Jurnal Teknologi

Full paper

On the Abelianization of a Torsion Free Crystallographic Group

Nor'ashiqin Mohd Idrus[a*], Nor Haniza Sarmin[b], Hazzirah Izzati Mat Hassim[b], Rohaidah Masri[a]

[a]Department of Mathematics, Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris, 35900 Tg. Malim, Perak, Malaysia
[b]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

*Corresponding author: norashiqin@fsmt.upsi.edu.my

Article history

Received :7 January 2014
Received in revised form :
3 July 2014
Accepted :15 August 2014

Graphical abstract

$1 \to L \to G \to P \to 1$

Abstract

A torsion free crystallographic group, which is also known as a Bieberbach group is a generalization of free abelian groups. It is an extension of a lattice group by a finite point group. The study of *n*-dimensional crystallographic group had been done by many researchers over a hundred years ago. A Bieberbach group has been characterized as a fundamental group of compact, connected, flat Riemannian manifolds. In this paper, we characterize Bieberbach groups with trivial center as exactly those with finite abelianizations. The abelianization of a Bieberbach group is shown to be finite if the center of the group is trivial.

Keywords: Torsion free; crystallographic group; Bieberbach group; abelianization

Abstrak

Satu kumpulan kristalografi bebas kilasan, yang juga dikenali sebagai kumpulan Bieberbach merupakan pengitlakan kumpulan abelan bebas. Ia adalah perluasan kepada kumpulan kekisi dengan kumpulan titik terhingga. Kajian mengenai kumpulan kristalografi berdimensi-*n* telah dijalankan oleh ramai penyelidik sejak beratus tahun dahulu. Satu kumpulan Bierberbach telah dicirikan sebagai satu kumpulan asas bagi manifold Riemannan rata yang padat dan berkait. Dalam kertas kerja ini, kami mencirikan kumpulan Bieberbach dengan pusat remeh sebagai kumpulan dengan abelanisasi terhingga. Telah ditunjukkan bahawa abelanisasi bagi kumpulan Bieberbach adalah terhingga jika pusat kumpulan tersebut adalah remeh.

Kata kunci: Kumpulan kristalografi; bebas kilasan; kumpulan Bieberbach; abelanisasi

1.0 INTRODUCTION

The study of *n*-dimensional crystallographic group particularly Bieberbach group had been done by many researchers over a hundred years ago. Farkas (1981) and Hiller (1986) completed the characterization of Bieberbach group by showing that a Bieberbach group is a torsion free crystallographic group G that fits into the short exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow P \longrightarrow 1$$

where P is a point group that is a finite group acting faithfully on a maximal normal free abelian subgroup L of G which is of finite rank. The subgroup L is called a lattice group. It follows that L is a Fitting subgroup of G and its rank or Hirsch length is referred to as the dimension of G. A crystallographic group is used in the mathematical approach in solving the problem involving the structure of a crystal by replacing the crystal pattern. Hence, any new properties or characterization concerning crystallographic groups, particularly Bieberbach groups might lead to new exploration of the groups by not only mathematicians but by physicists and chemists too.

Auslander and Lyndon (1955), Auslander and Kuranishi (1957) and Szczepanski (1996) characterize a Bieberbach group as a fundamental group of compact, connected, flat Riemannian manifolds. Auslander and Lyndon (1955) have also characterized a Bieberbach group in term of its center and the finiteness of its point group. Malfait and Szczepanski (2003) characterized Bieberbach groups in terms of the finiteness of the outer automorphism of the groups. They gave necessary and sufficient conditions on outer automorphism of the groups to be infinite. Putryez (2007) characterized Bieberbach groups of dimension n (n odd) with point group $Z_2^{\,n-1}$ in terms of their commutator subgroup, lattice subgroup and the abelianization of the groups. He proved that for any *n*-dimensional ($n > 3$) Bieberbach group with point group $Z_2^{\,n-1}$, the commutator subgroup is equal to its lattice subgroup and hence the abelianization of the group is isomorphic to the point group itself. In addition, Basri *et al.* (2013) computed the abelianization of the finite metacyclic 2-

groups.

In this paper, we present a new characterization of any Bieberbach group with finite point group where the characterization is based on the structure of the abelianization of a centerless Bieberbach group.

2.0 PRELIMINARIES

In this section, some basic concepts and preliminary results that are used in computing the abelianization of a centerless Bieberbach group are given.

Definition 2.1 Hirsch Length (Hungerford, 1974) The Hirsch length of a polycyclic group is the number of infnite factors in a polycyclic series for the group. The Hirsch length of a group G is denoted by $h(G)$.

Definition 2.2 Lifting (Hungerford, 1974) If $\pi: G \to Q$ is surjective, then a lifting of $x \in Q$ is an element $l(x) \in G$ with $\pi(l(x)) = x$.

Lemma 2.1 (Segal, 1983) *Let G be an extension of two polycyclic groups K by N. Then the Hirsch length of G is the sum of the Hirsch length of K and N, namely,*

$$h(G) = h(K) + h(N).$$

From Lemma 2.1, we prove the following lemma to be used in finding the results.

Lemma 2.2 *Let N and M be free abelian groups with $N \le M$. Then $N \cong M$ if and only if M/N is finite.*

Proof. Let N and M be free abelian groups with $N \le M$. Hence N and M satisfy the exact sequence $1 \to N \to M \to M/N \to 1$. Suppose $N \cong M$. With the exact sequence above, Lemma 2.1 gives us that $h(M) = h(N) + h(M/N)$. Since $N \cong M$, we have $h(M) = h(N)$ and hence $h(M/N) = 0$. This gives us M/N is finite.

Now suppose M/N is finite. So we have $h(M/N) = 0$. Then, since $h(M) = h(N) + h(M/N)$, we have $h(M) = h(N)$. So this conclude that $M \cong N$.

Theorem 2.2 (Rotman, 1995) *Two free abelian groups are isomorphic if and only if they have the same rank.*

3.0 MAIN RESULTS

The main objective of this paper is to prove the following theorem.

Main Theorem. *Let G be any Bieberbach group with finite point group. The group G has trivial center if and only if the abelianization of G is finite.*

Some preparatory lemmas that are vital in proving the Main Theorem are listed in this section.

Lemma 3.1. *Let G be a Bieberbach group with non-trivial finite point group P and lattice group L of rank n and $\phi: G \to P$ is an epimorphism with kernel L. Let $\bar{a}$ be any lifting of any non-trivial element a of P. Then $\bar{a}$ is not in the center of G.*

Proof. *Let $\bar{a}$ be the lifting of $a \ne 1$ in P. By Definition 2.2, we have $\phi(\bar{a}) = a \ne 1$ for the epimorphism ϕ. Now suppose $\bar{a}$ is in the center of G and let $L' = \langle \bar{a}, L \rangle$. It follows that L' is normal in G since for $g \in G$, $\bar{a}^g = \bar{a} \in L'$. Since the lattice group L is a maximal normal abelian subgroup of G, we have $L' \le L$. So we have $\phi(\bar{a}) = 1$, a contradiction. Hence $\bar{a}$ cannot be in the center of G.* □

Lemma 3.2. *Let G be a Bieberbach group with non-trivial finite point group P and lattice subgroup L of rank n and $\phi: G \to P$ is an epimorphism with kernel L. Then the action of P on L in G is faithful.*

Proof. *Suppose the action of P on L is not faithful. Then there exists a non-trivial $\bar{a}$ of $a \ne 1$ in P such that $\bar{a}$ has trivial action on L by conjugation in G, that is $l^{\bar{a}} = l$ for all $l \in L$. So $\bar{a}$ commutes with all elements of L. Therefore as in the proof of Lemma 3.1, $L' = \langle \bar{a}, L \rangle$ is a normal abelian subgroup of G. Since L is a maximal normal abelian subgroup of G, then $L' \le L$. Hence we have $\phi(\bar{a}) = 1$. This contradicts the fact that $a \ne 1$. Hence the action of P on L is faithful.* □

Following Lemma 3.2, we have this corollary.

Corollary 3.3. *Let G be a Bieberbach group with non-trivial finite point group P and lattice subgroup L and $\phi: G \to P$ is an epimorphism with kernel L. Then $Z(G)$, the center of G, is a proper subgroup of L.*

Proof. *First we show that $Z(G)$ is a subset of L. Lemma 3.1 gives us that none of the non-trivial liftings of generators of P are in $Z(G)$. That is any non-trivial liftings cannot be in the kernel of ϕ. Hence all elements of $Z(G)$ are elements of L.*

Next we show that $Z(G) = L$. Then we have $\bar{a}l = l\bar{a}$ for all lifts $\bar{a}$ of a in P and for all l in L. Since P acts faithfully on L, this implies that for all $\bar{a}$, $\phi(\bar{a}) = a = 1$ and it follows that P is trivial. This contradicts the hypothesis that P is a nontrivial finite group. Hence $Z(G) \ne L$. This conclude that $Z(G)$ is a proper subgroup of L. □

Next we prove Lemma 3.4 through Lemma 3.6. These lemmas will support our proof in main theorem.

Lemma 3.4. *Let G be a Bieberbach group with non-trivial finite point group P and lattice subgroup L and $\phi: G \to P$ is an epimorphism with kernel L. Then*

$$(G' \cap L) \cap Z(G) = 1.$$

Proof. Let G be the Bieberbach group as mentioned above. Since we have both $(G' \cap L) \leq L$ by definition and $Z(G) \lneq L$ by Corollary 3.3, then $(G' \cap L) \cap Z(G) \leq L.$ To prove $(G' \cap L) \cap Z(G) = 1,$ we show that all elements of $G' \cap L$ cannot be in $Z(G)$. To show this, it is enough to show that it is true for an arbitrary generator of $G' \cap L$ since $G' \cap L$ is abelian. We first compute generators of $G' \cap L$. For $\bar{a}l = g \in G$ and $\bar{b}l' = h \in G$, $\bar{a}, \bar{b} \notin L$, $[g,h] \in G'$, where

$$\begin{aligned}[g,h] &= [\bar{a}, \bar{b}l'] \cdot [[\bar{a}, \bar{b}l'], l] \cdot [l, \bar{b}l'] \\ &= [\bar{a}, l'] \cdot [\bar{a}, \bar{b}] \cdot [[\bar{a}, \bar{b}], l'] \cdot [[\bar{a}, l'] \cdot [\bar{a}, \bar{b}] \cdot [[\bar{a}, \bar{b}], l'], l] \cdot \\ &\quad [l, l'] \cdot [l, \bar{b}] \cdot [[l, \bar{b}], l'] \\ &= [\bar{a}, l'] \cdot [\bar{a}, \bar{b}] \cdot [[\bar{a}, \bar{b}], l'] \cdot [[\bar{a}, l'] \cdot [\bar{a}, \bar{b}] \cdot [[\bar{a}, \bar{b}], l'], l] \cdot [l, \bar{b}].\end{aligned}$$

For $[g,h]$ to be in L, we take $[\phi(\bar{a}), \phi(\bar{b})] = 1$, so that we have $[\bar{a}, \bar{b}] \in L$. So we have

$$[g,h] = [\bar{a}, l'] \cdot [\bar{a}, \bar{b}] \cdot [l, \bar{b}] = [l', \bar{a}]^{-1} [\bar{a}, \bar{b}][l, \bar{b}] \in G' \cap L.$$

Hence,

$$G' \cap L = \left\langle [l, \bar{a}], [\bar{a}, \bar{b}] \,\middle|\, [\phi(\bar{a}), \phi(\bar{b})] = 1 \text{ in } P,\ l \in L,\ \bar{a}, \bar{b} \notin L \right\rangle.$$

We show that the arbitrary generators $[l, \bar{a}] \neq 1$ and $[\bar{a}, \bar{b}] \neq 1$ cannot be in $Z(G)$. The definition $[l, \bar{a}] = l^{-1} l^{\bar{a}}$ gives us that the action of $\bar{a}$ will not fix l, otherwise $[l, \bar{a}] = 1$. Suppose $[l, \bar{a}]$ is in $Z(G)$. Hence $[l, \bar{a}]^g = [l, \bar{a}]$ for all lifts $g \in G$. Particularly, we choose $g = \bar{a}$. Hence we have

$$[l, \bar{a}]^{\bar{a}} = (l^{-1} l^{\bar{a}})^{\bar{a}} = (l^{-1})^{\bar{a}} (l^{\bar{a}})^{\bar{a}} = [l, \bar{a}] = l^{-1} l^{\bar{a}}.$$

This gives us that $\bar{a}$ fixes l^{-1}. This is a contradiction since $\bar{a}$ does not fix l, and therefore cannot fix l^{-1}. Hence $[l, \bar{a}]$ is not in $Z(G)$. By definition, $[\bar{a}, \bar{b}] = \bar{a}^{-1} \bar{a}^{\bar{b}}$. This gives us that the action of $\bar{b}$ will not fix $\bar{a}$, otherwise $[\bar{a}, \bar{b}] = 1$. Suppose now, $[\bar{a}, \bar{b}]$ is $Z(G)$. Hence $[\bar{a}, \bar{b}]^g = [\bar{a}, \bar{b}]$ for all lifts g in G. We choose $g = \bar{b}$. Hence we have

$$[\bar{a}, \bar{b}]^{\bar{b}} = (\bar{a}^{-1} \bar{a}^{\bar{b}})^{\bar{b}} = (\bar{a}^{-1})^{\bar{b}} (\bar{a}^{\bar{b}})^{\bar{b}} = [\bar{a}, \bar{b}] = \bar{a}^{-1} \bar{a}^{\bar{b}}.$$

This gives us that $\bar{b}$ fixes $\bar{a}^{-1}$. This is a contradiction since $\bar{b}$ does not fix $\bar{a}$. Hence $[\bar{a}, \bar{b}]$ cannot be in $Z(G)$. So we conclude that $(G' \cap L) \cap Z(G) = 1.$ □

Lemma 3.5. Let G be a Bieberbach group with point group P and lattice subgroup L of rank n. Then

$$(G' \cap L) \times Z(G) \cong L.$$

Proof. Let G be the Bieberbach group with point group P and lattice subgroup L of rank n. The result is immediate if P is trivial, $G = L$ and hence G' is trivial and $Z(G) = L.$

Suppose P is not trivial. We have $(G' \cap L) \times Z(G)$ is a free abelian subgroup of G since the direct product of free abelian groups is free abelian. To show $(G' \cap L) \times Z(G) \cong L,$ by Theorem 2.2, it is enough to show that the rank of $(G' \cap L) \times Z(G)$ is equal to the rank of L. However the rank of the groups is given by the infinite factors in a polycyclic series for the groups, that is, the rank of a group G is equal to the Hirsch length of the group $h(G)$ hence we show that the Hirsch length of $(G' \cap L) \times Z(G)$ is equal to the Hirsch length of L. Corollary 3.3 gives us that $Z(G) \subsetneq L$, so we only need to show that $L/Z(G)$ has the same Hirsch length as $G' \cap L$. This is because $Z(G) \to L \to L/Z(G)$ gives

$$h(L) = h(Z(G)) + h\left(L/Z(G)\right).$$

If $h\left(L/Z(G)\right) = h(G' \cap L)$ and since by Lemma 3.4, we have $(G' \cap L) \cap Z(G) = 1,$ then

$$h(L) = h(G' \cap L) + h(Z(G)) = h((G' \cap L) \times Z(G))$$

as needed.

So we can assume that L contains no central elements and hence by Corollary 3.3, G is centerless. Given the rank of L is n, hence the $h(L) = n$. Let $\{l_1, \ldots, l_n\}$ be a basis for L. That is $L \cong \langle l_1 \rangle \times \ldots \times \langle l_n \rangle$, where each $\langle l_i \rangle$ is isomorphic to C_0, the infinite cyclic group. None of these basis elements are in the center, so there exists a g_i in G such that $l_i^{g_i} = \bar{l}_i$ for some $\bar{l}_i \neq l_i$ in L. Hence $\bar{l}_i$ and l_i are conjugate, so there exists a $\bar{g}_i$ in G such that $l_i = \bar{l}_i^{\bar{g}_i} = \bar{l}_i[\bar{l}_i, \bar{g}_i]$.

Since L is normal, we have $[\bar{l}_i, \bar{g}_i]$ is an element of L. Multiplying both sides with $\bar{l}_i^{-1}$, we obtain $\bar{l}_i^{-1} l_i = [\bar{l}_i, \bar{g}_i]$.

The goal now is to show that the generators $\bar{l}_i^{-1} l_i$ for $i \in \{1, ..., n\}$ form a basis for a subgroup of $G' \cap L$. As the indices of $\bar{l}_i^{-1} l_i$ are from the indices of a basis of L, these generators have the same number as the dimension of L. Thus they span $G' \cap L$. So we only need to show that they are independent,

We are given that l_i for $i \in \{1, \ldots, n\}$ is a basis but $l_i = \bar{l}_i^{\bar{g}_i}$ and hence $\bar{l}_i^{\bar{g}_i} = \bar{l}_i[\bar{l}_i, \bar{g}_i]$ for $i \in \{1, \ldots, n\}$ is a basis for L. None of the $\bar{l}_i$ equal l_i since each $\bar{g}_i$ was picked not to commute with l_i.

For each $l_i \in L$, $\bar{l}_i$ can be written uniquely as a product of the basis elements $l_1, l_2, \ldots, l_n$, that is for each $\alpha_j \in$ □,

$$\bar{l}_i = \prod_{j=1}^{n} (l_j)^{\alpha_j} = (l_1)^{\alpha_1} (l_2)^{\alpha_2} \ldots (l_n)^{\alpha_n}.$$

Hence we have:

$$\bar{l}_i^{-1} l_i = (l_1)^{-\alpha_1} (l_2)^{-\alpha_2} \ldots (l_i)^{-\alpha_i + 1} \ldots (l_n)^{-\alpha_n}.$$

In other words, each $\bar{l}_i^{-1} l_i$ for $i \in \{1, \ldots, n\}$ can be written *uniquely as a product of basis elements* l_i. *Hence they are independent. So* $\bar{l}_i^{-1} l_i$ *for* $i \in \{1, \ldots, n\}$ *form a basis for a subgroup of* $G' \cap L$. *So with this, we proved that* $h\left(L / Z(G) \right) = h\,(G' \cap L)$ *and hence* $(G' \cap L) \times Z(G) \cong L$ *also holds when* P *is not trivial.* □

Lemma 3.6. *Let* G *be a Bieberbach group with lattice subgroup* L *and with any finite point group* P. *The abelianization of* G *is finite if and only if* $G' \cap L$ *is isomorphic to* L.

Proof. *Suppose* $G' \cap L$ *is isomorphic to* L. *Since we also have* $G' \cap L \le L$, *hence* $L / (G' \cap L)$ *is finite by Lemma 2.1. Let* $g \in G$ *and for some integer* m, *we have* $g^m \in L$ *since* $G/L \cong P$ *is finite. So for some integer* k, *we would have* $(g^m)^k$ *is in* $G' \cap L$ *since from above* $L / (G' \cap L)$ *is finite. But* $(g^m)^k = g^{mk}$, *hence this gives us that* $G / (G' \cap L)$ *is finite. Moreover we have* $G' \cap L \le G'$ *and since* $G / (G' \cap L)$ *is finite, then* G/G' *is finite.*

Now suppose G/G' *is finite. We show that* $L / (G' \cap L)$ *is finite. Let* $l \in L$, $l \notin (G' \cap L)$, *then for some integer* k, *we have* $l^k \in G'$ *since* G/G' *is finite. So we have* $l^k \in G' \cap L$. *This gives us that* $L / (G' \cap L)$ *is finite. Since we have* $G' \cap L \le L$, *then by Lemma 2.1,* $G' \cap L \cong L$ *as needed.* □

The proof of our main theorem is given in the following:

Proof of Main Theorem *Suppose the subgroup* G/G' *of* G *is finite, then by Lemma 3.6, we have* $G' \cap L$ *is isomorphic to* L. *Hence by Lemma 3.5,* $Z(G)$ *is trivial. Suppose now the center* $Z(G)$ *of* G *is trivial. By Lemma 3.5, the subgroup* $G' \cap L$ *is isomorphic to* L. *Hence by Lemma 3.6, the subgroup* G/G' *is finite.* □

■4.0 CONCLUSION

In this paper, we characterized any Bieberbach group with finite point group based on the structure of the abelianization of a centerless Bieberbach group. We proved that any Bieberbach group with finite point group has trivial center if and only if its abelianization is finite.

Acknowledgement

The authors would like to thank Ministry of Education (MOE) and Universiti Pendidikan Sultan Idris for the financial support through the Research Acculturation Collaborative Effort (RACE) grant Vote No. 2012-0144-101-62 from Universiti Pendidikan Sultan Idris and Vote No. 00M04 from Universiti Teknologi Malaysia.

References

[1] L. Auslander, M. Kuranishi. 1957. *Annals of Mathematics (3)*. 65: 411.
[2] M. Auslander, R. C. Lyndon. 1955. *American Journal of Mathematics*. 77: 929.
[3] A. M. Basri, N. H. Sarmin, N. M. Mohd Ali, J. R. Beuerle. 2013. *International Journal of Applied Mathematics and Statistics*. 45(15): 150.
[4] B. Eick, W. Nickel. 2002. *Polycyclic-Computing with Polycyclic Groups*. A GAP package.
[5] D. R. Farkas. 1981. *Journal of Mathematics*. 11: 511.
[6] H. Hiller. 1986. *The American Mathematical Monthly*. 93: 765.
[7] T. W. Hungerford. 1974. *Graduate Texts in Mathematics: Algebra*. New York: Springer-Verlag.
[8] W. Malfait, A. Szczepanski. 2003. *Compositio Mathematica*. 135: 89.
[9] B. Putryez. 2007. *Journal of Group Theory*. 10: 401.
[10] J. J. Rotman. 1995. *An Introduction to The Theory of Groups*. 4th ed. New York: Springer-Verlag.
[11] D. Segal. 1983. *Polycyclic Groups*. Cambridge: Cambridge University Press.
[12] A. Szczepanski. 1996. *Bull. Belg. Math. Soc. (3)*. 3: 585.