$$\zeta = 0.05 \sin(100\pi t) \tag{55}$$

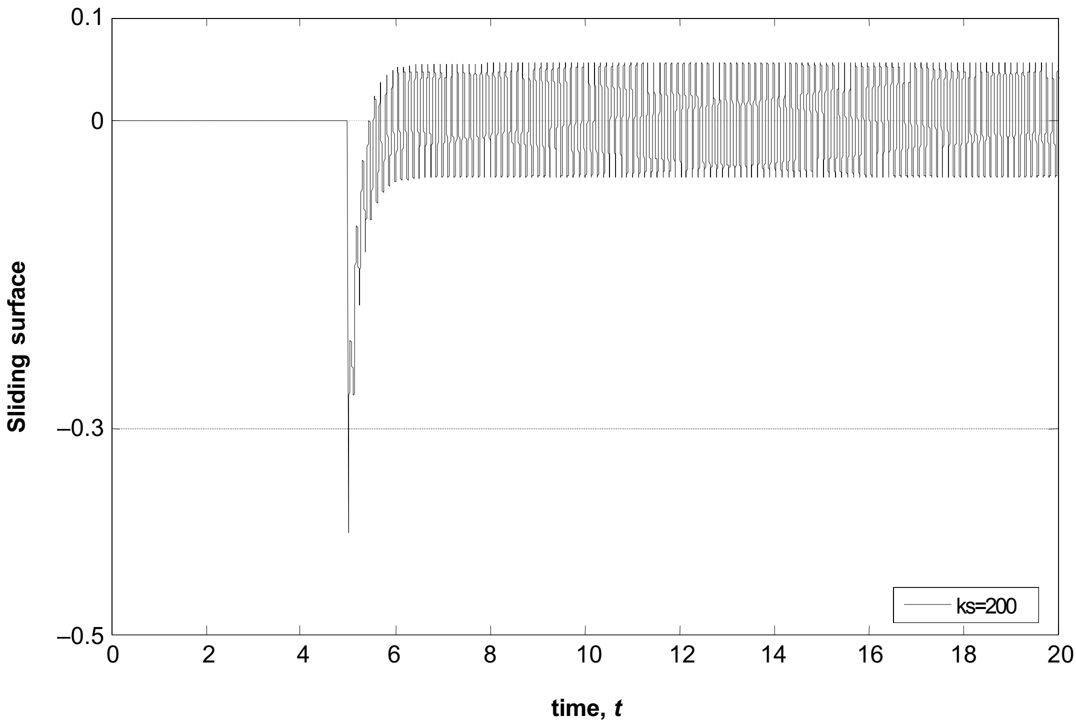
The simulation results in Figures 2(a) and 2(b) show that the US-SMC renders the system response invariant to matched uncertainty and also capable of controlling the system inherent with mismatched uncertainty, while maintaining the system stability, giving good tracking performance and reaching the sliding surface within the specific period (Figure 2(c)). The simulation result (Figure 2(d)) also shows that the chattering problem is completely eliminated with the used of the proposed US-SMC.



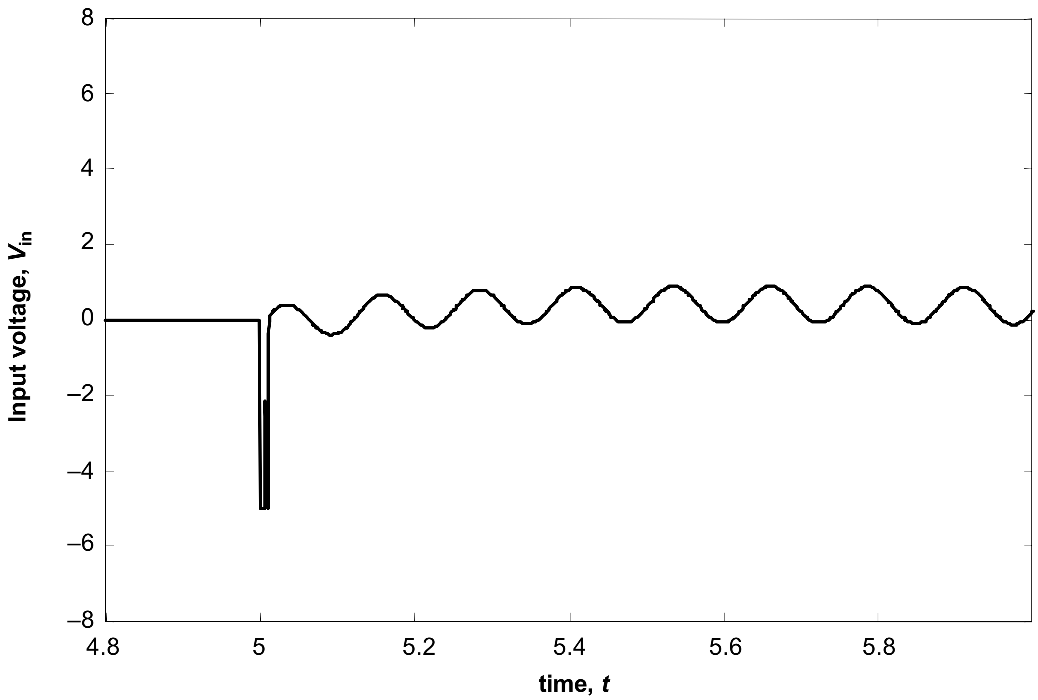
(a) Response of Rotary Arm with Different E_{ff}, $\kappa_s = 200$ and $\zeta = 0.05 \sin(100\pi t)$



(b) Response of Inverted Pendulum with Different E_{ff} , $\kappa_s = 200$ and $\zeta = 0.05 \sin(100\pi t)$



(c) Sliding Surface with $E_{ff} = 0.85$, $\kappa_s = 200$ and $\zeta = 0.05 \sin(100\pi t)$



(d) System Input Voltage with E_{ff} = 0.85, κ_s = 200 and ζ = 0.05 sin(100πt)

Figure 2 The performance of the unified smooth sliding mode controller

6.0 CONCLUSION

In this paper, a unified smooth sliding mode controller named US-SMC has been developed in which the sliding surface is designed with the integration of an optimal guaranteed cost H_∞ criterion to attenuate the mismatched disturbances and at the same time providing a chattering free control input signal. The LMI technique has been utilised in solving the derived guaranteed cost surface from a convex optimisation procedure. Thus, as proven mathematically and through the computer simulations that the proposed US-SMC is robust with respect to system with matched and mismatched uncertainties, nonlinearities and disturbances, and at the same time eliminating the chattering control input signal prevalent in most other sliding mode control techniques.

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