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EXPLICIT COMPUTATION OF COMMUTATOR SUBGROUPS IN SIX-DIMENSIONAL TORSION-FREE BIEBERBACH GROUPS WITH QUATERNION POINT GROUP OF ORDER EIGHT

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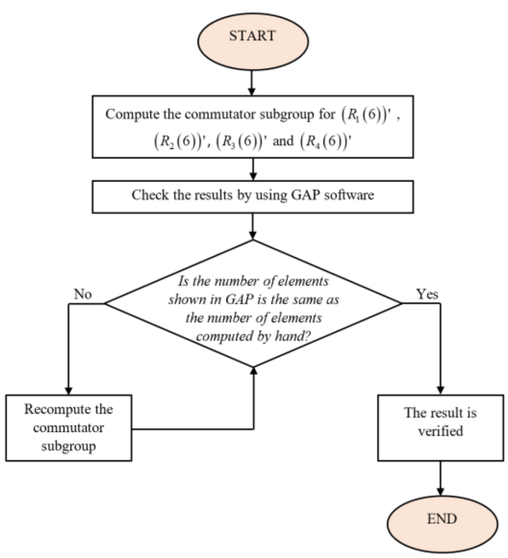
Graphical abstract



Abstract

The findings in the study of crystal structure and their properties, also known as crystallography, have many importance in practical applications. Based on a mathematical approach, several information on the crystallographic groups can be extracted by explicating their algebraic properties. One of the important parameters to compute the algebraic properties is by finding the commutator subgroup. This paper focuses on the Bieberbach groups, which are one of the torsion-free crystallographic groups. The Bieberbach groups of dimension six with the quaternion point group of order eight were found to be isomorphic to four polycyclic groups. However, previous studies have shown that the number of elements of the commutator subgroup for the first and second group, which consists of five elements, is found to be inaccurate with the aid of Groups, Algorithms and Programming (GAP) software. Furthermore, the commutator subgroup for the third and fourth group is yet to be computed. Thus, this paper includes an update on the computation of the commutator subgroup for the first and second groups, and the commutator subgroup for the third as well as the fourth group will be shown.

Keywords: Crystallography, Bieberbach group, polycyclic groups, commutator subgroup, quaternion point group

Abstrak

Penemuan dalam kajian struktur hablur dan sifatnya yang juga dikenali sebagai kristalografi, mempunyai banyak kepentingan dalam aplikasi praktikal. Berdasarkan pendekatan matematik, beberapa maklumat tentang kumpulan kristalografi boleh diekstrak dengan menghuraikan sifat algebra kumpulan tersebut. Salah satu parameter penting untuk mengira sifat algebra adalah dengan mencari subkumpulan penukar tertib. Kertas kerja ini memberi tumpuan kepada kumpulan Bieberbach, yang merupakan salah satu kumpulan kristalografi bebas kilasan. Kumpulan Bieberbach berdimensi enam dengan kumpulan titik kuaternion peringkat lapan didapati berisomorfisma kepada empat kumpulan polisiklik. Walau bagaimanapun, kajian lepas menunjukkan bahawa bilangan unsur subkumpulan penukar tertib bagi kumpulan pertama dan kedua yang terdiri daripada lima unsur didapati tidak tepat dengan bantuan perisian Kumpulan, Algoritma dan Pengaturcaraan (GAP). Tambahan pula, subkumpulan penukar tertib untuk kumpulan ketiga dan keempat masih belum dikira. Oleh itu, kertas kerja ini merangkum kemas kini pengiraan subkumpulan penukar tertib untuk kumpulan pertama dan kedua, dan subkumpulan penukar tertib untuk kumpulan ketiga serta keempat akan ditunjukkan.

Kata kunci: Kristalografi, kumpulan Bieberbach, kumpulan polisiklik, subkumpulan penukar tertib, titik kumpulan kuaternion

1.0 INTRODUCTION

For many years, the study of crystal structure and its properties has been carried out, as there are some interrelations with fields such as physics, chemistry, material science, and others. Crystal can be viewed as a repetition of atoms, also known as the basis, in a regular periodic pattern of points (or lattice points) [1]. The formation of a crystal structure is based on the different crystallographic space groups, which are defined by the symmetrical properties of a crystal. According to Newnham [2], there are four primary types of symmetry: translation, reflection, rotation, and inversion, where these symmetry operators transform a point with respective coordinates into a new position. Since the study of crystals is done at a macroscopic level, the study depends on Neumann's principle, which states "the symmetry elements of any physical property of a crystal must include the symmetry elements of the point group of the crystal." [3]. In this study, the quaternion representations in the form of symmetrical operations (i.e., quaternion rotation, quaternion identity and inversion, inverse rotation, and quaternion mirror plane) are used as the crystallographic point groups, as they can be directly applied to vectors in the crystal reference sets of axes for any crystallographic system [4]. Moreover, the use of quaternions in crystallographic studies offers numerous advantages and acts as a better alternative in the exploration of crystal properties.

This study also focuses on one of the crystallographic subgroups, which is the Bieberbach group of dimension six. A group *G* is said to be a Bieberbach if the group satisfies an exact sequence,

$$L \to G \to P \qquad (1)$$

where *L* is the lattice group of rank n, which consists of all translational shifts that describe *G*, and *P* represents the point group that describes the symmetrical operators [5]. Based on the exact sequence, it can be assumed that *G* is being built based on L and P.

The configuration and the properties of such a group can be further studied by inducing mathematical abstraction into the crystallographic studies. For years, group theorists have studied the homological invariants for various types of groups, such as the nonabelian tensor square, Schur multiplier, and Whitehead functor. Most homological invariants require the commutator subgroup (also known as the derived subgroup) to be computed. According to Robinson [6], the commutator subgroup is a group generated by all commutators in the form of

$$G' = \langle [g,h] \mid g,h \in G \rangle, \text{ where } [g,h] = g^{-1}h^{-1}gh \qquad (2)$$

Some of the previous studies that deal with the commutator subgroup can be found in Heras and Fernández-Alcober [7] and Shamsaki *et al.* [8]. In [7], the authors proved the capability in the construction of 2-generators commutator subgroups in a commutator form for a finite *p*-group. Then, the study in [8] presents the improved theorems in determining the order of the Schur multiplier of p-groups with the commutator subgroup of maximal order.

Kumar et al. [9] have also studied the presentation of the commutator subgroups for the virtual braid group, virtual twin group, and virtual triple group, which are the extensions of the symmetric group S_n. It is found that the quotient group between the group and their commutative subgroup is crystallographic for $n \geq 2$ and $n \geq 4$, respectively.

There are also several studies on the computation of the homological invariants of a group using the computation of the commutator subgroup as the building blocks in the computation process. One of which is the computation of a group abelianization, where it is defined as the quotient group *G*/[*G*, *G*] and is abelian [10, 11]. The computation result for both commutator subgroup and its abelianization can generalize some of these results:

(i) the homological invariants up to six-dimensional Bieberbach groups with abelian point group, cyclic point group of order two, three, and five [12, 13], and
(ii) the homological invariants of *n*-dimensional Beberbach groups with a symmetric point group of order six [14, 15].

Therefore, this study focuses on exploring the behaviour and the complexity of the Bieberbach groups of higher dimension with another nonabelian point group, the quaternion point group. By focusing on the Bieberbach groups, Mohammad *et al.* [16] state that there are four Bieberbach groups, which are denoted as $R_n(6)$ where n = 1, 2, 3, and 4, which are isomorphic to the quaternion point group. Mohammad [17] has successfully explicated the homological invariants for the first Bieberbach group of dimension six with the quaternion extension, $R_1(6)$. The explication of the homological invariants, such as the nonabelian tensor square of the group, used the results of the elements within the commutator subgroup, as there exists a homomorphism $\kappa : G \otimes G \to G'$ [18].

Initially, the study presents 72 commutator elements of the commutator subgroup. The simplification of the elements is done to eliminate repeated, inverse, and identity commutators. By using several theorems, the computation is done by hand. It is found that $\left(R_1(6)\right)'$ contain five commutators listed as the elements of the commutator subgroup. Then, A Rahman *et al.* [19] extend the study by Mohammad [17] by working on the computation of the commutator subgroup for the second Bieberbach group of dimension six with the quaternion extension, denoted as $\left(R_2(6)\right)'$. Both $\left(R_1(6)\right)'$ and $\left(R_2(6)\right)'$ are found to have five elements.

In this study, the assistance from Groups, Algorithms and Programming (GAP) software is needed to complement the results of $\left(R_1(6)\right)'$ and $\left(R_2(6)\right)'$. This is because GAP software can list the number of elements in the commutator subgroup by displaying the order of each element, but not the elements themselves. However, the results from the GAP software show otherwise, where there are 11 elements for both $\left(R_1(6)\right)'$ and $\left(R_2(6)\right)'$. Hence, this study aims to recompute $\left(R_1(6)\right)'$ and $\left(R_2(6)\right)'$, as well as finding the commutator elements of the commutator subgroup for the remaining groups (i.e., n = 3 and 4).

2.0 METHODOLOGY

In this section, the theorems and basic definitions used to generate the elements of the commutator subgroup are presented. The results of this study are then discussed in the latter part.

2.1 Preliminaries

In 2018, Mohammad [17] used the algorithms developed by Blyth & Morse [20] to compute the homological invariants for $R_1(6)$. Before the computation is done, it is a necessary step suggested by Blyth & Morse to transform the matrix representation of a group into a polycyclic presentation. The polycyclic presentation needs to be a consistent presentation, in the sense that every generator must be able to be computed in the form of conjugation.

Initially, all four Bieberbach groups of dimension six with the quaternion extension, denoted as $G_n(6)$ where n = 1, 2, 3, and 4, are generated by eight generators in the form of matrices that consist of 2 basis generators (denoted as r_0 and r_1), and six lattice generators ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, and ℓ_6. Under isomorphic mapping γ, these groups are found to be isomorphic to the newly polycyclic group which is generated by eight generators with an additional arbitrary generator, that is $\gamma : G_n(6) \to R_n(6)$. Let the mapping $\gamma(r_0) = a$ and $\gamma(r_1) = a$. Thus, the consistent polycyclic presentation for all four $R_n(6)$ are given in the following four theorems:

Theorem 1 [17]
Let $G_1 = \langle r_0, r_1, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \rangle$ be the first Bieberbach group of dimension six with the quaternion point group of order eight, then the polycyclic presentation of $R_1(6)$ is established as:

$$
\begin{aligned}
R_1(6) = \langle a, b, c, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \mid\ & a^2 = c\ell_6, b^2 = c, \\
& c^2 = \ell_5^{-1}\ell_6^{-1}, b^a = bc^{-1}\ell_5^{-1}, c^a = c, c^b = c, \\
& \ell_1^a = \ell_3, \ell_1^b = \ell_4, \ell_1^c = \ell_1^{-1}, \ell_2^a = \ell_4, \ell_2^b = \ell_3^{-1}, \\
& \ell_2^c = \ell_2^{-1}, \ell_3^a = \ell_1^{-1}, \ell_3^b = \ell_2, \ell_3^c = \ell_3^{-1}, \\
& \ell_4^a = \ell_2^{-1}, \ell_4^b = \ell_1^{-1}, \ell_4^c = \ell_4^{-1}, \ell_5^a = \ell_5, \\
& \ell_5^b = \ell_6, \ell_5^c = \ell_5, \ell_6^a = \ell_6, \ell_6^b = \ell_5, \ell_6^c = \ell_6, \\
& \ell_2^{\ell_1} = \ell_2, \ell_3^{\ell_1} = \ell_3, \ell_4^{\ell_1} = \ell_4, \ell_5^{\ell_1} = \ell_5, \\
& \ell_6^{\ell_1} = \ell_6, \ell_3^{\ell_2} = \ell_3, \ell_4^{\ell_2} = \ell_4, \ell_5^{\ell_2} = \ell_5, \\
& \ell_6^{\ell_2} = \ell_6, \ell_4^{\ell_3} = \ell_4, \ell_5^{\ell_3} = \ell_5, \ell_6^{\ell_3} = \ell_6, \\
& \ell_5^{\ell_4} = \ell_5, \ell_6^{\ell_4} = \ell_6, \ell_6^{\ell_5} = \ell_6, \ell_2^{\ell_1^{-1}} = \ell_2, \\
& \ell_3^{\ell_1^{-1}} = \ell_3, \ell_4^{\ell_1^{-1}} = \ell_4, \ell_5^{\ell_1^{-1}} = \ell_5, \ell_6^{\ell_1^{-1}} = \ell_6, \\
& \ell_3^{\ell_2^{-1}} = \ell_3, \ell_4^{\ell_2^{-1}} = \ell_4, \ell_5^{\ell_2^{-1}} = \ell_5, \ell_6^{\ell_2^{-1}} = \ell_6, \\
& \ell_4^{\ell_3^{-1}} = \ell_4, \ell_5^{\ell_3^{-1}} = \ell_5, \ell_6^{\ell_3^{-1}} = \ell_6, \ell_5^{\ell_4^{-1}} = \ell_5, \\
& \ell_6^{\ell_4^{-1}} = \ell_6, \ell_6^{\ell_5^{-1}} = \ell_6 \rangle.
\end{aligned}
\tag{3}
$$

Theorem 2 [21]
Let $G_2 = \langle r_0, r_1, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \rangle$ be the second Bieberbach group of dimension six with the quaternion

point group of order eight, then the polycyclic presentation of $R_2(6)$ is established as:

$$\begin{aligned}
R_2(6)=\langle & a,b,c,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6 \mid a^2=c\ell_6, b^2=c,\\
& c^2=\ell_5^{-1}\ell_6^{-1}, b^a=bc^{-1}\ell_5^{-1}, c^a=c, c^b=c,\\
& \ell_1^{a}=\ell_3, \ell_1^{b}=\ell_1\ell_2^{-1}\ell_4, \ell_1^{c}=\ell_1^{-1}, \ell_2^{a}=\ell_1\ell_3\ell_4,\\
& \ell_2^{b}=\ell_1\ell_2^{-1}\ell_3^{-1}, \ell_2^{c}=\ell_2^{-1}, \ell_3^{a}=\ell_1^{-1},\\
& \ell_3^{b}=\ell_2\ell_3\ell_4, \ell_3^{c}=\ell_3^{-1}, \ell_4^{a}=\ell_1\ell_2^{-1}\ell_3^{-1},\\
& \ell_4^{b}=\ell_1^{-1}\ell_3^{-1}\ell_4^{-1}, \ell_4^{c}=\ell_4^{-1}, \ell_5^{a}=\ell_5, \ell_5^{b}=\ell_6,\\
& \ell_5^{c}=\ell_5, \ell_6^{a}=\ell_6, \ell_6^{b}=\ell_5, \ell_6^{c}=\ell_6, \ell_2^{\ell_1}=\ell_2,\\
& \ell_3^{\ell_1}=\ell_3, \ell_4^{\ell_1}=\ell_4, \ell_5^{\ell_1}=\ell_5, \ell_6^{\ell_1}=\ell_6,\\
& \ell_3^{\ell_2}=\ell_3, \ell_4^{\ell_2}=\ell_4, \ell_5^{\ell_2}=\ell_5, \ell_6^{\ell_2}=\ell_6,\\
& \ell_4^{\ell_3}=\ell_4, \ell_5^{\ell_3}=\ell_5, \ell_6^{\ell_3}=\ell_6, \ell_5^{\ell_4}=\ell_5,\\
& \ell_6^{\ell_4}=\ell_6, \ell_6^{\ell_5}=\ell_6, \ell_2^{\ell_1^{-1}}=\ell_2, \ell_3^{\ell_1^{-1}}=\ell_3,\\
& \ell_4^{\ell_1^{-1}}=\ell_4, \ell_5^{\ell_1^{-1}}=\ell_5, \ell_6^{\ell_1^{-1}}=\ell_6, \ell_3^{\ell_2^{-1}}=\ell_3,\\
& \ell_4^{\ell_2^{-1}}=\ell_4, \ell_5^{\ell_2^{-1}}=\ell_5, \ell_6^{\ell_2^{-1}}=\ell_6, \ell_4^{\ell_3^{-1}}=\ell_4,\\
& \ell_5^{\ell_3^{-1}}=\ell_5, \ell_6^{\ell_3^{-1}}=\ell_6, \ell_5^{\ell_4^{-1}}=\ell_5, \ell_6^{\ell_4^{-1}}=\ell_6,\\
& \ell_6^{\ell_5^{-1}}=\ell_6\rangle.
\end{aligned} \tag{4}$$

Theorem 3 [21]

Let $G_3=\langle r_0,r_1,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6\rangle$ be the third Bieberbach group of dimension six with the quaternion point group of order eight, then the polycyclic presentation of $R_3(6)$ is established as:

$$\begin{aligned}
R_3(6)=\langle & a,b,c,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6 \mid a^2=c\ell_3^{-1}\ell_6, b^2=c,\\
& c^2=\ell_5^{-1}\ell_6^{-1}, b^a=\ell_4^{-1}\ell_5b^{-1}c^2, c^a=a^2\ell_6^{-1}\ell_1^{-1},\\
& c^b=c, \ell_1^{a}=\ell_3, \ell_1^{b}=\ell_1\ell_2^{-1}\ell_4, \ell_1^{c}=\ell_1^{-1},\\
& \ell_2^{a}=\ell_1\ell_3\ell_4, \ell_2^{b}=\ell_1\ell_2^{-1}\ell_3^{-1}, \ell_2^{c}=\ell_2^{-1},\\
& \ell_3^{a}=\ell_1^{-1}, \ell_3^{b}=\ell_2\ell_3\ell_4, \ell_3^{c}=\ell_3^{-1},\\
& \ell_4^{a}=\ell_1\ell_2^{-1}\ell_3^{-1}, \ell_4^{b}=\ell_1^{-1}\ell_3^{-1}\ell_4^{-1}, \ell_4^{c}=\ell_4^{-1},\\
& \ell_5^{a}=\ell_5, \ell_5^{b}=\ell_6, \ell_5^{c}=\ell_5, \ell_6^{a}=\ell_6, \ell_6^{b}=\ell_5,\\
& \ell_6^{c}=\ell_6, \ell_2^{\ell_1}=\ell_2, \ell_3^{\ell_1}=\ell_3, \ell_4^{\ell_1}=\ell_4,\\
& \ell_5^{\ell_1}=\ell_5, \ell_6^{\ell_1}=\ell_6, \ell_3^{\ell_2}=\ell_3, \ell_4^{\ell_2}=\ell_4,\\
& \ell_5^{\ell_2}=\ell_5, \ell_6^{\ell_2}=\ell_6, \ell_4^{\ell_3}=\ell_4, \ell_5^{\ell_3}=\ell_5,\\
& \ell_6^{\ell_3}=\ell_6, \ell_5^{\ell_4}=\ell_5, \ell_6^{\ell_4}=\ell_6, \ell_6^{\ell_5}=\ell_6,\\
& \ell_2^{\ell_1^{-1}}=\ell_2, \ell_3^{\ell_1^{-1}}=\ell_3, \ell_4^{\ell_1^{-1}}=\ell_4, \ell_5^{\ell_1^{-1}}=\ell_5,\\
& \ell_6^{\ell_1^{-1}}=\ell_6, \ell_3^{\ell_2^{-1}}=\ell_3, \ell_4^{\ell_2^{-1}}=\ell_4, \ell_5^{\ell_2^{-1}}=\ell_5,\\
& \ell_6^{\ell_2^{-1}}=\ell_6, \ell_4^{\ell_3^{-1}}=\ell_4, \ell_5^{\ell_3^{-1}}=\ell_5, \ell_6^{\ell_3^{-1}}=\ell_6,\\
& \ell_5^{\ell_4^{-1}}=\ell_5, \ell_6^{\ell_4^{-1}}=\ell_6, \ell_6^{\ell_5^{-1}}=\ell_6\rangle.
\end{aligned} \tag{5}$$

Theorem 4 [21]

Let $G_4=\langle r_0,r_1,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6\rangle$ be the fourth Bieberbach group of dimension six with the quaternion point group of order eight, then the polycyclic presentation of $R_4(6)$ is established as:

$$\begin{aligned}
R_4(6)=\langle & a,b,c,\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6 \mid a^2=\ell_2\ell_5c, b^2=c,\\
& c^2=\ell_1^{-1}\ell_3\ell_5^{-1}\ell_6, b^a=\ell_6^{-1}b^{-1}c^2, c^a=\ell_1^{-1}\ell_2b^2,\\
& c^b=c, \ell_1^{a}=\ell_4, \ell_1^{b}=\ell_3, \ell_1^{c}=\ell_1^{-1}, \ell_2^{a}=\ell_3,\\
& \ell_2^{b}=\ell_4^{-1}, \ell_2^{c}=\ell_2^{-1}, \ell_3^{a}=\ell_2^{-1}, \ell_3^{b}=\ell_1^{-1},\\
& \ell_3^{c}=\ell_3^{-1}, \ell_4^{a}=\ell_1^{-1}, \ell_4^{b}=\ell_2, \ell_4^{c}=\ell_4^{-1},\\
& \ell_5^{a}=\ell_1\ell_3^{-1}\ell_5, \ell_5^{b}=\ell_2^{-1}\ell_3^{-1}\ell_6^{-1},\\
& \ell_5^{c}=\ell_1\ell_2\ell_3^{-1}\ell_4\ell_5, \ell_6^{a}=\ell_2\ell_4\ell_6,\\
& \ell_6^{b}=\ell_2^{-1}\ell_3\ell_5^{-1}, \ell_6^{c}=\ell_1^{-1}\ell_2\ell_3\ell_4\ell_6, \ell_2^{\ell_1}=\ell_2,\\
& \ell_3^{\ell_1}=\ell_3, \ell_4^{\ell_1}=\ell_4, \ell_5^{\ell_1}=\ell_5, \ell_6^{\ell_1}=\ell_6, \ell_3^{\ell_2}=\ell_3,\\
& \ell_4^{\ell_2}=\ell_4, \ell_5^{\ell_2}=\ell_5, \ell_6^{\ell_2}=\ell_6, \ell_4^{\ell_3}=\ell_4,\\
& \ell_5^{\ell_3}=\ell_5, \ell_6^{\ell_3}=\ell_6, \ell_5^{\ell_4}=\ell_5, \ell_6^{\ell_4}=\ell_6,\\
& \ell_6^{\ell_5}=\ell_6, \ell_2^{\ell_1^{-1}}=\ell_2, \ell_3^{\ell_1^{-1}}=\ell_3, \ell_4^{\ell_1^{-1}}=\ell_4,\\
& \ell_5^{\ell_1^{-1}}=\ell_5, \ell_6^{\ell_1^{-1}}=\ell_6, \ell_3^{\ell_2^{-1}}=\ell_3, \ell_4^{\ell_2^{-1}}=\ell_4,\\
& \ell_5^{\ell_2^{-1}}=\ell_5, \ell_6^{\ell_2^{-1}}=\ell_6, \ell_4^{\ell_3^{-1}}=\ell_4, \ell_5^{\ell_3^{-1}}=\ell_5,\\
& \ell_6^{\ell_3^{-1}}=\ell_6, \ell_5^{\ell_4^{-1}}=\ell_5, \ell_6^{\ell_4^{-1}}=\ell_6, \ell_6^{\ell_5^{-1}}=\ell_6\rangle.
\end{aligned} \tag{6}$$

As mentioned in the previous section, the computation of the homological invariants of a group requires the commutator subgroup to be listed in the commutator form, which satisfies the following definition:

Definition 1 [6]

For elements *x* and *y* in a group *G*, the conjugate of *x* by *y* is $x^y=y^{-1}xy$, whereas the commutator of x and *y* is $[x,y]=x^{-1}x^y$.

The commutator subgroup is then generated, respectively, based on (3), (4), (5), and (6). The computation to simplify and eliminate identity and repeated terms is carried out. Mohammad [17] has presented the commutator subgroup for $R_1(6)$. Five commutator elements are listed as the commutator subgroup, and it is presented in the following lemmas:

Lemma 1 [17]

The commutator subgroup of $R_1(6)$ is

$$\begin{aligned}(R_1(6))'=\{&[a,b],[a,\ell_1],[a,\ell_3],[b,\ell_4],\\ &[b,\ell_5]\}\end{aligned} \tag{7}$$

where each of the commutators is given by

$$(R_1(6))'=\{c\ell_5,\ \ell_1\ell_3^{-1},\ \ell_1\ell_3,\ \ell_1\ell_4,\ \ell_5\ell_6^{-1}\}. \tag{8}$$

In 2024, A Rahman et al. [19] have presented the computation for all 72 commutators for $(R_2(6))'$ where the simplified result is shown in the following lemma:

Lemma 2 [19]

The commutator subgroup of $R_2(6)$ is

$$(R_2(6))' = \{[a, b], [a, \ell_1], [a, \ell_3], [\ell_3, b], [\ell_6, b]\} \quad (9)$$

where each of the commutators is given by

$$(R_2(6))' = \{c\ell_5, \ell_1\ell_3^{-1}, \ell_1\ell_3, \ell_2\ell_4, \ell_5\ell_6^{-1}\}. \quad (10)$$

The validity of both lemmas is verified by using GAP software. The GAP software can generate the number of elements in the commutator subgroup with their respective order. However, the elements of the commutator subgroup are not shown. In addition, the number of elements of the commutator subgroup does not correspond to the number of commutator subgroups listed in Lemma 1 and Lemma 2. This implies that both lemmas are listed inaccurately.

3.0 RESULTS AND DISCUSSION

The commutator subgroups for $R_1(6)$, $R_2(6)$, $R_3(6)$, and $R_4(6)$ are written in the form of commutators as in Definition 1. A total of 72 possible commutators can be formed as the elements in the commutator subgroup for all four groups. The commutator subgroup is presented as follows:

$$\begin{aligned}&(R_1(6))' = (R_2(6))' = (R_3(6))' = (R_4(6))' = \\ &\{[a, b], [a, c], [b, c], [a, \ell_1], [a, \ell_2], \\ &[a, \ell_3], [a, \ell_4], [a, \ell_5], [a, \ell_6], [b, \ell_1], \\ &[b, \ell_2], [b, \ell_3], [b, \ell_4], [b, \ell_5], [b, \ell_6], \\ &[c, \ell_1], [c, \ell_2], [c, \ell_3], [c, \ell_4], [c, \ell_5], \\ &[c, \ell_6], [b, a], [c, a], [c, b], [\ell_1, a], \\ &[\ell_2, a], [\ell_3, a], [\ell_4, a], [\ell_5, a], [\ell_6, a], \\ &[\ell_1, b], [\ell_2, b], [\ell_3, b], [\ell_4, b], [\ell_5, b], \\ &[\ell_6, b], [\ell_1, c], [\ell_2, c], [\ell_3, c], [\ell_4, c], \\ &[\ell_5, c], [\ell_6, c], [\ell_1, \ell_2], [\ell_1, \ell_3], [\ell_1, \ell_4], \\ &[\ell_1, \ell_5], [\ell_1, \ell_6], [\ell_2, \ell_3], [\ell_2, \ell_4], [\ell_2, \ell_5], \\ &[\ell_2, \ell_6], [\ell_3, \ell_4], [\ell_3, \ell_5], [\ell_3, \ell_6], [\ell_4, \ell_5], \\ &[\ell_4, \ell_6], [\ell_5, \ell_6], [\ell_2, \ell_1], [\ell_3, \ell_1], [\ell_4, \ell_1], \\ &[\ell_5, \ell_1], [\ell_6, \ell_1], [\ell_3, \ell_2], [\ell_4, \ell_2], [\ell_5, \ell_2], \\ &[\ell_6, \ell_2], [\ell_4, \ell_3], [\ell_5, \ell_3], [\ell_6, \ell_3], [\ell_5, \ell_4], \\ &[\ell_6, \ell_4], [\ell_6, \ell_5]\}.\end{aligned} \quad (11)$$

3.1 Commutator Subgroup for $R_1(6)$

The computation is done by expanding the commutator into the conjugation form. By relating both the conjugation form of the commutator and the polycyclic presentation, the formation of the commutators, as in (8), is obtained. Some of the computations that lead to the result of one and inverse commutators are eliminated from the subgroup. Thus, $(R_1(6))'$ is then reduced to only consisting of five elements, as in Lemma 1. However, GAP shows different results for $(R_1(6))'$.

As mentioned in the previous section, all four Bieberbach groups of dimension six consist of eight generators and are initially given in matrix form. Firstly, every matrix is inserted into GAP and declared as a group named `G1` by the following prompt:

```
gap> l1 := [
> [4 / 4,0,0,0,0,0,0,4 / 4],
> [0,4 / 4,0,0,0,0,0,0],
> [0,0,4 / 4,0,0,0,0,0],
> [0,0,0,4 / 4,0,0,0,0],
> [0,0,0,0,4 / 4,0,0,0],
> [0,0,0,0,0,4 / 4,0,0],
> [0,0,0,0,0,0,4 / 4]];

gap> l2 := [
> [4 / 4,0,0,0,0,0,0,0],
> [0,4 / 4,0,0,0,0,0,4 / 4],
> [0,0,4 / 4,0,0,0,0,0],
> [0,0,0,4 / 4,0,0,0,0],
> [0,0,0,0,4 / 4,0,0,0],
> [0,0,0,0,0,4 / 4,0,0],
> [0,0,0,0,0,0,0,4 / 4]];


gap> l3 := [
> [4 / 4,0,0,0,0,0,0,0],
> [0,4 / 4,0,0,0,0,0,0],
> [0,0,4 / 4,0,0,0,4 / 4],
> [0,0,0,4 / 4,0,0,0,0],
> [0,0,0,0,4 / 4,0,0,0],
> [0,0,0,0,0,4 / 4,0,0],
> [0,0,0,0,0,0,0,4 / 4]];

gap> l4 := [
> [4 / 4,0,0,0,0,0,0,0],
> [0,4 / 4,0,0,0,0,0,0],
> [0,0,4 / 4,0,0,0,0,0],
> [0,0,0,4 / 4,0,0,4 / 4],
> [0,0,0,0,4 / 4,0,0,0],
> [0,0,0,0,0,4 / 4,0,0],
> [0,0,0,0,0,0,0,4 / 4]];

gap> l5 := [
> [4 / 4,0,0,0,0,0,0,0],
> [0,4 / 4,0,0,0,0,0,0],
> [0,0,4 / 4,0,0,0,0,0],
> [0,0,0,4 / 4,0,0,0,0],
> [0,0,0,0,4 / 4,0,4 / 4],
> [0,0,0,0,0,4 / 4,0,0],
> [0,0,0,0,0,0,0,4 / 4]];

gap> l6 := [
> [4 / 4,0,0,0,0,0,0,0],
> [0,4 / 4,0,0,0,0,0,0],
> [0,0,4 / 4,0,0,0,0,0],
> [0,0,0,4 / 4,0,0,0,0],
> [0,0,0,0,4 / 4,0,0,0],
> [0,0,0,0,0,4 / 4,4 / 4],
> [0,0,0,0,0,0,0,4 / 4]];
```

```
gap> r0 := [
> [0/4,0/4,4/4,0/4,0/4,0/4,0/4],
> [0/4,0/4,0/4,4/4,0/4,0/4,0/4],
> [-4/4,0/4,0/4,0/4,0/4,0/4,0/4],
> [0/4,-4/4,0/4,0/4,0/4,0/4,0/4],
> [0/4,0/4,0/4,0/4,4/4,0/4,-1/4],
> [0/4,0/4,0/4,0/4,0/4,4/4,1/4],
> [0/4,0/4,0/4,0/4,0/4,0/4,4/4]];

gap> r1 := [
> [0/4,0/4,0/4,4/4,0/4,0/4,0/4],
> [0/4,0/4,-4/4,0/4,0/4,0/4,0/4],
> [0/4,4/4,0/4,0/4,0/4,0/4,0/4],
> [-4/4,0/4,0/4,0/4,0/4,0/4,0/4],
> [0/4,0/4,0/4,0/4,0/4,4/4,-2/4],
> [0/4,0/4,0/4,0/4,4/4,0/4,0/4],
> [0/4,0/4,0/4,0/4,0/4,0/4,4/4]];

gap> G1 := Group(r0,r1,l1,l2,l3,l4,l5,l6 );
<matrix group with 8 generators>
```

Since the polycyclic presentation is consistent as in Theorem 1, the isomorphism of `G1` needs to be transformed onto polycyclically presented groups (pcp-groups) before finding the commutator subgroup. The following prompt shows how the transformation is done and the findings of the commutator subgroups.

```
gap> Gpcp:=Image(IsomorphismPcpGroup(G1));
Pcp-group with orders [ 2, 4, 3, 3, 3, 3, 3,
3, 0, 0, 0, 0, 0, 0 ]
gap> Gd:=DerivedSubgroup(Gpcp);
Pcp-group with orders [ 2, 3, 3, 3, 3, 3, 0,
0, 0, 0, 0 ]
```

The GAP output showed that the commutator subgroup for $R_1(6)$ consists of 11 elements, one of which has the order of two, five of them have the order of three, and another five elements have the order of zero. Notice that the results show the drawbacks of using GAP, where only the order for each element is given, but not the elements themselves. According to the elements of $(R_1(6))'$ listed in Mohammad [17], five commutators in the last phase of simplification are computed inaccurately.

In the last phase of the computations, the study shows that some commutators are made up of possible multiplication between two commutators within the commutator subgroups. However, that is not the case for several commutators, for instance $[b, \ell_3] = \ell_2^{-1}\ell_3$. The commutator is said to be simplified from the multiplication between the inverse of both $[a, \ell_2] = \ell_2\ell_4^{-1}$ and $[b, \ell_4] = \ell_1\ell_4$ with $[a, \ell_3] = \ell_1\ell_3$. However, the simplification of $[a, \ell_2]$ also includes the multiplication between $[b, \ell_3]$ and other commutators. This implies redundancy in the computation. Thus, both commutators should not be excluded from the commutator subgroup. By removing the redundant computation, it is found that it has the following commutator subgroup:

Theorem 5

The commutator subgroup of $R_1(6)$ is

$$(R_1(6))' = \{[a, b], [a, \ell_1], [a, \ell_2], [a, \ell_3], [a, \ell_4], [b, \ell_1], [b, \ell_2], [b, \ell_3], [b, \ell_4], [b, \ell_5], [b, \ell_6]\} \quad (12)$$

where each of the commutators is given by

$$(R_1(6))' = \{c\ell_5, \ell_1\ell_3^{-1}, \ell_2\ell_4^{-1}, \ell_1\ell_3, \ell_2\ell_4, \ell_1\ell_4^{-1}, \ell_2\ell_3, \ell_2^{-1}\ell_3, \ell_1\ell_4, \ell_5\ell_6^{-1}, \ell_5^{-1}\ell_6\}. \quad (13)$$

3.2 Commutator Subgroup for $R_2(6)$

The computation of the commutator subgroup depends on the group's respective polycyclic presentation, even though the number of commutator elements is the same. The computation is done by expanding the commutator into the conjugation form. By relating both the conjugation form of the commutator and the polycyclic presentation, the formation of the commutators, as in (10), is obtained.

For $(R_2(6))'$, it is found that 42 commutators are the identity element (i.e., resulting in 1), 15 commutators are the inverse of other commutators, and the remaining commutators are made up of a multiplication of different commutators. Thus, $(R_2(6))'$ is then reduced to only consisting of five elements, as in Lemma 2. However, similar to $R_1(6)$, the GAP shows different results for $(R_2(6))'$. By using a similar approach and GAP prompt as the one that has been done for $R_1(6)$, the GAP shows the following output:

```
gap> Gpcp:=Image(IsomorphismPcpGroup(G2));
Pcp-group with orders [ 2, 4, 3, 3, 3, 3, 3,
3, 0, 0, 0, 0, 0, 0 ]
gap> Gd:=DerivedSubgroup(Gpcp);
Pcp-group with orders [ 2, 3, 3, 3, 3, 3, 0,
0, 0, 0, 0 ]
```

Similarly, $(R_2(6))'$ also consists of 11 elements, where one of which has order two, five of them have order of three, and another five elements have the order of zero. According to the elements of $(R_2(6))'$ listed in A Rahman *et al.* [19], five commutators in the last phase of simplification are computed inaccurately.

For example, the element $[\ell_2, b] = \ell_1\ell_2^{-2}\ell_3$ is said to be made up of the combination of $[a, \ell_1] = \ell_1\ell_3^{-1}$,

$[\ell_3, b]^{-1} = (\ell_2\ell_4)^{-1}$ and $[b, \ell_1]^{-1} = (\ell_2\ell_4^{-1})^{-1}$. However, the simplification of $[b, \ell_1]$ contains the combination of $[\ell_2, b]$ and $[\ell_4, a]$. This shows that the computation forms a redundancy, which implies that it is unnecessary for both commutators $[\ell_2, b]$ and $[b, \ell_1]$ be eliminated from the list of elements in the commutator subgroup. Thus, by choosing the suitable inverse commutator and eliminating redundant computation, the commutator subgroup of is given as follows:

Theorem 6

The commutator subgroup of $R_2(6)$ is

$$(R_2(6))' = \{[a, b], [a, \ell_1], [a, \ell_2], [a, \ell_3], [a, \ell_4], [b, \ell_1], [b, \ell_2], [b, \ell_3], [b, \ell_4], [b, \ell_5], [b, \ell_6]\} \quad (14)$$

where each of the commutators is given by

$$(R_2(6))' = \{c\ell_5, \ell_1\ell_3^{-1}, \ell_1^{-1}\ell_2\ell_3^{-1}\ell_4^{-1}, \ell_1\ell_3, \ell_1^{-1}\ell_2\ell_3\ell_4, \ell_2\ell_4^{-1}, \ell_1^{-1}\ell_2^{2}\ell_3, \ell_2^{-1}\ell_4^{-1}, \ell_1\ell_3\ell_4^{2}, \ell_5\ell_6^{-1}, \ell_5^{-1}\ell_6\}. \quad (15)$$

3.3 Commutator Subgroup for $R_3(6)$

In this section, the computation of the commutator subgroup for the third Bieberbach group of dimension six with the quaternion extension is presented. As shown in (11), the initial number of elements in the commutator subgroup for $R_3(6)$ are the same as the second Bieberbach group of dimension six with the quaternion extension. The commutator subgroup for $R_3(6)$ is given in the following theorem.

Theorem 7

The commutator subgroup of $R_3(6)$ is

$$(R_3(6))' = \{[a, b], [a, c], [a, \ell_1], [a, \ell_2], [a, \ell_4], [b, \ell_1], [b, \ell_2], [b, \ell_3], [b, \ell_4], [b, \ell_5], [b, \ell_6]\} \quad (16)$$

where each of the commutators is given by

$$(R_3(6))' = \{c^{-1}\ell_1^{-1}\ell_3^{-1}\ell_4^{-1}\ell_6^{-1}, \ell_1\ell_3, \ell_1\ell_3^{-1}, \ell_1^{-1}\ell_2\ell_3^{-1}\ell_4^{-1}, \ell_1^{-1}\ell_2\ell_3\ell_4, \ell_2\ell_4^{-1}, \ell_1^{-1}\ell_2^{2}\ell_3, \ell_2^{-1}\ell_4^{-1}, \ell_1\ell_3\ell_4^{2}, \ell_5\ell_6^{-1}, \ell_5^{-1}\ell_6\}. \quad (17)$$

Proof. The computation is done by using the polycyclic presentation in Theorem 3 and the conjugation between generators. Some of the computations on the simplification of each generator are shown as follows:

(i) To show $[a, b] = c^{-1}\ell_1^{-1}\ell_3^{-1}\ell_4^{-1}\ell_6^{-1}$.

By right conjugation, $b^a = a^{-1}ba$. By (5) as in Theorem 3, $b^a = \ell_4^{-1}\ell_5 b^{-1}c^2$. Relating both equations will imply to $a^{-1}ba = \ell_4^{-1}\ell_5 b^{-1}c^2$. Multiplying both sides of the equation by $a^{-1}b^{-1}a$ will give $1 = \ell_4^{-1}\ell_5 b^{-1}c^2 a^{-1}b^{-1}a$. Then, multiplying both sides by b leads to $b = \ell_4^{-1}\ell_5 b^{-1}c^2 a^{-1}b^{-1}ab$. Thus, $c^{-2}b\ell_5^{-1}\ell_4 b = a^{-1}b^{-1}ab$ or it can be simplified as $c^{-2}b\ell_5^{-1}\ell_4 b = [a, b]$. Thus, by rearranging the generators based on their polycyclic presentation, we have $[a, b] = c^{-1}\ell_1^{-1}\ell_3^{-1}\ell_4^{-1}\ell_6^{-1}$.

(ii) To show $[a, c] = \ell_1\ell_3$.

By right conjugation, $c^a = a^{-1}ca$. By (5) as in Theorem 3, $c^a = a^2\ell_6^{-1}\ell_1^{-1}$. Relating both expressions will imply to $a^{-1}ca = a^2\ell_6^{-1}\ell_1^{-1}$. By a similar approach in the previous computation, we have $[a, c] = \ell_1\ell_3$.

The same method of computation is applied to the remaining 70 commutators. There are 38 commutators resulting in 1, acting as the identity commutator, where 30 of them involve commutators with lattice generators. Those identity generators are excluded from the commutator subgroup. The computation for one of the commutators, which leads to 1, is shown as follows:

(iii) To show $[b, c] = 1$.

By right conjugation, $c^b = b^{-1}cb$. By (5) as in Theorem 3, $c^b = c$. Relating both expressions will imply to $b^{-1}cb = c$. Multiplying both sides of the equation by $b^{-1}c^{-1}$ will give $b^{-1} = cb^{-1}c^{-1}$. Then, multiplying both sides by b leads to $1 = b^{-1}c^{-1}bc$ and by (2), $[b, c] = b^{-1}c^{-1}bc$. Thus, $[b, c] = 1$

Some of the commutators are made up of the inverse of other commutators. It is also possible that any of the commutators is made up from the combination of other commutators.

(iv) To show $[c, \ell_1] = \ell_1^{2}$.

By right conjugation, $\ell_1^{c} = c^{-1}\ell_1 c$. By (5) as in Theorem 3, $\ell_1^{c} = \ell_1^{-1}$. Relating both expressions will imply to $c^{-1}\ell_1 c = \ell_1^{-1}$. By multiplying both sides by $c^{-1}\ell_1^{-1}$,

$c^{-1} = \ell_1^{-1}c^{-1}\ell_1^{-1}$. Next, multiplying both sides by $c\ell_1$ and ℓ_1 during the latter part, the equation will be $\ell_1^2 = c^{-1}\ell_1^{-1}c\ell_1 = [c, \ell_1]$. However, the simplification of commutators $[a, c]$ and $[a, \ell_1]$ lead to the result of $\ell_1\ell_3$ and $\ell_1\ell_3^{-1}$, respectively. By multiplying both commutators, the results will be ℓ_1^2. Thus, it implies that $[a, c]$ and $[a, \ell_1]$ able to form $[c, \ell_1]$. Hence, $[c, \ell_1]$ is eliminated from the list of the commutator subgroup.

(v) To show $[c, a] = \ell_1^{-1}\ell_3^{-1}$.

By the same approach in simplifying $[a, c]$, the aim now is to express $[c, a]$ as $c^{-1}a^{-1}ca$ instead of $[a, c] = a^{-1}c^{-1}ac$. So, $a^{-1}ca = a^2\ell_6^{-1}\ell_1^{-1}$. Multiplying c^{-1} to both side resulting to $c^{-1}a^{-1}ca = c^{-1}a^2\ell_6^{-1}\ell_1^{-1}$. By (5) as in Theorem 3, $a^2 = c\ell_3^{-1}\ell_6$. Substituting a^2 into the previous equation resulting to $[c, a] = c^{-1}c\ell_3^{-1}\ell_6\ell_6^{-1}\ell_1^{-1} = \ell_3^{-1}\ell_1^{-1}$. Thus, it implies that $[c, a] = [a, c]^{-1}$ and $[c, a]$ can be eliminated in the commutator subgroup.

By simplifying the remaining commutators, the commutator subgroup can be reduced to have 11 independent results. The validity of the result is checked by GAP, and it produced the same output (i.e., the commutator subgroup of $R_3(6)$ have 11 elements) with the same order as $R_1(6)$, and $R_2(6)$. ∎

3.4 Commutator Subgroup for $R_4(6)$

The computation for the fourth Bieberbach group of dimension six with the quaternion extension is shown in this section. By referring to (11), $(R_4(6))'$ consists of 72 possible commutators which some of which can be excluded from the list. The commutator subgroup for $R_4(6)$ is given in the following theorem.

Theorem 8

The commutator subgroup of $R_4(6)$ is

$$(R_4(6))' = \{[a, b], [a, c], [a, \ell_1], [a, \ell_2], [a, \ell_3], [a, \ell_4], [a, \ell_5], [a, \ell_6], [b, \ell_2], [b, \ell_3], [b, \ell_5]\} \quad (18)$$

where each of the commutators is given by

$$(R_4(6))' = \{c^{-1}\ell_2^{-1}\ell_3\ell_5^{-1}, \ell_1^{-1}\ell_2, \ell_1\ell_4^{-1}, \ell_2\ell_3^{-1}, \ell_2\ell_3, \ell_1\ell_4, \ell_1^{-1}\ell_3, \ell_2^{-1}\ell_4, \ell_2\ell_4, \ell_1\ell_3, \ell_2\ell_3\ell_5\ell_6\}. \quad (19)$$

Proof. The computation is done by using the polycyclic presentation in Theorem 4 and the conjugation between generators. Some of the computations on the simplification of each generator are shown as follows:

(i) To show $[a, b] = c^{-1}\ell_2^{-1}\ell_3\ell_5^{-1}$.

By right conjugation, $b^a = a^{-1}ba$. By (6) as in Theorem 4, $b^a = \ell_6^{-1}b^{-1}c^2$. Relating both expressions will imply to $a^{-1}ba = l_6^{-1}b^{-1}c^2$. By rearranging the generators for both sides of the equations, we have $[a, b] = c^{-1}\ell_2^{-1}\ell_3\ell_5^{-1}$.

(ii) To show $[a, c] = \ell_1^{-1}\ell_2$.

By right conjugation, $c^a = a^{-1}ca$. By (6) as in Theorem 4, $c^a = \ell_1^{-1}\ell_2b^2$. Relating both expressions will imply to $a^{-1}ca = \ell_1^{-1}\ell_2b^2$. By using a similar approach as shown in the previous computation, we have $[a, c] = \ell_1^{-1}\ell_2$.

(iii) To show $[a, \ell_2] = \ell_2\ell_3^{-1}$.

By right conjugation, $\ell_2^a = a^{-1}\ell_2a$. By (6) as in Theorem 4, $\ell_2^a = \ell_3$. Relating both expressions will imply to $a^{-1}\ell_2a = \ell_3$. Multiplying both sides of the equation by $a^{-1}\ell_2^{-1}a$ will give $1 = \ell_3a^{-1}\ell_2^{-1}a$. Then, multiplying both sides by ℓ_3^{-1} and ℓ_2 leads to $\ell_2\ell_3^{-1} = a^{-1}\ell_2^{-1}a\ell_2$ and by (2), $[a, \ell_2] = a^{-1}\ell_2^{-1}a\ell_2$. Thus, $[a, \ell_2] = \ell_2\ell_3^{-1}$.

The same method of computation is applied to the remaining 70 commutators. There are 32 commutators resulting in 1, acting as the identity commutator, where 30 of them involve commutators with lattice generators. Those identity generators are excluded from the commutator subgroup. The computation for one of the commutators, which leads to 1, is shown as follows:

(iv) To show $[b, c] = 1$.

By right conjugation, $c^b = b^{-1}cb$. By (6) as in Theorem 4, $c^b = c$. Relating both expressions will imply to $b^{-1}cb = c$. Multiplying both sides of the equation by $b^{-1}c^{-1}$ will give $b^{-1} = cb^{-1}c^{-1}$. Then, multiplying both sides by b leads to $1 = b^{-1}c^{-1}bc$ and by (2), $[b,c] = b^{-1}c^{-1}bc$. Thus, $[b,c] = 1$.

Some of the commutators are made up of the inverse of other commutators. It is also possible that any of the commutators is made up from the combination of other commutators.

(v) To show $[c, \ell_1] = \ell_1^2$.

By right conjugation, $\ell_1^c = c^{-1}\ell_1 c$. By (6) as in Theorem 4, $\ell_1^c = \ell_1^{-1}$. Relating both expressions will imply to $c^{-1}\ell_1 c = \ell_1^{-1}$. By multiplying both sides by $c^{-1}\ell_1^{-1}$, $c^{-1} = \ell_1^{-1}c^{-1}\ell_1^{-1}$. Next, multiplying both sides by $c\ell_1$ and ℓ_1 during the latter part, the equation will be $\ell_1^2 = c^{-1}\ell_1^{-1}c\ell_1 = [c, \ell_1]$. However, the simplification of commutators $[a, \ell_1]$ and $[a, \ell_4]$ lead to the result of $\ell_1\ell_4^{-1}$ and $\ell_1\ell_4$, respectively. By multiplying both commutators, the results will be ℓ_1^2. Thus, it implies that $[a, \ell_1]$ and $[a, \ell_4]$ able to form $[c, \ell_1]$. Hence, $[c, \ell_1]$ is eliminated from the list of the commutator subgroup.

(vi) To show $[b, \ell_6] = \ell_2\ell_3^{-1}\ell_5\ell_6$.

By right conjugation, $\ell_6^b = b^{-1}\ell_6 b$. By (6) as in Theorem 4, $\ell_6^b = \ell_2^{-1}\ell_3\ell_5^{-1}$. Relating both expressions will imply to $b^{-1}\ell_6 b = \ell_2^{-1}\ell_3\ell_5^{-1}$. By multiplying both sides by ℓ_6^{-1}, $\ell_6^{-1}b^{-1}\ell_6 b = \ell_6^{-1}\ell_2^{-1}\ell_3\ell_5^{-1}$. By (2), it can be expressed as $[\ell_6, b] = \ell_2^{-1}\ell_3\ell_5^{-1}\ell_6^{-1}$. However, inversing both sides of the equation, $\left(\ell_6^{-1}b^{-1}\ell_6 b\right)^{-1} = \left(\ell_2^{-1}\ell_3\ell_5^{-1}\ell_6^{-1}\right)^{-1}$, it leads to $b^{-1}\ell_6^{-1}b\ell_6 = \ell_2\ell_3^{-1}\ell_5\ell_6$, and thus, $[b, \ell_6] = \ell_2\ell_3^{-1}\ell_5\ell_6$. This implies that $[\ell_6, b] = [b, \ell_6]^{-1}$. In addition to this, the simplification of commutators $[a, \ell_2]$, $[a, \ell_3]$ and $[b, \ell_5]$ lead to the result of $\ell_2\ell_3^{-1}$, $\ell_2\ell_3$ and $\ell_2\ell_3\ell_5\ell_6$, respectively. Thus, $[b, \ell_6]$ can be expressed as $[b, \ell_5][a, \ell_3]^{-1}[a, \ell_2]$. In this case, both $[\ell_6, b]$ and $[b, \ell_6]$ are eliminated as the elements in $\left(R_4(6)\right)'$ as both commutators are formed from other commutators.

The validity of the result is checked by GAP, and it produced the same output (i.e., the commutator subgroup of $R_4(6)$ have 11 elements) with the same order as the previous three groups. ■

4.0 CONCLUSION

Since the explication of the homological invariants of a particular group, for instance, the group that has been transformed into a polycyclic presentation, uses commutator calculus as part of the computation, the correct version of the commutator subgroup will be useful to ease the computing process. Furthermore, for a group with a higher dimension, by hand computation in finding the specified elements of the homological invariants can be tedious. Thus, by eliminating and excluding repeated terms and identity terms in the commutator subgroup allow one to explicate the homological invariants efficiently.

The computation of the commutator subgroup for the Bieberbach groups of dimension six with the quaternion extension, $R_n(6)$, where n = 1, 2, 3, and 4, yields a similar number of commutator elements with different formations of generators. With computer aid, this study was able to find the correct number of commutator elements for each commutator subgroup after the elements were simplified. However, we would like to highlight the limited capabilities of GAP software. In this case, GAP is only capable of presenting the order for each element in the group, but not the elements themselves. Hence, the computation by hand is also needed to further understand the construction, as well as the configuration for each commutator element that is listed in the commutator subgroup. It is recommended that the GAP algorithm be constructed to identify the exact elements for the orders that have been presented. Future researchers may also use the commutator subgroup to explicate the homological invariants of a group, such as the nonabelian tensor square, the Schur multiplier, and the exterior square.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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