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Full Paper

BIFURCATION ANALYSIS OF TEMPERATURE EFFECTS IN LAKE EUTROPHICATION

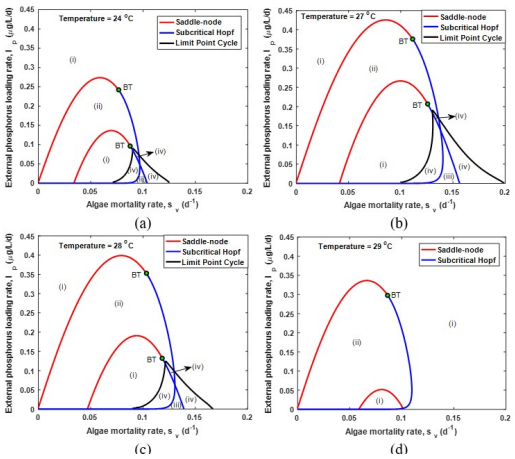
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Graphical abstract



Abstract

Over the past 20 years, rising global temperatures have accelerated the biological processes in lake systems and potentially intensify the lake eutrophication. This poses a challenge to establishing sustainable lake management strategies. To address this issue, this paper develops an algae-phosphorus model which incorporates the temperature effects to assess how warming influences lake eutrophication dynamics. We incorporate differential equations in formulating the model. A eutrophic lake, Tasik Harapan is chosen as the case study. Both XPPAUT and MatCont are used in conducting co-dimension one and co-dimension two bifurcation analyses. The bifurcation results revealed that rising temperatures would lead to an abrupt increase in algae concentration, shifting the lake to eutrophic state. Regime shift thresholds of external phosphorus loading are identified at various temperatures. A higher flushing rate is required to restore the lake when the temperature increases. Variation in algae mortality rate at different temperatures would lead to the occurrence of bistability and Hopf bifurcations. In two-parameter space of external phosphorus loading and algae mortality rates, variation in temperature causes various model dynamics, including saddle-node bifurcation, Bogdanov-Takens bifurcation and Hopf bifurcation. The complex dynamics and regime shift thresholds simulated from the model can guide early interventions to prevent irreversible degradation.

Keywords: Lake dynamics; temperature; regime shift; tropical lake; Bogdanov-Takens bifurcation

Abstrak

Sepanjang 20 tahun yang lalu, peningkatan suhu global telah mempercepatkan proses biologi dalam sistem tasik dan berpotensi meningkatkan eutrofikasi tasik. Ini menimbulkan cabaran untuk mewujudkan strategi pengurusan tasik yang mampan. Untuk menangani isu ini, makalah ini membangunkan model alga-fosforus yang menggabungkan kesan suhu untuk menilai bagaimana pemanasan mempengaruhi dinamik eutrofikasi tasik. Kami menggabungkan persamaan pembezaan dalam merumuskan model. Tasik eutrofik, Tasik Harapan dipilih sebagai kajian kes. Kedua-dua XPPAUT dan MatCont digunakan dalam menjalankan analisis percabangan dimensi satu dan dimensi dua. Keputusan analisis percabangan mendedahkan bahawa peningkatan suhu akan membawa kepada peningkatan mendadak dalam kepekatan

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alga, mengalihkan tasik kepada keadaan eutrofik. Ambang anjakan rejim pemuatan fosforus luaran dikenal pasti pada pelbagai suhu. Kadar penyiraman yang lebih tinggi diperlukan untuk memulihkan tasik apabila suhu meningkat. Perubahan dalam kadar kematian alga pada suhu yang berbeza akan membawa kepada berlakunya kebolehbiruan dan percabangan Hopf. Dalam ruang dua parameter pemuatan fosforus luaran dan kadar kematian alga, variasi suhu menyebabkan pelbagai model dinamik, termasuk percabangan nod pelana, percabangan Bogdanov-Takens dan percabangan Hopf. Dinamik kompleks dan ambang anjakan rejim yang disimulasikan daripada model boleh membimbing intervensi awal untuk mengelakkan kemerosotan yang tidak dapat dipulihkan.

***Kata kunci*:** Dinamik tasik, suhu, peralihan rejim, tasik tropika, percabangan Bogdanov-Takens

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1.0 INTRODUCTION

Lake eutrophication is a critical environmental issue, characterised by excessive nutrient enrichment, mainly nitrogen and phosphorus, that leads to excessive algae and plant growth, oxygen depletion, and degradation of water quality [1]. This process disrupts aquatic ecosystems, resulting in biodiversity loss and economic damage, particularly in fisheries and tourism, while also increasing water treatment costs [2]. There is also a risk to human health, as toxins from harmful algal blooms such as cyanobacteria can cause illness [3]. Although eutrophication is a natural process, it is often accelerated by specific factors such as nutrient load, primarily from agricultural runoff, wastewater discharge, industrial effluents, and climate change [1]. Signs of eutrophication include water becoming green or murky with taste, odour, and quality of water being severely affected.

Phosphorus is an essential nutrient for aquatic ecosystems [4]. It promotes the algae growth and other aquatic plants, forming the foundation of the food web. In balanced quantities, phosphorus helps to maintain the health and productivity of a lake. Lakes with low phosphorus levels typically feature clear water and support a diverse range of aquatic life. Conversely, high phosphorus levels result in uncontrolled algae growth. This rapid algae proliferation causes the water to become cloudy, obstructing sunlight and leading to the death of aquatic plants. The decomposition of algae depletes oxygen and creates dead zones [5].

Lakes are commonly classified as either temperate or tropical lakes based on their climatic zones. Temperate lakes are found in regions with distinct seasons, experiencing significant seasonal temperature variations [6]. In contrast, tropical lakes are located within the tropical climate zone and the lakes experience warm temperatures throughout the year [6]. Most studies on lake eutrophication have focused on temperate lakes, while tropical lakes have received less attention. This has resulted in a significant gap in our understanding of tropical lakes, especially given their distinct thermal conditions compared to temperate systems.

Bifurcation analysis aims to understand how a system's behavior changes when a parameter is varied [7]. In lake research, various bifurcations such as saddle-node, transcritical and Hopf bifurcations are related with the transitions of different lake trophic status [8-12]. In transcritical bifurcation, intersection of a stable state and an unstable state occurs. The stable state becomes unstable while unstable state change to stable [8]. Saddle-node bifurcation is often associated with the occurrence of regime shift in a lake, where a clean lake suddenly shifts to a turbid and eutrophic lake when a certain nutrient level is crossed [9-12]. Hopf bifurcation occurs when an equilibrium becomes unstable and a periodic solution emerges from it [8], [11], [12].

Global warming has raised surface temperatures over the past few decades, impacting lake ecosystems. The increasing temperatures accelerate biological and chemical processes in aquatic systems, particularly in lakes. Elevated temperatures have intensified carbon dioxide emissions by enhancing microbial respiration and primary production [13]. Additionally, higher temperatures hasten phosphorus release by stimulating microbial processes [4]. Elevated temperatures promote algal blooms by speeding up metabolic and photosynthetic rates while reducing oxygen levels [14,15]. This phenomenon exacerbates lake eutrophication.

Wang et al. [4] developed a mathematical model to simulate phosphorus interactions and transport at the lake sediment, incorporating temperature as a key factor. The diffusion rate of phosphorus varies with temperature and is modelled using the Arrhenius equation. Additionally, microbial phosphorus uptake rates are also formulated as temperature-dependent. In another study, [13] presented a mathematical model to explore the interactions

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between eutrophication, temperature, and carbon dioxide fluxes in shallow lakes. The rate of microbial respiration and the carbon dioxide flux equation depend on temperature. Salk et al. [14] developed a mathematical model that simulates phytoplankton growth under conditions of warming and nutrient enrichment. Temperature influences the rate of phytoplankton biomass change and nutrient dynamics.

Based on published literature, most of the studies on the temperature effects are conducted for temperate lakes [17,18]. Limited research focuses on the impact of temperature in tropical lakes. The knowledge on managing temperature effects in temperate lakes might not be applicable to tropical lakes. Further, limited studies considered bifurcation analysis to illustrate the temperature effects on tropical lake dynamics. This paper aims to formulate an algae-phosphorus model that incorporates the effects of temperature, to investigate the impact of temperature on lake eutrophication dynamics, and to provide insights into lake management through bifurcation results. The case study focuses on Tasik Harapan, a tropical lake experiencing severe eutrophication. Both XPPAUT and MatCont are employed to analyse lake dynamics in one-parameter space and two-parameter space. The thresholds identified in the bifurcation results are crucial in lake management under future warming scenarios. We organize the paper as follows: introduction, mathematical model, model analysis results, discussion, and concluding remarks.

2.0 MATHEMATICAL MODEL

This study focuses on Tasik Harapan which located at Universiti Sains Malaysia. The depth of Tasik Harapan is ranging from 1.0 to 1.5 m. The surface area is 6,070 m^2. The lake temperature for Tasik Harapan is 30°C. We formulated a mathematical model which comprises of algae and phosphorus for Tasik Harapan, as shown in Equations (1) and (2). Table 1 displays the parameters used in the model.

$$\frac{dA}{dt} = e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2} bA\frac{P-p_aA}{h_a+P-p_aA} - hA - \theta_1^{T-20}gA - \theta_2^{T-20}sA$$

(Equation 1)

$$\frac{dP}{dt} = l_p + re^{0.099(T-20)}\frac{P^q}{P^q+n^q} + \theta_1^{T-20}e_pgA - e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2} bA\frac{P-p_aA}{h_a+P-p_aA} - hP$$

(Equation 2)

2.1 Nonnegativity of Algae Growth Function

In the algae-phosphorus model, we consider the Monod function for algae growth, $bA(P - p_aA)/(h_a + P - p_aA)$. In this function, $P - p_aA$ represents the remaining phosphorus in the lake that is available for algae growth. For the model, we assume the initial conditions $P(0) \geq p_aA(0)$. Then, we define Equation 3:

$$Q(t) = P(t) - p_aA(t)$$

(Equation 3)

Differentiate Equation (3) to get Equation (4):

$$\frac{dQ}{dt} = \frac{d}{dt}\left[P(t) - p_aA(t)\right] = \frac{dP}{dt} - p_a\frac{dA}{dt}$$

(Equation 4)

Substitute Equations (1) and (2) into Equation (4), we have Equation (5):

$$\begin{aligned}\frac{dQ}{dt} &= \left[\begin{array}{l} l_p + re^{0.099(T-20)}\frac{P^q}{P^q+n^q} + \theta_1^{T-20}e_pgA - \\ e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2} bA\frac{P-p_aA}{h_a+P-p_aA} - hP \end{array}\right] \\ &\quad - p_a\left[e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2} bA\frac{P-p_aA}{h_a+P-p_aA} - hA - \theta_1^{T-20}gA - \theta_2^{T-20}sA \right] \\ &= l_p + re^{0.099(T-20)}\frac{P^q}{P^q+n^q} + \theta_1^{T-20}e_pgA - hQ \\ &\quad -(1+p_a)e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2} bA\frac{Q}{h_a+Q} + p_a\left(\begin{array}{l} hA + \theta_1^{T-20}gA \\ +\theta_2^{T-20}sA \end{array}\right)\end{aligned}$$

(Equation 5)

When Q→0 (remaining phosphorus in lake almost zero), we have Equation (6):

$$\begin{aligned}\frac{dQ}{dt} &\approx l_p + re^{0.099(T-20)}\frac{P^q}{P^q+n^q} + \theta_1^{T-20}e_pgA - h(0) \\ &\quad -(1+p_a)e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2} bA\frac{0}{h_a+0} + p_a\left(\theta_1^{T-20}gA + \theta_2^{T-20}sA\right) \\ &= l_p + re^{0.099(T-20)}\frac{P^q}{P^q+n^q} + \theta_1^{T-20}e_pgA + p_a\left(+\theta_1^{T-20}gA + \theta_2^{T-20}sA\right) \geq 0\end{aligned}$$

(Equation 6)

If Q approaches zero, we have $dQ/dt \geq 0$. This suggests that $Q(t) = 0$ is the lower bound and Q is nonnegative. Hence, Equation (7) implies the nonnegativity of the algae growth function:

$$P(t) - p_aA(t) \geq 0 \text{ for all } t \geq 0, \text{ given } P(0) - p_aA(0) \geq 0$$

(Equation 7)

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2.2 Temperature Functions on Biological Processes

To determine the temperature influences on lake eutrophication, temperature functions are incorporated on biological processes in the lake, including internal phosphorus recycling, algae growth, zooplankton grazing and algal mortality rates.

The term $e^{-2.3\left(\frac{T_{opt}-T}{T_{opt}-T_{\min}}\right)^2}$ is included in the model to capture the algae growth pattern at various temperatures. This term is estimated to fit the algae growth pattern in the laboratory studies by [19]. The choice of a rational function in this term will produce an algae growth rate which changes with temperature in a nonlinear but bounded way. In lake ecosystem, the algae growth rate often increases rapidly with temperature but slows down after a certain temperature. The use of a rational function can capture this algae growth pattern, and the algae cannot grow infinitely as temperature increases. This reflects the reality of algae growth.

Table 1 Definitions and units of the parameters in the model

Parameters	Definition	Unit	Value	Source
A	Algal concentration	µg/L	—	—
P	Phosphorus concentration	µg/L	—	—
t	Time	d	—	—
b	Algal growth rate	d^{-1}	0.7	[11]
h_a	Half saturation constant	µg/L	25	[20]
p_a	Phosphorus content percentage	—	1	[11]
h	Flushing rate	d^{-1}	0	—
g	Zooplankton grazing rate	d^{-1}	0.0011	[11]
s	Algal mortality rate	d^{-1}	0.0019	[11]
I_p	External phosphorus loading rate	µg/L/d	0.3	[11]
r	Internal phosphorus recycling rate	µg/L/d	0.245	[11]
n	Half saturation value of recycling function	µg/L	10	[21]
q	Parameter for steepness of sigmoid function near n	—	20	[22]
e_p	Phosphorus excretion associated with grazing	µg/µg	0.65	[23]
T	Lake temperature	°C	Different scenarios	—
T_{opt}	Optimum temperature	°C	27	[24]
T_{min}	Minimum temperature	°C	20	[24]
θ_1	Temperature coefficient for zooplankton grazing	—	1.05	[24]
θ_2	Temperature coefficient for algae mortality	—	1.02	[24]

In the lake ecosystem, the zooplankton grazing rate at various temperatures is described by the Arrhenius equation in Equation (8) [25]:

$$g(T) = Ve^{-\frac{E_a}{RT}} \quad \text{(Equation 8)}$$

In Equation (8), $g(T)$ is the zooplankton grazing rate, V is the pre-exponential factor, E_a is the activation energy, R is the universal energy (J mol^{-1}), T is the absolute temperature (K). We compare the rate at two different temperatures, T and T_{ref}, as shown in Equation (9):

$$\frac{g(T)}{g(T_{ref})} = \frac{Ae^{-\frac{E_a}{RT}}}{Ae^{-\frac{E_a}{RT_{ref}}}} = e^{-\frac{E_a}{R}\left(\frac{1}{T}-\frac{1}{T_{ref}}\right)} \quad \text{(Equation 9)}$$

When the temperature varies in the range 0 to 40°C, the Arrhenius equation can be approximated by an empirical rule, where the zooplankton grazing rate increases by a roughly constant for each 10°C rise in the temperature. This leads to the Q_{10} model in Equation (10):

$$\frac{g(T)}{g(T_{ref})} = Q_{10}^{\frac{T-T_{ref}}{10}} \quad \text{(Equation 10)}$$

$$g(T) = g(T_{ref}) \times Q_{10}^{\frac{T-T_{ref}}{10}} \quad \text{(Equation 11)}$$

In Equation (11), Q_{10} is the factor at which the zooplankton grazing rate increases for every 10°C rise in the temperature. To find the zooplankton grazing rate for every 1°C increase in the temperature (instead of every 10°C), we define Equation (12):

$$\theta_1 = Q_{10}^{1/10} \quad \text{(Equation 12)}$$

$$g(T) = g(T_{ref}) \times \left(Q_{10}^{1/10}\right)^{T-T_{ref}} = g(T_{ref}) \times \theta_1^{T-T_{ref}} \quad \text{(Equation 13)}$$

In lake ecosystem, 20°C is commonly used as a reference temperature in metabolism studies [26-28]. Setting T_{ref} = 20°C, we have Equation (14):

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$$g(T) = g(20) \times \theta_1^{T-20}$$

(Equation 14)

In Equation (14), $g(20)$ represents the zooplankton grazing rate at 20°C. The term θ_1^{T-20} is the temperature coefficient and it provides the fractional increase in the zooplankton grazing rate per °C. This temperature coefficient θ_1^{T-20} is also incorporated in other study [24] to demonstrate the temperature effects in lake ecosystems. In this study, we include θ_1^{T-20} for zooplankton grazing rate and θ_2^{T-20} for algae mortality rate.

We plotted the graphs of temperature functions for internal phosphorus recycling, algae growth, zooplankton grazing and algae mortality as illustrated in Figure 1. From Figure 1, the internal phosphorus recycling, zooplankton grazing rate and algal mortality rate increase exponentially as temperature increases. The temperature function for algae growth shows a bell-shaped curve (Figure 1(a)). In a lake, temperature regulates algae growth through its effect on the metabolic processes in algae. At low temperature, the metabolic processes slow down and algae growth is limited. As temperature increases, metabolic processes such as nutrient uptake and photosynthesis accelerate, leading to faster algae growth rates. At optimal temperature (27°C), algae will grow rapidly [21]. Beyond the optimal temperature, enzyme denaturation occurs and reduces the metabolic activities. Algae growth then starts to decline and eventually approaches zero at 40°C.

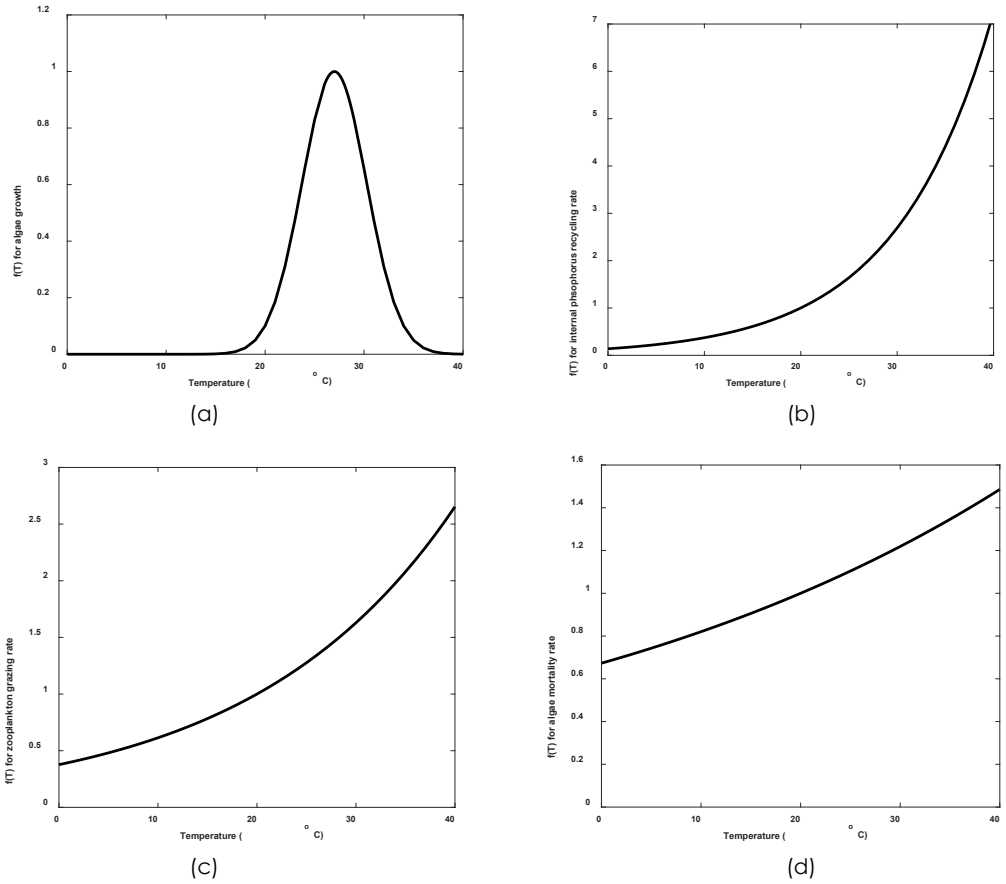


Figure 1 Graphs of temperature functions for (a) internal phosphorus recycling, (b) algae growth, (c) zooplankton grazing and (d) algae mortality

3.0 RESULTS AND DISCUSSION

The temperature effects in lake eutrophication are assessed in one-parameter space and two-parameter space with the use of XPPAUT and MatCont [29,30]. First, external phosphorus loading rate, flushing rate and algae mortality rate are analyzed at various temperatures. The bifurcation results are then further investigated in two-parameter space, where external phosphorus loading rate and algae mortality rate are chosen at different temperatures

3.1-Co-dimension One Bifurcation Analysis

We plotted algae, A (μg/L) versus external phosphorus loading rate, I_P (μg/L/d) at 24°C, 27°C, 28°C, 29°C, 30°C and 32°C as shown in Figure 2. From

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Figure 2(a), bistability of two stable steady states is detected in the range 0 μg/L/d < l_p < 0.0173 μg/L/d, where two stable equilibria (E1 and E3) and one unstable equilibrium (E2) exist. An increase in l_p will lead to an increase in *A* along the stable steady state E3, reaching a critical point at l_p = 0.0173 μg/L/d. Beyond this point, a further increase in l_p will cause a sudden shift in *A* to 151 μg/L (stable state at E1) and the lake becomes eutrophic (*A* > 10 μg/L). The value of *A* then remains in E1. The value of l_p = 0.0173 μg/L/d is recognized as the threshold for a sudden shift in lake eutrophication. When l_p exceeds this threshold, *A* will increase abruptly, causing the lake to become eutrophic. Based on Figure 2, when temperature increases, the regime shift threshold increases slightly and the sudden jump in *A* is more abrupt. For example, at 32°C, when the l_p value exceeds 0.0196 μg/L/d, the value of *A* increases greatly from 6.7 μg/L to 266 μg/L. The increase in *A* is larger at 32°C compared to that at 24°C.

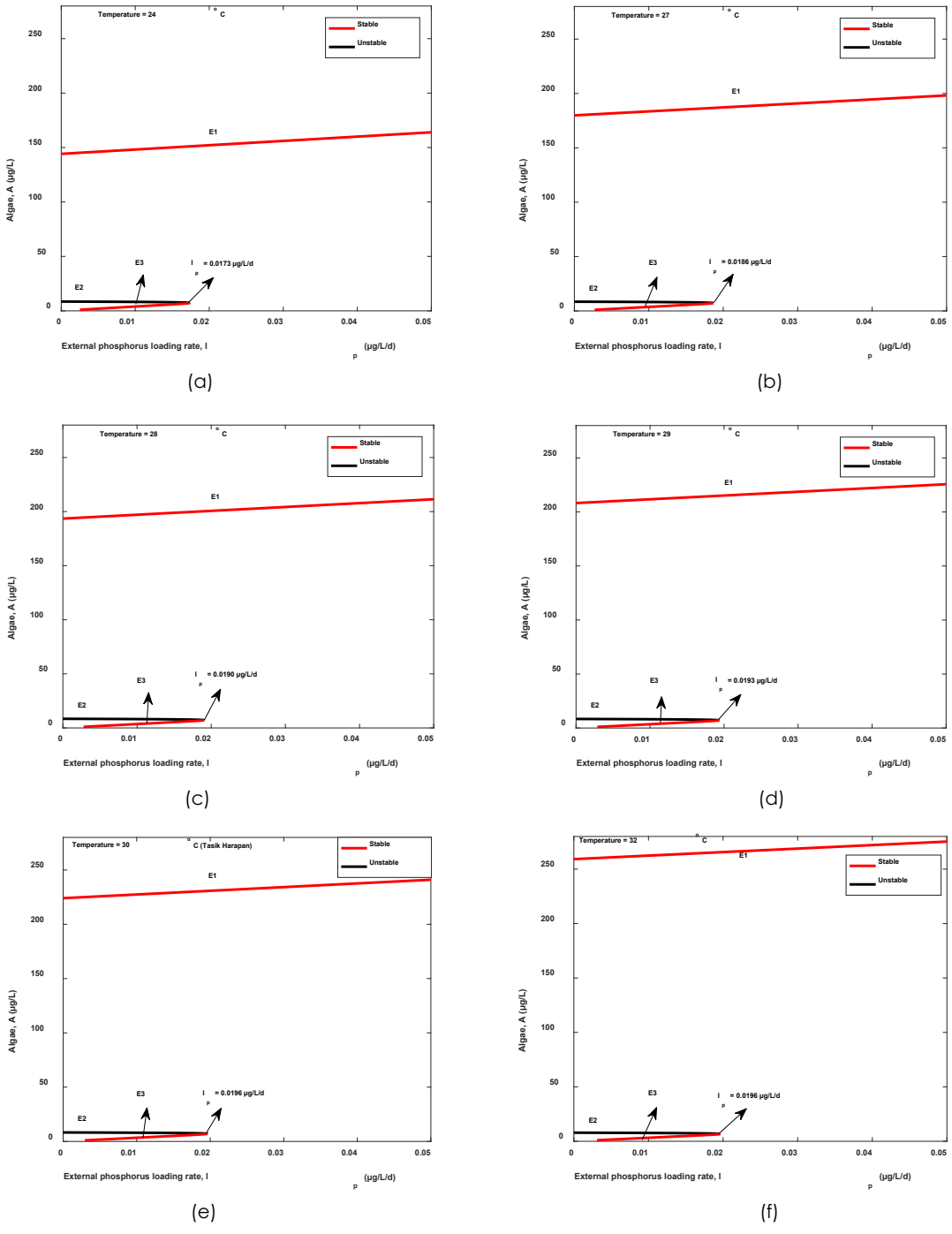
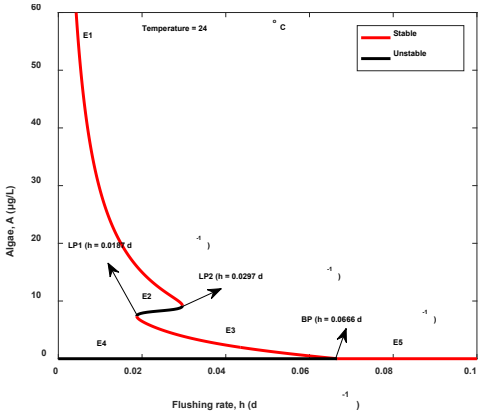


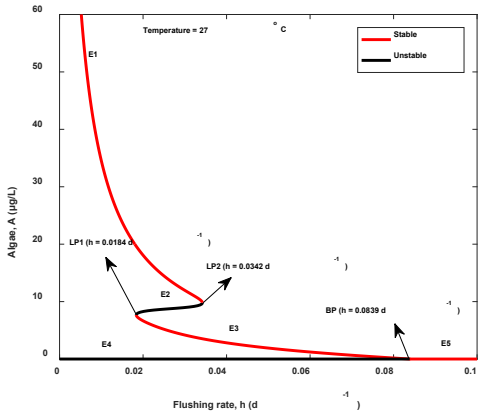
Figure 2 Graphs of algae, *A* (μg/L) versus external phosphorus loading, l_p (μg/L/d) at (a) 24°C, (b) 27°C, (c) 28°C, (d) 29°C, (e) 30°C and (f) 32°C

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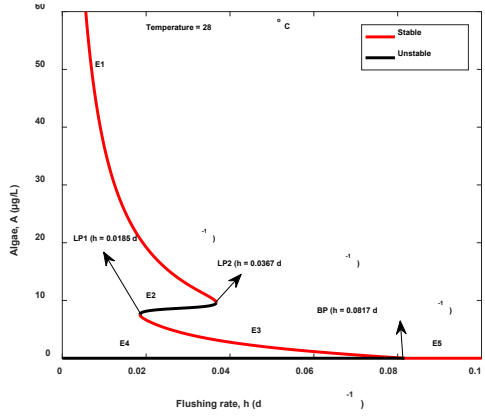
To determine how temperature affects the flushing rate required for lake restoration, we plotted the graph of algae, *A* (µg/L) versus flushing rate, *h* (d^{-1}) at 24°C, 27°C, 28°C, 29°C, 30°C and 32°C as displayed in Figure 3. In Figure 3(a), at 24°C, bistability of two stable steady states is observed for 0.0187 d^{-1} < *h* < 0.0297 d^{-1}, indicating the existence of two stable equilibria (E1 and E3) and one unstable equilibrium (E2). The value of *A* will be attracted to either E1 or E3, depending on the initial condition. As expected, the value of *A* decreases when a higher flushing rate is applied to the lake. In Figure 3(a), transcritical bifurcation occurs at BP (*h* = 0.0666 d^{-1}), where stable steady state E3 collides with unstable steady state E4 and exchanges stability. E3 now becomes unstable (not shown in Figure 3(a) due to negative values of *A*) and E4 becomes stable steady state E5. From Figure 3(e), for Tasik Harapan with *A* = 300 µg/L, the value of *A* will decrease along E1 when *h* increases until LP2 (*h* > 0.045 d^{-1}). A further increase in *h* beyond 0.045 d^{-1} will cause *A* to "jump" to stable steady state E3. Tasik Harapan becomes oligotrophic with *A* less than 2.4 µg/L. In Figure 3(f), when the temperature increases to 32°C, the value of *A* will decrease gradually along E1 until BP3 (*h* = 0.0755 d^{-1}). Eventually, the algae population becomes extinct (*A* = 0 µg/L) when *h* ≥ 0.0755 d^{-1}. Based on the bifurcation diagram, as temperature increases, the range of bistability increases (LP1 < *h* < LP2), indicating that restoring a eutrophic lake to an oligotrophic state requires a higher flushing rate (refer to the value of LP2).



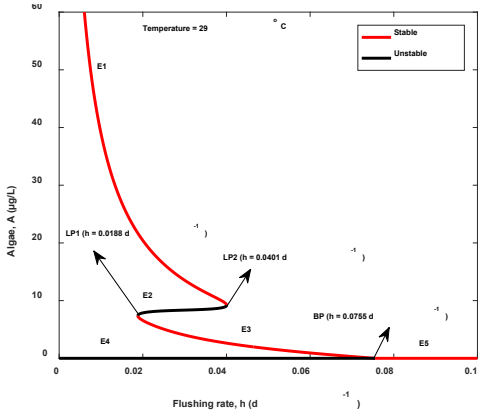
(a)



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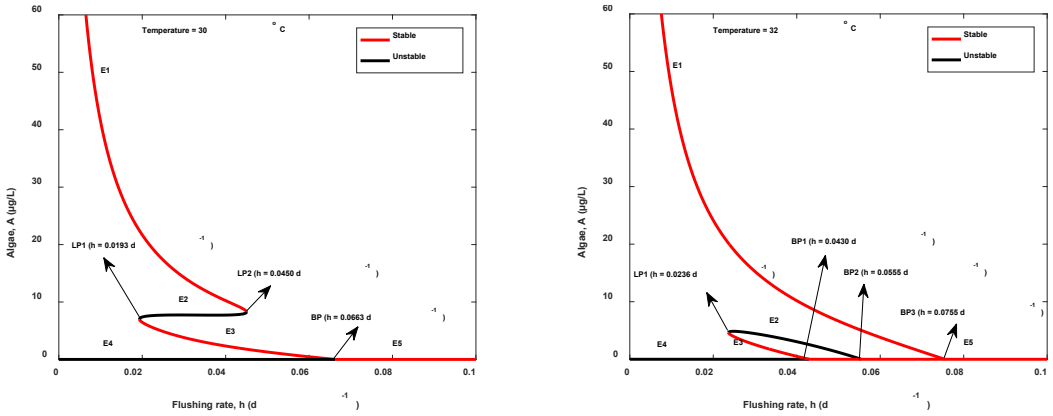


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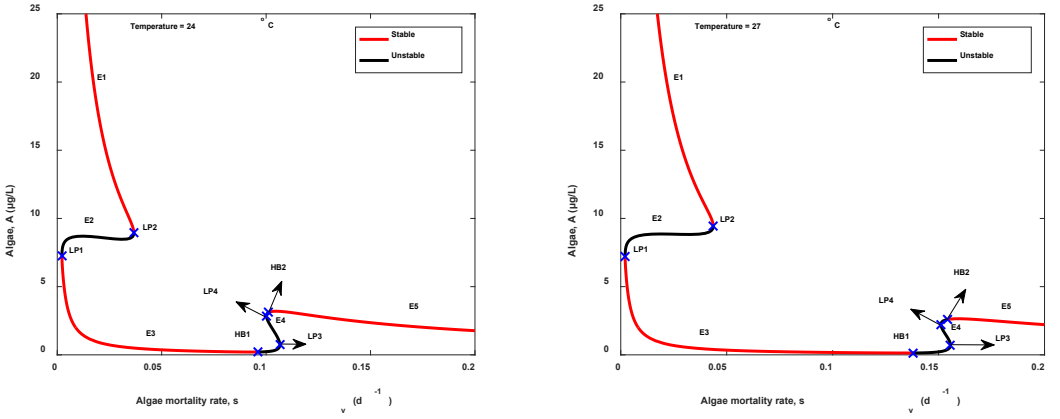
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(e) (f)

Figure 3 Graphs of algae, *A* (μg/L) versus flushing rate, *h* (d^{-1}) at (a) 24°C, (b) 27°C, (c) 28°C, (d) 29°C, (e) 30°C and (f) 32°C

Next, by considering I_P = 0.02 μg/L/d, we plotted the graph of algae, *A* (μg/L) versus zooplankton mortality rate, s_V (d^{-1}) at 24°C, 27°C, 28°C, 29°C, 30°C and 32°C as indicated in Figure 4. In Figure 4(a), both saddle-node (LP points) and Hopf bifurcations (HB points) are detected as s_V varies. Bistability of two stable steady states (E1 and E3) occurs in LP1 < s_V < LP2. Within this range of s_V, if *A* falls between E1 and E2, *A* will converge to the stable steady state E1. If *A* falls below E2, it will approach to stable steady state E3. Further increase in s_V will decrease the value of *A* until at HB1, where Hopf bifurcation occurs. From Figure 4, as temperature increases, the limit points LP2 and LP4 are moving nearer until a closed curve is formed at 30°C and 32°C. At 30°C and 32°C, the closed curve and stable steady state E1 form bistability of two stable steady states when LP1 < s_V < HB. A s_V value within this range together with an initial value of *A* which is lower than E3 will cause the value of *A* to be attracted to E3. An initial value of *A* which is higher than E2 will approach E1. These dynamics are further investigated in co-dimension two bifurcation analysis.



(a) (b)

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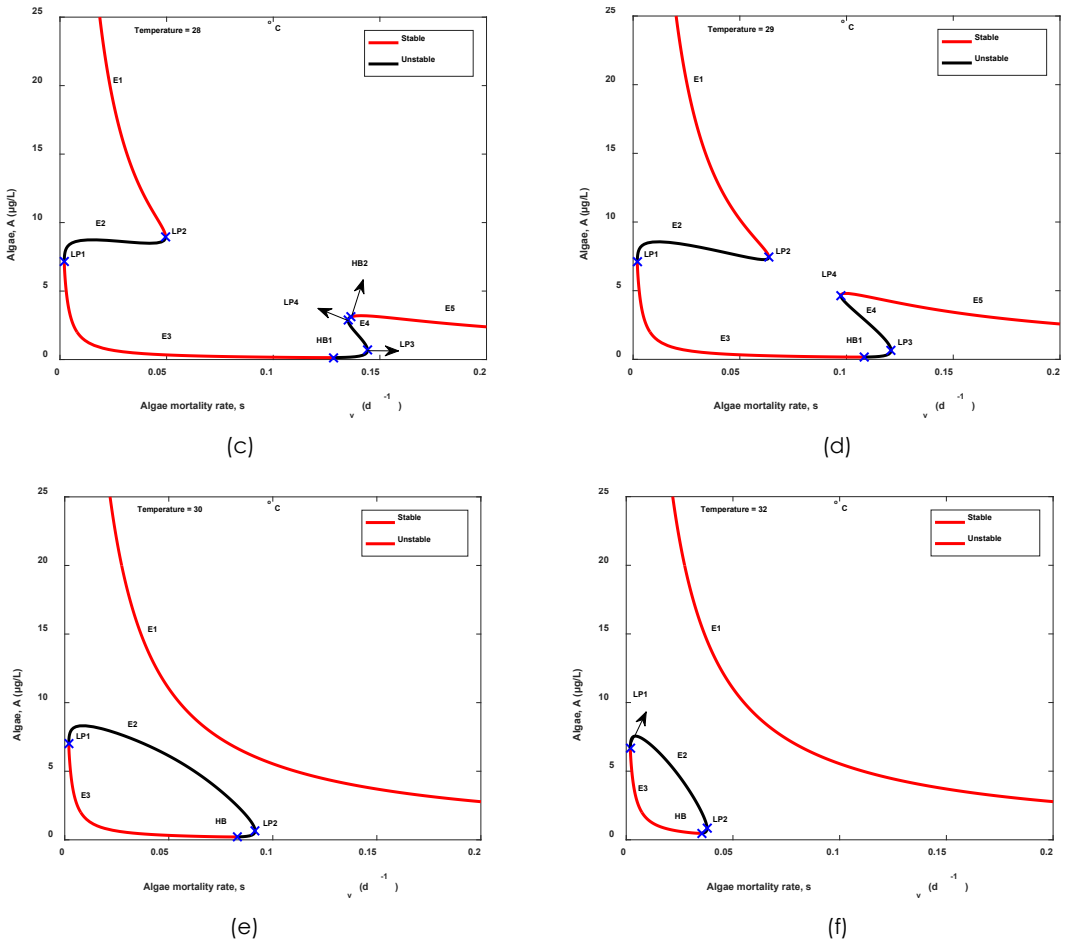


Figure 4 Graphs of algae, *A* (μg/L) versus algae mortality rate, s_v (d^{-1}) at (a) 24°C, (b) 27°C, (c) 28°C, (d) 29°C, (e) 30°C and (f) 32°C

3.2 Co-dimension Two Bifurcation Analysis

To further explore the lake dynamics, bifurcation diagrams of external phosphorus loading rate, l_p (μg/L/d) versus algae mortality rate, s_v (d^{-1}) are plotted at 24°C, 27°C, 28°C, 29°C, 30°C and 32°C as shown in Figure 5. Figures 5(a) to 5(c) illustrate three types of bifurcation curves, i.e., saddle-node curve, (ii) subcritical Hopf curve and (iii) limit point of cycles (LPC) curve. Only saddle-node and subcritical Hopf curves are detected at 29°C, 30°C and 32°C (Figures 5(d) to 5(f)). A Bogdanov-Takens (BT) bifurcation occurs at the intersection of Hopf and branch point curves. Some parts of the saddle-node curve vanish due to this interaction. These curves partition the parameter space into four different dynamic regions, namely (i) a stable equilibrium, (ii) bistability of two stable steady states, (iii) stable limit cycle and (iv) a stable equilibrium coexists with an unstable and a stable limit cycle. In region (i), only one stable steady state exists, where both algae and phosphorus co-exist. The trophic status of the lake can be determined based on algae concentration, i.e., oligotrophic (A < 2.6 μg/L), mesotrophic (2.6 μg/L ≤ A < 10 μg/L), or eutrophic (A ≥ 10 μg/L). In region (ii), regime shift in lake eutrophication will occur. Small changes in l_p or s_v will lead to eutrophication if the threshold is surpassed. Within the region exhibiting a stable limit cycle, the algae concentration displays oscillations. In region (iv), the algae concentration will either approach to a stable steady state or oscillate, depending on its initial value.

Based on Figure 5, the lake dynamics are complex at 24°C, 27°C and 28°C due to the occurrence of four different types of lake dynamics. The region of these complex dynamics is larger at 27°C and 28°C compared to that at 24°C. At temperatures 29°C, 30°C and 32°C, the lake dynamics are rather simple: only regions with a stable equilibrium and bistability of two stable steady states exist. At 32°C, the narrow bistable region (ii) implies a low chance of the system shifting between the two stable steady states.

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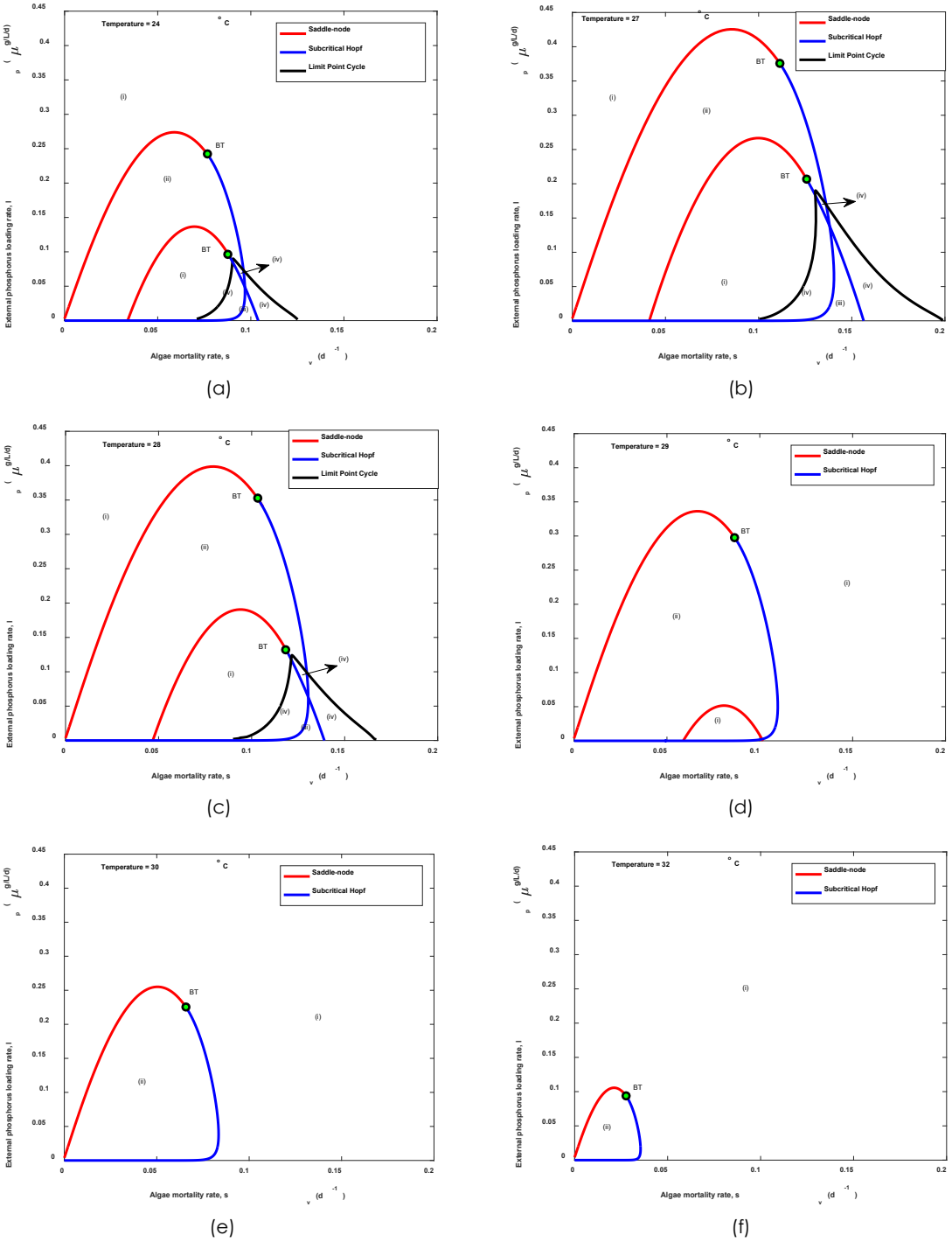


Figure 5 Graphs of external phosphorus loading, l_p (μg/L/d) versus algae mortality rate, s_v (d^{-1}) at (a) 24°C, (b) 27°C, (c) 28°C, (d) 29°C, € 30°C and (f) 32°C

3.3 Discussion

Temperature is one of the key environmental factors that drive algae growth in a lake. Identifying temperature effects in lake eutrophication is crucial in lake eutrophication management. Based on the bifurcation results, we provide three significant biological implications: (1) an increase in temperature would induce a larger sudden increase in algae concentration, (2) a higher temperature necessitates a greater flushing rate to achieve lake restoration, and (3) a moderate temperature would induce complex lake dynamics.

First, our bifurcation results demonstrated that an increase in temperature would induce a larger sudden increase in algae concentration (Figure 2).

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One possible explanation for this observation is that higher temperatures enhance internal phosphorus recycling, which supports increased algae growth. Lake sediments naturally store phosphorus, where certain forms are permanently buried while others may be remobilized into the overlying water [31]. Higher temperatures enhance the phosphorus release from sediment into the overlying water column, serving as a nutrient source and hence stimulating the algae growth. Such observation on a larger sudden increase in algae concentration triggered by internal phosphorus recycling has been demonstrated in the previous work [32]. In addition to the significant contribution of internal phosphorus recycling at warmer temperatures, the shallow depth of Tasik Harapan further enhances the availability of internal phosphorus recycling for algae growth. During high wind events, sediment resuspension occurs, introducing phosphorus back into the lake water and increasing the availability of phosphorus for algae uptake.

Second, a higher temperature necessitates a greater flushing rate to achieve lake restoration. During the flushing process, fresh and clean water enters the lake and dilutes the algae and phosphorus concentrations. Eventually, the entire lake, containing algae and phosphorus, is flushed and replaced with fresh water. The purpose of flushing is to reduce the water residence time. At high temperature, phosphorus is released from the sediment at a faster rate. A higher flushing rate is essential to replace the lake water at a faster rate and to reduce the time for algae to proliferate.

Third, from the co-dimension two bifurcation results, we also observe that a moderate temperature (for example 27°C and 28°C) would induce complex lake dynamics. Small variations in phosphorus loading or algal mortality may lead to a sudden transition in lake dynamics. These changes in lake dynamics have large impacts on the lake ecosystem. For example, in the region of bistability of two stable steady states, the lake typically stays at one stable state unless a threshold is exceeded. Small changes in external phosphorus loading or algae mortality rate might shift the lake to another stable state, causing difficulty in lake restoration. Additional lake restoration efforts and higher costs are often required to remediate the lake [33]. Hence, quantifying the temperature effects in lake eutrophication is crucial for providing insights on lake eutrophication management.

The bifurcation results in this study reveals the possible thresholds where small variation in the parameters can cause a sudden shift from a clear to a eutrophic state. This insight enables the lake managers to design monitoring programs and implement early interventions to maintain lake conditions well below these thresholds. Additionally, the bifurcation results also provide an estimate of the nutrient reduction required to restore the eutrophic lake. This offers guidance for policymakers in setting the nutrient reduction targets. By incorporating temperature effects, this study demonstrates how temperature influences lake eutrophication. The results could serve as an early warning for the sudden change in lake conditions, enabling policymakers to adjust management strategies in response to changing temperatures.

4.0 CONCLUSION

We formulated a mathematical model of algae and phosphorus that incorporates the influence of temperature in Tasik Harapan. The temperature effects are assessed by using co-dimension one and co-dimension two bifurcation analyses. The bifurcation results demonstrate that higher temperatures would lead to a larger sudden increase in algae concentration. A higher flushing rate is required to restore a eutrophic lake at higher temperature. A moderate temperature (for example 27°C and 28°C) would induce complex lake dynamics. While most published studies on temperate lakes primarily focus on the impact of temperature on external phosphorus loading, our paper uniquely highlights the critical interplay between temperature, external phosphorus loading, and algal mortality, demonstrating that this interplay is significant in managing lakes. Small perturbation in these ecological parameters can lead to abrupt change in lake eutrophication dynamics when critical thresholds are surpassed. These possible thresholds and lake dynamic changes could assist lake managers in designing lake management strategies. Timely lake remediation is possible to avoid undesired shift in lake eutrophication.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Ethics declaration

Formal ethical approval was not obtained for this study. However, the authors confirm that the research was conducted in accordance with established ethical principles and good research practices, with due consideration given to participant rights, informed consent, confidentiality, and responsible data handling.

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