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ESTIMATION OF *p*-ADIC SIZES OF COMMON ZEROS OF PARTIAL DERIVATIVE POLYNOMIALS ASSOCIATED WITH A QUINTIC FORM

SAPAR S.H[1] & MOHD. ATAN K.A[2]

Abstract. Let $\underline{x} = \{x_i, x_2, \ldots, x_n\}$ be a vector in a space Z^n with Z ring of integers and let q be a positive integer, f a polynomial in $\underline{x}$ with coefficients in Z. The exponential sum associated with f is defined as $S(f;q) = \sum \exp(2\pi i f(x)/q)$ where the sum is taken over a complete set of residues modulo q. The value of $S(f;q)$ has been shown to depend on the estimate of the cardinality $|V|$, the number of elements contained in the set $V = \{\underline{x} \bmod q \mid \underline{f_x} \equiv \underline{0} \bmod q\}$ where $\underline{f_x}$ is the partial derivatives of f with respect to $\underline{x}$. To determine the cardinality of V, the information on the p-adic sizes of common zeros of the partial derivatives polynomials need to be obtained. This paper discusses a method of determining the p-adic sizes of the components of (ξ, η) a common root of partial derivative polynomials of $f(x, y)$ in $Z_p[x, y]$ of degree five based on the p-adic Newton polyhedron technique associated with the polynomial. The quintic polynomial is of the form $f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$.

Keywords: Exponential sums, cardinality, p-adic sizes, Newton polyhedron

Abstrak. Katakan $\underline{x} = \{x_i, x_2, \ldots, x_n\}$ vektor dalam ruang Z^n dengan Z menandakan gelanggang integer dan q integer positif, f polinomial dalam $\underline{x}$ dengan pekali dalam Z. Hasil tambah eksponen yang disekutukan dengan f ditakrifkan sebagai $S(f;q) = \sum \exp(2\pi i f(x)/q)$ yang dinilaikan bagi semua nilai x di dalam reja lengkap modulo q. Nilai $S(f;q)$ adalah bersandar kepada penganggaran bilangan unsur $|V|$, yang terdapat dalam set $V = \{\underline{x} \bmod q \mid \underline{f_x} \equiv \underline{0} \bmod q\}$ dengan $\underline{f_x}$ menandakan polinomial-polinomial terbitan separa f terhadap $\underline{x}$. Untuk menentukan kekardinalan bagi V, maklumat mengenai saiz p-adic pensifar sepunya perlu diperolehi. Makalah ini membincangkan suatu kaedah penentuan saiz p-adic bagi komponen (ξ,η) pensifar sepunya pembezaan separa $f(x,y)$ dalam $Z_p[x, y]$ berdarjah lima berasaskan teknik polihedron Newton yang disekutukan dengan polinomial terbabit. Polinomial berdarjah lima yang dipertimbangkan berbentuk $f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$.

Kata kunci: Hasil tambah eksponen, kekardinalan, saiz p-adic, polihedron Newton

[1] Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia
Email: sitihas@fsas.upm.edu.my

[2] Laboratory of Theoretical Mathematics, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia
Email: kamel@inform.upm.edu.my

1.0 INTRODUCTION

In our discussion, we use notation Z_p, Ω_p and $ord_p x$ to denote respectively the ring of p-adic integers, completion of the algebraic closure of Q_p the field of rational p-adic numbers and the highest power of p which divides x. For each prime p, let $\underline{f} = (f_1, f_2, \ldots, f_n)$ be an n-tuple polynomials in $Z_p[\underline{x}]$ where Z_p is the ring of p-adic intergers and $\underline{x} = \{x_i, x_2, \ldots, x_n\}$.

The estimation of $|V|$ has been the subject of many research in number theory one of which is in finding the best possible estimates to multiple exponential sums of the form $S(f;q) = \sum_{\underline{x} \bmod q} \exp\left(\frac{2\pi i f}{q}\right)$ where $f(\underline{x})$ is a polynomial in $Z[\underline{x}]$ and the sum taken over a complete set of residues x modulo a positive integer q.

Loxton and Vaughn [1] are among the researchers who investigated $S(f;q)$ where f is a non-linear polynomial in $Z[\underline{x}]$ and they found that the estimate of $S(f;q)$ depends on the value of $|V|$ the number of common zeros of the partial derivatives of f with respect to $\underline{x}$ modulo q. By using this result, the estimate of $S(f;q)$ was found by them in terms of invariants related to f. In his quest to find a more explicit estimate of $S(f;q)$, Mohd Atan [2] began by investigating the sum associated with lower degree polynomials. He considered in particular the non-linear polynomial $f(x,y) = ax^3 + bx^2y + cx + dy + e$ with coefficients in Z_p. He found that the p-adic sizes for the zero (ξ,η) of this polynomial is $ord_p\xi \geq \frac{1}{2}(\alpha - \delta)$ and $ord_p\eta \geq \frac{1}{2}(\alpha - \delta)$ with $\delta = \max\left\{ord_p 3a, \frac{3}{2} ord_p b\right\}$.

Later, Mohd. Atan and Abdullah [3] considered a cubic polynomial of the form $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + kx + my + n$ and obtained the p-adic sizes for the root (ξ,η) of this polynomial as $ord_p\xi \geq \frac{1}{2}(\alpha - \delta)$ and $ord_p\eta \geq \frac{1}{2}(\alpha - \delta)$ with $\delta = \max\{ord_p 3a, ord_p b, ord_p c, ord_p 3d\}$.

Subsequently, in 1997 Chan and Mohd. Atan [4] investigated a polynomial of a higher degree then the one considered above in $Z_p[x,y]$ of the form

$$f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + mx + ny + k$$

and showed that for (ξ,η) a root of $f(x,y)$, $ord_p\xi \geq \frac{1}{3}(\alpha - \delta)$ and $ord_p\eta \geq \frac{1}{3}(\alpha - \delta)$ with $\delta = \max\{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e\}$.

Heng and Mohd. Atan [5] determined the cardinality associated with the partial derivatives functions of the cubic form $f(x,y) = ax^3 + bx^2y + cx + dy + e$. In their work, they attempt to find a better estimate by looking at the maximum number of common zeros associated with the partial derivatives $f_x(x, y)$ and $f_y(x, y)$. A sharper result was obtained with δ similar to the one considered by Mohd. Atan [2]. However, results for two-variable polynomials of higher degrees are less complete.

In this paper, we will discuss a method of determining explicitly the p-adic sizes of the components (ξ,η) a common root of partial polynomial of $f(x, y)$ in $Z_p[x, y]$ of degree five. The polynomial that we consider in this paper is of the form

$$f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + mx^5 + nx + ty + k$$

where the dominant terms are complete.

Our approach entails examination of combinations of indicator diagrams associated with the Newton polyhedrons of f_x and f_y.

2.0 NEWTON POLYHEDRON

In this section, we give a brief description of the polyhedron as developed by [6]. It is a two-variable analogue of the p-adic Newton polygon in single variable as developed by [7].

Definition 2.1:

Let p be a prime and $f(x,y) = \Sigma\, a_{ij}x^iy^j$ a polynomial in $Z_p[x, y]$. We map the term $T_{ij} = a_{ij}x^iy^j$ of $f(x, y)$ to the points $P_{ij} = (i, j, ord_pa_{ij})$ in the three dimensional Euclidean space and call this set of points Newton diagram of $f(x, y)$. Below is an example of a Newton diagram for a lower degree polynomial.

Example 2.1:

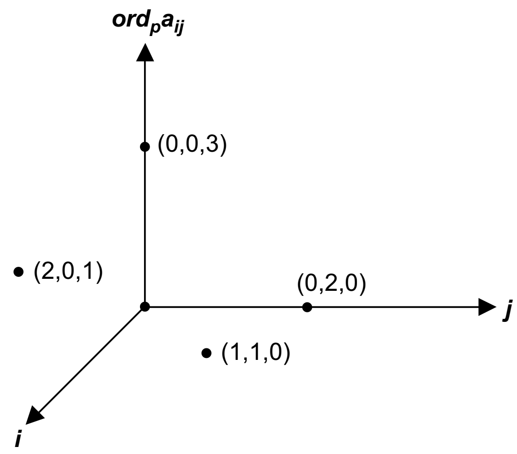

Figure 1 Newton diagram of $f(x,y) = 3x^2 + 2xy - y^2 + 27$ with $p = 3$

Definition 2.2:

Let p be a prime and $f(x,y) = \Sigma\, a_{ij} x^i y^j$ a polynomial in $\Omega_p[x,y]$. The Newton polyhedron of $f(x,y)$ is the lower convex hull of the Newton diagram of $f(x,y)$. It is the highest convex connected surface which passes through or below the points P_{ij} in the Newton diagram of $f(x,y)$. If $a_{ij} = 0$ then the associated point is omitted, since it lies at infinity above the i-j plane. Below is the Newton polyhedron associated with the polynomial in Example 2.1.

Example 2.2:

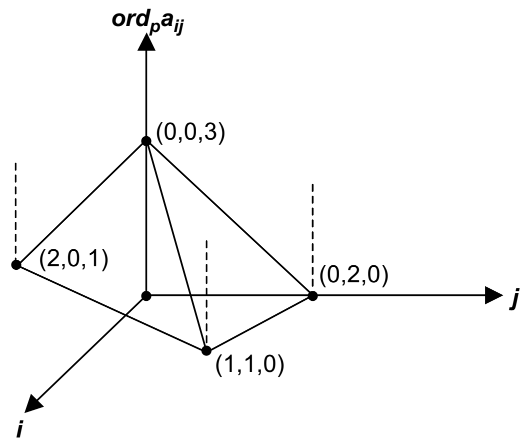

Figure 2 The Newton polyhedron of $f(x,y) = 3x^2 + 2xy - y^2 + 27$ with $p = 3$

Definition 2.3:

Let $(\mu_i, \lambda_i, 1)$ be the normalized upward-pointing normals to the faces F_i of N_f, the Newton polyhedron of a polynomial $f(x,y)$ in $Q_p[x,y]$. We map $(\mu_i, \lambda_i, 1)$ to the points (μ_i, λ_i) in the x-y plane. If F_r and F_s are adjacent faces in N_f, sharing a common edge, we construct the straight line joining (μ_r, λ_r) and (μ_s, λ_s). If F_r has a common edge with the vertical face F say in N_f, we construct the straight line segment joining (μ_r, λ_r) and the appropriate point at infinity that corresponds to the normal of F, that is the segment along a line with slope $-\alpha/\beta$. We call the set of lines so obtained the indicator diagram associated with the Newton polyhedron of $f(x,y)$ [2]. The indicator diagram associated with the Newton polyhedron in Example 2.2 is as shown in the following example.

Example 2.3:

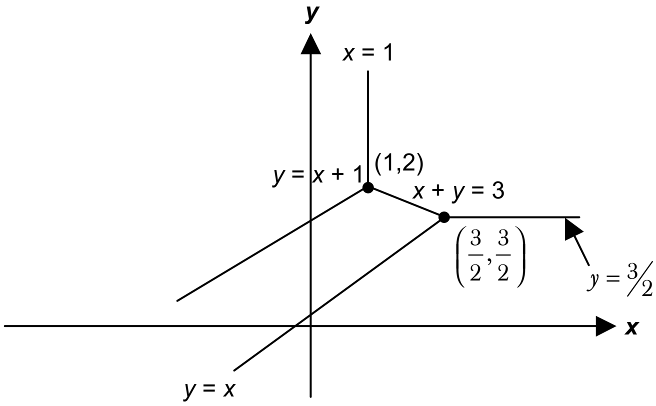

Figure 3 Indicator diagram associated with the polynomial $f(x, y) = 3x^2 + 2xy - y^2 + 27$ with $p = 3$

3.0 p-ADIC ORDERS OF ZEROS OF A POLYNOMIAL

In 1986 Mohd. Atan and Loxton conjectured that to every point of intersection of the combination of the indicator diagrams associated with the Newton polyhedrons of a pair of polynomials in $Z_p[x, y]$ there exist common zeros of both polynomials whose p-adic orders correspond to this point [6]. The conjecture is as follows:

Conjecture

Let p be a prime. Suppose f and g are polynomials in $Z_p[x, y]$. Let (μ, λ) be a point of intersection of the indicator diagrams associated with f and g. Then there are ξ and η in Ω_p satisfying $f(\xi, \eta) = g(\xi, \eta) = 0$ and $ord_p\xi = \mu$, $ord_p\eta = \lambda$.

A special case of this conjecture was proved by Mohd. Atan and Loxton [6]. Sapar and Mohd. Atan [8] improved this result and it is written as follows:

Theorem 3.1

Let p be a prime. Suppose f and g are polynomials in $Z_p[x, y]$. Let (μ, λ) be a point of intersection of the indicator diagrams associated with f and g at the vertices or simple points of intersections. Then there are ξ and η in Ω_p satisfying $f(\xi,\eta) = g(\xi,\eta) = 0$ and $ord_p\xi = \mu$, $ord_p\eta = \lambda$.

In Theorem 3.2 we give the p-adic sizes of common zeros of partial derivatives of the polynomial $f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$. First, we have the assertion as in Lemma 3.1. In this lemma and the theorem that follows,

$$\alpha_1 = \frac{4b + 2\lambda_1 c}{4(5a + \lambda_1 b)}, \alpha_2 = \frac{4b + 2\lambda_2 c}{4(5a + \lambda_2 b)}, \text{ with } \lambda_1, \lambda_2 \text{ zeros of}$$

$k(\lambda) = (10dm - 4e^2)\lambda^2 + (10cm - 2de)\lambda + 2ce - d^2$. We note that clearly $\alpha_1 \neq \alpha_2$ if $\lambda_1 \neq \lambda_2$.

Lemma 3.1

Suppose U, V in Ω_p with $U = x + \alpha_1 y$ and $V = x + \alpha_2 y$. Let $p > 5$ be a prime, a, b, c, d, e and m in Z_p, $\delta = max\{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e, ord_p m\}$, $ord_p s, ord_p t \geq \alpha > \delta$ and $ord_p b^2 > ord_p ac$. If $ord_p U = \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b}, ord_p V = \frac{1}{4} ord_p \frac{s + \lambda_2 t}{5a + \lambda_2 b}$ and $ord_p (10cm - 2de)^2 > ord_p (10dm - 4e^2)(2ce - d^2)$ then $ord_p x \geq \frac{1}{4}(\alpha - \delta)$ and $ord_p y \geq \frac{1}{4}(\alpha - \delta)$.

Proof:

From $U = x + \alpha_1 y$ and $V = x + \alpha_2 y$, we have

$$x = \frac{\alpha_2 U - \alpha_1 V}{\alpha_2 - \alpha_1} \text{ and } y = \frac{U - V}{\alpha_1 - \alpha_2}$$

Then,

$$ord_p x = ord_p (\alpha_1 V - \alpha_2 U) - ord_p (\alpha_1 - \alpha_2) \quad (1)$$

and

$$ord_p y = ord_p (U - V) - ord_p (\alpha_1 - \alpha_2) \quad (2)$$

with $$ord_p (\alpha_1 - \alpha_2) = ord_p \frac{(2b^2 - 5ac)(\lambda_2 - \lambda_1)}{2(5a + \lambda_1 b)(5a + \lambda_2 b)} \quad (3)$$

and $$\lambda_2 - \lambda_1 = -\frac{\sqrt{(10cm - 2de)^2 - 4(10dm - 4e^2)(2ce - d^2)}}{10dm - 4e^2}$$

Since $ord_p (10cm - 2de)^2 > ord_p (10dm - 4e^2)(2ce - d^2)$, we have $\lambda_1 \neq \lambda_2$ and

$$ord_p (\lambda_1 - \lambda_2) = \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2}$$

Hence, from (1) and (3),

$$ord_p x = ord_p\left(\alpha_2 U - \alpha_1 V\right) - ord_p \frac{\left(2b^2 - 5ac\right)\left(\lambda_2 - \lambda_1\right)}{2\left(5a + \lambda_1 b\right)\left(5a + \lambda_2 b\right)}$$

Suppose min $\{ord_p\alpha_2 U, ord_p\alpha_1 V\} = ord_p\alpha_2 U$, we have

$$ord_p x \geq ord_p U + ord_p \frac{4b + 2\lambda_2 c}{4\left(5a + \lambda_2 b\right)} - ord_p \frac{\left(2b^2 - 5ac\right)\left(\lambda_2 - \lambda_1\right)}{2\left(5a + \lambda_1 b\right)\left(5a + \lambda_2 b\right)}$$

Thus, we obtain

$$ord_p x \geq ord_p U + ord_p(2b + \lambda_2 c) - ord_p\,(2b^2 - 5ac) - ord_p\,(\lambda_1 - \lambda_2) + ord_p\,(5a + \lambda_1 b)$$

That is,

$$ord_p x \geq \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} + ord_p\left(2b + \lambda_2 c\right) - ord_p\left(2b^2 - 5ac\right)$$
$$- \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + ord_p\left(5a + \lambda_1 b\right)$$

since $ord_p U = \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b}$

Suppose min$\{ord_p 2b, ord_p\lambda_2 c\} = ord_p b$. Since $ord_p b^2 > ord_p ac$, we have

$$ord_p x \geq \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} + ord_p b - ord_p ac - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + ord_p\left(5a + \lambda_1 b\right)$$

Suppose min$\{ord_p 5a, ord_p\lambda_1 b\} = ord_p\lambda_1 b$ and since $ord_p b^2 > ord_p ac$, we have

$$ord_p x \geq \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} + ord_p b - ord_p b^2 - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + ord_p \lambda_1 b$$

Since $ord_p(10cm - 2de)^2 > ord_p(10dm - 4e^2)(2ce - d^2)$, we find that

$$ord_p x \geq \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2}$$
$$= \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b}$$

Suppose $\min\{ord_p s, ord_p \lambda_1 t\} = ord_p s$ and $\min\{ord_p 5a, ord_p \lambda_1 b\} = ord_p \lambda_1 b$. Then,

$$ord_p x \geq \frac{1}{4}\left(ord_p s - ord_p \lambda_1 b\right)$$
$$\geq \frac{1}{4}\left(ord_p s - ord_p a\right)$$

By hypothesis,

$$ord_p x \geq \frac{1}{4}(\alpha - \delta)$$

Now from (2) and (3), we have

$$ord_p y = ord_p (U - V) - ord_p \frac{\left(2b^2 - 5ac\right)(\lambda_2 - \lambda_1)}{2(5a + \lambda_1 b)(5a + \lambda_2 b)}$$

Suppose $\min\{ord_p U, ord_p V\} = ord_p U$ and since $ord_p (5a + \lambda_1 b) = ord_p (5a + \lambda_2 b)$, we obtain

$$ord_p y > ord_p U - ord_p (2b^2 - 5ac) - ord_p (\lambda_2 - \lambda_1) + 2 ord_p (5a + \lambda_1 b)$$

That is,

$$ord_p y \geq \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} - ord_p ac - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + 2 ord_p (5a + \lambda_1 b)$$
$$= \frac{1}{4} ord_p (s + \lambda_1 t) - ord_p ac - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{7}{4} ord_p (5a + \lambda_1 b)$$

Suppose $\min\{ord_p 5a, ord_p \lambda_1 b\} = ord_p \lambda_1 b$. Since $ord_p b^2 > ord_p ac$, we have

$$ord_p y \geq \frac{1}{4} ord_p (s + \lambda_1 t) - ord_p b^2 - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{7}{4} ord_p b$$
$$+ \frac{7}{4}\left(\frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2}\right)$$
$$\geq \frac{1}{4} ord_p (s + \lambda_1 t) - \frac{1}{4} ord_p b - \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{1}{2} ord_p \frac{2ce - d^2}{10dm - 4e^2}$$
$$\geq \frac{1}{4}\left(ord_p (s + \lambda_1 t) - ord_p b\right)$$

By hypothesis,

$$ord_p y \geq \frac{1}{4}(\alpha - \delta)$$

We will get the same result if

$$\min\{ord_p U, ord_p V\} = ord_p V,\ \min\{ord_p 2b, ord_p \lambda_2 c\} = ordp\lambda_2 c,$$
$$\min\{ord_p 5a, ord_p \lambda_1 b\} = ord_p a \text{ and } \min\{ord_p s, ord_p \lambda_1 t\} = ord_p \lambda_1 t$$

Therefore, we have

$$ord_p x \geq \frac{1}{4}(\alpha - \delta) \text{dan } ord_p y \geq \frac{1}{4}(\alpha - \delta)$$

as asserted.

Theorem 3.2

Let $f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$ be a polynomial in $Z_p[x,y]$ with $p>5$. Suppose $\alpha > 0$, $\delta = max\ \{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e, ord_p m\}$, $ord_p b^2 > ord_p ac$ and $ord_p(10cm - 2de)^2 > ord_p(10dm - 4e^2)(2ce - d^2)$.

If $ord_p f_x(0,0)$, $ord_p f_y(0,0) \geq \alpha > \delta$ there exists (ξ,η) in Ω_p^2 such that $f_x(\xi,\eta) = 0$, $f_y(\xi,\eta) = 0$ and $ord_p \xi \geq \frac{1}{4}(\alpha - \delta), ord_p \eta \geq \frac{1}{4}(\alpha - \delta)$.

Proof:

Let $g = f_x$ and $h = f_y$, and λ a constant.

Then,

$$(g + \lambda h)(x,y) = (5a + \lambda b)x^4 + (4b + 2\lambda c)x^3y + (3c + 3\lambda d)x^2y^2 + (2d + 4\lambda e)xy^3$$
$$+(e + 5\lambda m)y^4 s + \lambda t$$

and

$$\frac{(g+\lambda h)(x,y)}{5a+\lambda b} = x^4 + \left(\frac{4b+2\lambda c}{5a+\lambda b}\right)x^3 y + \left(\frac{3c+3\lambda d}{5a+\lambda b}\right)x^2 y^2 + \left(\frac{2d+4\lambda e}{5a+\lambda b}\right)xy^3$$
$$+\left(\frac{e+5\lambda m}{5a+\lambda b}\right)y^4 + \frac{s+\lambda t}{5a+\lambda b} \qquad (4)$$

Let α_{ij} denote the coefficients of $x^i y^j$ in the completed quartic form of Equation (4), $0 \leq i \leq 4$, $0 \leq j \leq 4$. By completing the quartic Equation (4) and by solving simultaneously equations $\alpha_{ij}(\lambda) = 0$, $i \neq 0$, $j \neq 0$, and $i + j = 4$, we obtain

$$\frac{(g+\lambda h)(x,y)}{5a+\lambda b}=\left(x+\frac{4b+2\lambda c}{4(5a+\lambda b)}y\right)^4+\frac{s+\lambda t}{5a+\lambda b}, \tag{5}$$

where λ satisfies the equation

$$\frac{e+5\lambda m}{5a+\lambda b}-\frac{1}{2}\frac{(d+2\lambda e)^2}{(c+\lambda d)(5a+\lambda b)}=0$$

That is,

$$(10dm - 4e^2)\lambda^2 + (10cm - 2de)\lambda + 2ce - d^2 = 0 \tag{6}$$

From (6), we have two values of λ, say λ_1, λ_2 where

$$\lambda_1=\frac{-\left(10cm-4e^2\right)+\sqrt{(10cm-2de)^2-4\left(10dm-4e^2\right)\left(2ce-d^2\right)}}{2\left(10dm-4e^2\right)}$$

and $$\lambda_2=\frac{-\left(10cm-4e^2\right)-\sqrt{(10cm-2de)^2-4\left(10dm-4e^2\right)\left(2ce-d^2\right)}}{2\left(10dm-4e^2\right)}$$

$\lambda_1 \neq \lambda_2$, because $ord_p(10cm - 2de)^2 > ord_p\,(10dm - 4e^2)(2ce - d^2)$.

Now, let

$$U = x+\frac{4b+2\lambda_1 c}{4(5a+\lambda_1 b)}y \tag{7}$$

$$V = x+\frac{4b+2\lambda_2 c}{4(5a+\lambda_2 b)}y \tag{8}$$

$$F(U,V) = (g + \lambda_1 h)(x,y) \tag{9}$$

and $$G(U,V) = (g + \lambda_2 h)(x,y) \tag{10}$$

By substituting U and V in (5), we obtain a polynomial in (U,V) as follows:

$$F(U,V) = (5a + \lambda_1 b)U^4 + s + \lambda_1 t \tag{11}$$

$$G(U,V) = (5a + \lambda_2 b)V^4 + s + \lambda_2 t \tag{12}$$

The combination of the indicator diagrams associated with Newton polyhedron of (11) and (12) is as shown below.

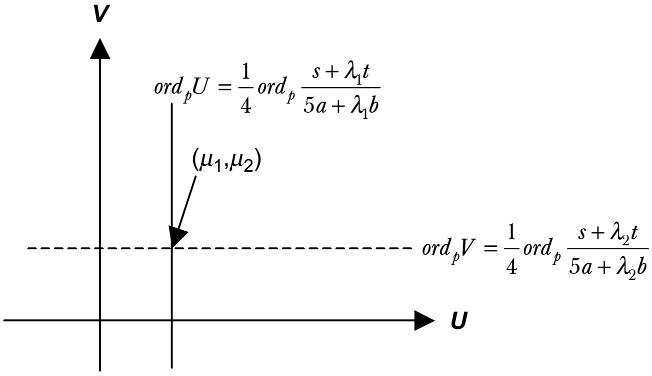

Figure 4 The indicator diagrams of $F(U,V) = (5a + \lambda_1 b)U^4 + s + \lambda_1 t$ and $G(U,V) = (5a + \lambda_2 b)U^4 + s + \lambda_2 t$

From Figure 4 and Theorem 3.1 there exists $(\hat{U},\hat{V})$ in Ω_p^2 such that $F_1(\hat{U},\hat{V}) = 0$, $G_1(\hat{U},\hat{V}) = 0$ and $ord_p\hat{U} = \mu_1$, $ord_p\hat{V} = \mu_2$ with $\mu_1 = \frac{1}{4} ord_p \frac{s+\lambda_1 t}{5a+\lambda_1 b}$ and $\mu_2 = \frac{1}{4} ord_p \frac{s+\lambda_2 t}{5a+\lambda_2 b}$. Let $U = \hat{U}$ and $V = \hat{V}$ in (7) and (8). There exists (x_0, y_0) in Ω_p^2 such that

$$x_0 = \frac{\alpha_2\hat{U} - \alpha_1\hat{V}}{\alpha_2 - \alpha_1} \text{ and } y_0 = \frac{\hat{U} - \hat{V}}{\alpha_1 - \alpha_2}.$$

Hence, $ord_p x_0 = ord_p(\alpha_1\hat{V} - \alpha_2\hat{U}) - ord_p(\alpha_1 - \alpha_2)$ and $ord_p y_0 = ord_p(V - U) - ord_p(\alpha_1 - \alpha_2)$.

From Lemma 3.1, we find that $ord_p x_0 \geq \frac{1}{4}(\alpha - \delta)$ and $ord_p y_0 \geq \frac{1}{4}(\alpha - \delta)$. Let $\xi = x_0$ and $h = y_0$. By back substitution in (9) and (10) and since $\lambda_1 \neq \lambda_2$ we have $g(\xi,\eta) = f_x(\xi,\eta) = 0$ and $h(\xi,\eta) = f_y(\xi,\eta) = 0$. Thus, $ord_p\xi = ord_p x_0 \geq \frac{1}{4}(\alpha - \delta)$ and $ord_p\eta = ord_p y_0 \geq \frac{1}{4}(\alpha - \delta)$ with (ξ,η) a common zero of g and h.

4.0 CONCLUSION

Our investigation observes that if p is an odd prime, $p > 5$, $f(x,y) = ax^5 + bx^4y + cx^3y^3 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$ a polynomial in $Z_p[x,y]$ with $ord_p b^2 > ord_p ac$ and $ord_p(10cm - 2de)^2 > ord_p(10dm - 4e^2)(2ce - d^2)$, then the p-adic sizes of common zeros of partial derivatives of this polynomial is

$$ord_p\xi \geq \frac{1}{4}(\alpha - \delta)\text{and } ord_p\eta \geq \frac{1}{4}(\alpha - \delta)$$

with $\xi = \max\{ord_pa, ord_pb, ord_pc, ord_pd, ord_pe, ord_pm\}$ and $ord_pf_x(0,0)$, $ord_pf_y(0,0) \geq \alpha > \xi$.

This work demonstrates that common zeros of certain p-adic orders of partial derivatives of a two-variable polynomial with coefficients in Z_p can be obtained through applications of the Newton polyheron technique. We have also shown that the p-adic orders of the zeros can be determined explicitly in terms of the p-adic orders of the coefficients of the dominant terms of the two-variable polynomial. This work extends future direction in finding explicit estimates of exponential sums associated with much higher degree of two-variable polynomials, which will in turn pave the way to finding better estimates of the sum associated with polynomials in several variables.

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