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STABILITY ASPECTS OF EXPLICIT RUNGE-KUTTA METHOD FOR DELAY DIFFERENTIAL EQUATIONS

FUDZIAH ISMAIL[1*], AUNG SAN LWIN[2] & MOHAMED SULEIMAN[3]

Abstract. The linear delay differential equations (DDEs) are solved by Runge-Kutta method using different types of interpolation to approximate the delay terms. The stability polynomials are derived and the respective regions of stability are presented.

Keywords: Runge-Kutta, delay differential equations, stability, interpolation

Abstrak. Persamaan pembezaan lengah linear (PPL) diselesaikan dengan kaedah Runge-Kutta menggunakan interpolasi yang berbeza bagi penghampiran sebutan lengahnya. Polinomial kestabilannya diterbitkan dan rantau kestabilannya dipersembahkan.

Kata kunci: Runge-Kutta, persamaan pembezaan lengah, kestabilan, interpolasi

1.0 INTRODUCTION

When dealing with real-world problems, the change of a process $y(t)$ and thus the derivative $y'(t)$ in its mathematical representation often not only depend on the value of the process at present, but also on the past values. The differential equations describing such processes are usually called delay or retarded differential equation, since they also involve terms of the form $y(t-\tau)$ with $\tau > 0$. DDE with a single delay can be written as:

$$y'(t) = f\left(t, y(t), y\left(t-\tau\left(t, y(t)\right)\right)\right) \quad for \quad t \in [a,b] \tag{1a}$$

If we seek the solution of (1a) for $a \leq t \leq b$, an initial function of the following form is required:

$$y(t) = \phi(t) \quad for \quad t \in [a^*, a] \tag{1b}$$

where $a^* = \min\{\tau(t, y(t))\}$.

There are many concepts of stability in numerical methods when applied to DDE, depending on the test equation as well as the delay term involved. The most commonly used test equation in the literature is the linear test equation of the form:

[1,2&3]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia
*Corresponding author: Email: fudziah@fsas.upm.edu.my

$$y'(t) = \lambda\ y(t) + \mu\ y(t-\tau) \qquad t \geq 0 \tag{2}$$
$$y(t) = \phi(t) \qquad t \in [-\tau, 0]$$

where $\lambda, \mu \in$ C, $\tau > 0$. Barwell [1] studied the stability properties of linear multistep methods and backward Euler method. Al-Mutib [2] examined the P-stability and Q-stability properties of the Kutta-Merson method, 3-stage fully implicit Runge-Kutta method and also the Trapezoidal rule. In't Hout [3] dealt with the stability properties of Runge-Kutta methods, especially the subclass collocation method with abscissa in (0,1). Zennaro [4] analyzed the P-stability properties of Runge-Kutta method in general. Ismail [5] analyzed the P-stability and Q-stability of singly diagonally implicit Runge-Kutta method of order five.

Runge-Kutta methods can be adapted to DDEs, when the method is used to solve DDE (1), $y(t-\tau(t, y(t))$ or $y(t_n+c_i h-\tau)$ can be approximated by using interpolation with previously computed values of $y(t)$. There are a number of techniques for obtaining the approximation of $y(t_n+c_ih-\tau)$. In this paper, we are concerned with Lagrange interpolation, Hermite interpolation and continuous extensions Runge-Kutta (CERK) formula; we will not be discussing the approximation of the solution y, but the stability properties and regions for method of order two. Stability properties are needed for consistency of the numerical solution.

2.0 STABILITY POLYNOMIAL OF RUNGE-KUTTA METHOD USING LAGRANGE INTERPOLATION FOR THE DELAY TERM

Assume that the numerical solution has been calculated up to the point t_n with uniform step size h, satisfying $h = \frac{t}{m}$, m is a positive integer. Lagrange interpolation is used to approximate the delay term using previously calculated values of y, giving

$$\begin{aligned} y(t_n + c_i h - \tau) &= y(t_{n-m} + c_i h) \\ &= \sum_{l=r1}^{s1} L_l(c_i)\, y_{n-m+l} \end{aligned} \tag{3}$$

y_{n-m+l} is the calculated value of $y(t_{n-m+l})$, and

$$L_l(c_i) = \prod_{j_1=r_1}^{s_1} \frac{c_i - j_1}{l - j_1}, \quad j_1 \neq l.$$

When s - stage Runge-Kutta method is applied to DDE (1) with constant delay $\tau = 1$, the following equations are obtained.

$$k_{n+1}^{(i)} = f\left(t_n + c_i h, \quad y_n + h\sum_{j=1}^{s} a_{ij} k_{n+1}^{(j)}, \quad \sum_{l=r_1}^{s_1} L_l\left(c_i\right) y_{n-m+l} \right), \tag{4}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_{n+1}^{(i)} \tag{5}$$

Define $e = (1, \ldots, 1)^T$ and for $n \geq 1$,

$$\underline{k}_n = \left(k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(s)}\right)^T$$

$$\underline{b} = \left(b_1, b_2, \ldots, b_s\right)^T$$

and $L_l(c) = \left(L_l(c_1), \ldots, L_l(c_s)\right)^T$

For $n \geq m$, taking f as in Equation (1a), (4) takes the form

$$\underline{k}_{n+1} = \lambda\left(y_n e + hA\underline{k}_{n+1}\right) + \mu\sum_{l=r_1}^{s_1} L_l(c)\, y_{n-m+l} \tag{6a}$$

$$y_{n+1} = y_n + h\underline{b}^T \underline{k}_{n+1} \tag{6b}$$

From Equation (6a) we have

$$(I\text{-}h\lambda A)\underline{k}_{n+1} = \lambda y_n e + \mu\sum_{l=r_1}^{s_1} L_l(c)\, y_{n-m+l}$$

or

$$h\underline{k}_{n+1} = \alpha(I\text{-}h\lambda A)^{-1} y_n e + \beta(I\text{-}h\lambda A)^{-1}\sum_{l=r_1}^{s_1} L_l(c)\, y_{n-m+l} \tag{7}$$

Here $\alpha = h\lambda$, $\beta = h\mu$ and I is the identity matrix.

Substituting Equation (7) into (6b) gives

$$y_{n+1} = y_n + \underline{b}^T\alpha(I\text{-}\alpha A)^{-1} y_n e + \beta\underline{b}^T(I\text{-}\alpha A)^{-1}\sum_{l=r_1}^{s_1} L_l(c)\, y_{n-m+l} \tag{8}$$

Taking $y_n = \left(y_n, h\underline{k}_n\right)^T$, Equations (8) and (7) can be written in the following compact form:

$$y_{n+1} = Xy_n + \sum_{l=r_1}^{s_1} Z_l\, y_{n-m+l} \tag{9}$$

$$X = \left(\begin{array}{c|c} 1+\alpha \underline{b}^T \eta e & \underline{0} \\ \hline \alpha \eta e & \bar{0} \end{array} \right) \quad Z = \left(\begin{array}{c|c} \beta \underline{b}^T \eta L_l & \underline{0} \\ \hline \beta L_l & \bar{0} \end{array} \right)$$

and $\eta = (I - \alpha A)^{-1}$, $\underline{0}$ is matrix 1 by (s-1) with all zero entries and $\bar{0}$ is matrix (s-1) by (s-1) with all zero entries. By putting $n - m + l = 0$, the stability polynomial will be in the standard form, and the recurrence is stable if the zeros ξ_i of the stability polynomial

$$S_p(\alpha, \beta, \xi) = \det \left[\xi^{m+2} I - \xi^{m+1} X - \sum_{l=r_1}^{s_1} \xi^{1+l} Z_l \right] \tag{10}$$

has magnitude less than or equal to one.

It is obvious that the order of the stability polynomial is high, depending on how big matrix A is. To illustrate the stability analysis, the improved Euler method which has lower order and stage will be used so that the order of the polynomial is lower and hence easier to solve. Improved Euler method can be considered as lower order explicit Runge-Kutta method.

Improved Euler method:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

When the method is substituted into Equation (10), the following is obtained.

$$\det \left\{ \xi^{m+2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \xi^{m+1} \begin{bmatrix} 1+\alpha \underline{b}^T \eta e & 0 & 0 \\ \alpha \eta e & 0 & 0 \\ & 0 & 0 \end{bmatrix} - \sum_{l=-1}^{1} \xi^{1+l} \begin{bmatrix} \beta \underline{b}^T \eta L_l(c) & 0 & 0 \\ \beta \eta L_l(c) & 0 & 0 \\ & 0 & 0 \end{bmatrix} \right\} = 0, \tag{11}$$

To obtain the stability regions of methods, the delay term is evaluated using three points Lagrange interpolation by taking $l = -1,0,1$ so that the order of the interpolation is equal to the order of the Runge-Kutta method itself. Stability polynomial of the method is

$$S_p(\alpha, \beta, \xi) = \xi^{m+2} - \left(1 + \alpha b^T \eta e\right) \xi^{m+1} - \beta b^T \eta \left(L_{-1}(c) + L_0(c)\xi + L_1(c)\xi^2\right) \tag{12}$$

In the standard form (the lowest degree of ξ is zero), stability polynomial yields

$$S_p(\alpha, \beta, \xi) = \xi^{m+1} - \left(1 + \alpha + \frac{\alpha^2}{2}\right)\xi^m - \frac{\beta}{2}\xi + \beta\left(\frac{1+\alpha}{2}\right) \tag{13}$$

3.0 STABILITY POLYNOMIAL OF RUNGE-KUTTA METHOD USING HERMITE INTERPOLATION FOR THE DELAY TERM

When *s* stage Runge-Kutta method is applied to DDE (1), using Hermite interpolation for the delay term, (see Ismail *et al.* [6]) the following equations are obtained.

$$k_{n+1}^{(i)} = f\left(t_n + c_i h, y_n + h\sum_{j=1}^{i} a_{ij} k_{n+1}^{(j)} + \sum_{l=r}^{s} H(c_i)\, y_{n-m+l} + \overline{H}(c_i)\, h\, y'_{n-m+l} \right) \quad (14a)$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i\, k_{n+1}^{(i)} \quad (14b)$$

where $H(\underline{c}) = H(c_1),\ldots,H(c_s)$ and $\overline{H} = \overline{H}(c_1),\ldots,\overline{H}(c_s)$ are the coefficients of Hermite interpolation. For $n \geq m$ (14) takes the form

$$\underline{K}_{n+1} = \lambda\left(y_n e + hA\underline{K}_{n+1}\right) + \mu\sum_{l=r}^{s}\left(H_1(\underline{c})\, y_{n-m+l} + h\,\overline{H}_l(\underline{c})\, h\, y'_{n-m+l}\right) \quad (15a)$$

$$y_{n+1} = y_n + h\underline{b}^T \underline{k}_{n+1} \quad (15b)$$

Replacing y'_{n-m+l} and $\lambda\, y_{n-m+l} + \mu\, y_{n-2m+l}$, we will have

$$\underline{K}_{n+1} = \lambda ye + h\lambda A\underline{K}_{n+1} + \mu\sum_{l=r}^{s} H_l(\underline{c}) y_{n-m+l} + \mu h\lambda\sum_{l=r}^{s}\overline{H_l}(\underline{c}) y_{n-m+l}$$
$$+ h\mu^2\sum_{l=r}^{s}\overline{H_l}(\underline{c}) y_{n-2m+l}$$

Hence, we have

$$(I - h\lambda A)\underline{K}_{n+1} = \lambda y_n e + \mu\sum_{l=r}^{s}\left[\left(H_l(\underline{c}) + h\lambda\overline{H_l}(\underline{cc})\right) y_{n-m+l} + h\mu\overline{H}(\underline{c})_l\, y_{n-2m+l}\right]$$

$$h\underline{K}_{n+1} = h\lambda(I - h\lambda A)^{-1} y_n e + h\mu(I - h\lambda A)^{-1}\sum_{l=r}^{s}\left[\left(H_l(\underline{c}) + h\lambda\overline{H_l}(\underline{c})\right) y_{n-m+l}\right.$$
$$\left. + h\mu\overline{H_l}(\underline{c})\, y_{n-2m+l}\right] \quad (16a)$$

Replacing the above equation into (15b) and taking $\alpha = h\lambda$, $\beta = h\mu$ and $\eta = (I - h\lambda A)^{-1}$ we have

$$y_{n+1} = y_n + \alpha \underline{b}^T \eta y_n e + \beta \underline{b}^T \eta \sum_{l=r}^{s} \left[\left(H_l(\underline{c}) + \lambda \overline{H_l}(\underline{c}) \right) y_{n-m+l} + h\mu \overline{H_l}(\underline{c})\, y_{n-2m+l} \right] \tag{16b}$$

Rewritting $Y_{n+1} = \begin{bmatrix} y_{n+1} \\ h\underline{k}_{n+1} \end{bmatrix}$, Equation (16) can be written as:

$$\begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} Y_{n+1} = \begin{bmatrix} 1+\alpha b^T \eta e & 0 \\ \lambda \alpha e & e \end{bmatrix} Y_n + \begin{bmatrix} \beta \underline{b}^T \eta \sum \left(H_l(\underline{c}) + \alpha \overline{H_l}(\underline{c}) \right) & 0 \\ \beta \eta \sum \left(H_l(\underline{c}) + \alpha \overline{H_l}(\underline{c}) \right) & \underline{0} \end{bmatrix} Y_{n-m+l}$$
$$+ \begin{bmatrix} \beta^2 \underline{b}^T \eta \sum \overline{H_l}(\underline{c}) & 0 \\ \beta^2 \eta \sum \overline{H_l}(\underline{c}) & \underline{0} \end{bmatrix} Y_{n-2m+l}$$

I is the identity matrix, replacing Y by ξ, the stability polynomial of the method is

$$S_p(\alpha, \beta, \xi) =$$
$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} \xi^{m+1} - \begin{bmatrix} 1+\alpha \underline{b}^T \eta e & 0 \\ \alpha \eta e & e \end{bmatrix} \xi^m - \begin{bmatrix} \beta \underline{b}^T \eta \sum \left(H_l(\underline{c}) + \alpha \overline{H_l}(\underline{c}) \right) & 0 \\ \beta \eta \sum \left(H_l(\underline{c}) + \alpha \overline{H_l}(\underline{c}) \right) & \underline{0} \end{bmatrix} \xi^{n-m+l} \right.$$
$$\left. - \begin{bmatrix} \beta^2 \underline{b}^T \eta \sum \overline{H_l}(\underline{c}) & 0 \\ \beta^2 \eta \sum \overline{H_l}(\underline{c}) & \underline{0} \end{bmatrix} \xi^{n-2m+l} \right)$$

Again, the improved Euler method is used to illustrate the stability polynomial when Runge-Kutta method is used to solve (1.1) using Hermite interpolation.

$$S_p(\alpha, \beta, \xi) = \xi^{m+1} - \left(1 + \alpha + \frac{\alpha^2}{2} \right) \xi^m - \beta \begin{bmatrix} \frac{1+\alpha}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xi^{n-m}$$
$$- \beta \begin{bmatrix} \frac{1+\alpha}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xi^{n-m+1} - \beta^2 \alpha \begin{bmatrix} \frac{1+\alpha}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$S_p(\alpha, \beta, \xi) = \xi^{m+1} - \left(1 + \alpha + \frac{\alpha^2}{2} \right) \xi^m - \beta \left(\frac{1+\alpha}{2} \right) \xi^{n-m} - \frac{\beta}{2} \xi^{n-m+1}$$

Letting $n = m = 1$ so that the polynomial is again in the standard form. Hermite interpolation polynomial yields:

$$S_p(\alpha, \beta, \xi) = \xi^2 - \left(1 + \alpha + \frac{\alpha^2}{2} \right) \xi - \frac{\beta}{2} \xi - \beta \left(\frac{1+\alpha}{2} \right) \tag{17}$$

4.0 STABILITY POLYNOMIAL OF CERK FORMULA FOR DDES

Paul and Baker [7] considered the stability analysis for CERK formula,

$$u_n(nh+\theta h) = u_n(nh) + h\sum_{r=1}^{s} b_r(\theta) Y'_{n,r}$$

$$Y'_{n,r} = \lambda Y_{n,r} + \mu u_m(nh + c_r h - \tau)$$

when $\theta = 0$, ($\tau = mh$). They considered the stability properties of a recurrence of the form

$$y_{n+1} = \left[1 + \lambda h b^T (I - \lambda h A)^{-1} e\right] y_n + \mu h \underline{b}^T (I - \lambda h A)^{-1} U_{n-m}, \tag{18}$$

in which U_{n-m} is a vector consisting of 'back-values' $u(nh+c_i h-\tau)$.

Taking $\theta = 0$ and $\tau = mh$ where

$$u(nh + c_i h - mh) = u((n-m)h + c_i h),$$

which we can use the internal values at the point t_{n-m} to t_{n-m+1}.

Taking $\eta = (I - \lambda h A)^{-1}$ and $r(\lambda h) = 1 + \lambda h \underline{b}^T \eta e$

$$y_{n+1} = r(\lambda h)\, y + \mu\ h \underline{b}^T\ \eta\ \phi_{n-m} \tag{19}$$

Introducing the vector

$$\Phi_n = \left[y_{n+1}, Y_{n,1}, Y_{n,2}, \ldots, Y_{n,s}\right]^T$$

where $y_{n,i},\ i = 1, \ldots, s$ are the internal values.

The recurrence equation can be expressed as:

$$\Phi_n = X\Phi_{n-1} + Z\Phi_{n-m} \tag{20}$$

where

$$X = \left(\begin{array}{c|c} r(\lambda h) & \underline{0} \\ \hline \eta e & \bar{0} \end{array}\right), \quad Z = \left(\begin{array}{c|c} \underline{0} & \mu h b^T \eta \\ \hline \bar{0} & \mu h \eta A \end{array}\right)$$

This recurrence equation is stable if the zeros ξ_i of the stability polynomial

$$S_p(\alpha, \beta, \xi) = \det\left[\xi^m I - \xi^{m-1} X - Z\right] \tag{21}$$

satisfy the root condition. Here $\lambda h = \alpha$, $\mu h = \beta$ and I is the identity matrix.

Equation (21) can be expressed as follows:

$$S_p(\alpha,\beta,\xi) = \det\left\{\xi^m \begin{bmatrix} 1 & \underline{0} \\ \underline{0} & I \end{bmatrix} - \xi^{m-1}\begin{bmatrix} r(\alpha) & \underline{0} \\ \eta e & \underline{0} \end{bmatrix} - \begin{bmatrix} \underline{0} & \beta \underline{b}^T \eta \\ \underline{0} & \beta\eta A \end{bmatrix}\right\},$$

Again, to illustrate the stability region, the improved Euler method is used.

$$\left[\xi^m - \left(1+\alpha+\frac{\alpha^2}{2}\right)\xi^{m-1}\right]\xi^m\xi^m - \frac{1}{2}\beta^2\xi^{m-1}$$

$$-\left[\frac{1}{2}\beta(1+\alpha)\xi^{m-1}\xi^m + \frac{1}{2}\beta(1+\alpha)\xi^m\xi^{m-1}\right] = 0$$

$$\xi^{3m} - \left(1+\alpha+\frac{\alpha^2}{2}\right)\xi^{3m-1} - \frac{1}{2}\beta^2\xi^{m-1} - \beta(1+\alpha)\xi^{2m-1} = 0$$

$$\xi^{3m} - \left(1+\alpha+\frac{\alpha^2}{2}\right)\xi^{3m-1} - \beta(1+\alpha)\xi^{2m-1} - \frac{1}{2}\beta^2\xi^{m-1} = 0$$

Taking $m = 1$, the stability polynomial of the method is

$$S_p(\alpha,\beta,\xi) = \xi^3 - \left(1+\alpha+\frac{\alpha^2}{2}\right)\xi^2 - \beta(1+\alpha)\xi - \frac{1}{2}\beta^2 \quad (22)$$

5.0 LOCATING STABILITY BOUNDARIES

In this paper, the Fortran 77 code of a local grid search algorithm (LGSA) developed by Paul [8] is used to find the stability regions.

The technique here is to determine whether $S_p(\lambda,\mu,\xi)$ satisfies the root-condition for each grid point. The strict root condition is usually satisfied in the interior of the stability region. The LGSA is to determine the boundary of the stability region, searching in the neighborhood of a known boundary-point. The basis of the algorithm is to impose a local grid about a known stability boundary point.

The root condition is then tested at all the grid points, going around the grid in either a clockwise or anti-clockwise direction, in order to determine the path of the stability boundary. Once the path of the stability boundary has been found, the search is stopped and the current stability boundary point updated. In the case that the stability region is closed, this process is repeated until the stability boundary returns to the starting point. In certain case, it is possible that due to rounding error, the stability boundary of a closed stability region does not return to the starting point. In this eventuality the LGSA needs to be stopped manually. The point $(\lambda h, \mu h) = (0,0)$ is always on the boundary and the search commenced by investigating points on a grid surrounding (0,0), which is appropriate to start searching in the direction of λh and μh in case of a consistent Runge-Kutta method.

To get the stability regions, the LGSA is used by applying it to the stability polynomials (13), (17) and (22). This is done when improved Euler method is used together with Lagrange interpolation, Hermite interpolation and continuous extensions Runge-Kutta formula to evaluate the delay term.

6.0 ILLUSTRATIVE RESULTS

The following figures present the stability regions of improved Euler method using Lagrange interpolation, Hermite interpolation and continuous extensions of Runge-Kutta formula to approximate the delay term.

For comparison purposes, they are all plotted on the same scale.

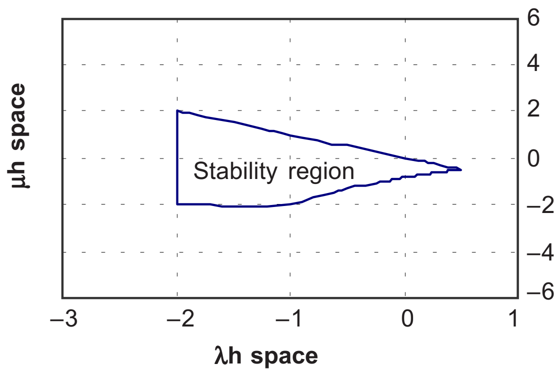

Figure 1 Stability region of improved Euler method using Lagrange interpolation to approximate the delay term

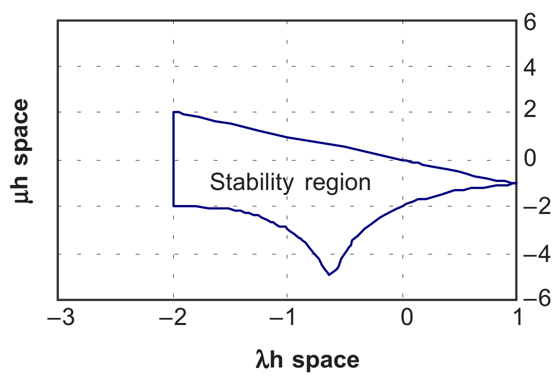

Figure 2 Stability region of improved Euler method using Hermite Interpolation to approximate the delay term

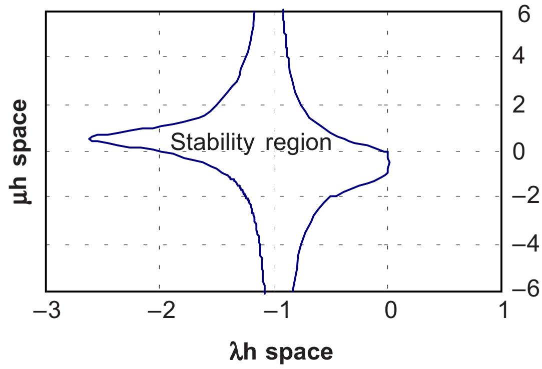

Figure 3 Stability region of improved Euler method using CERK formula to approximate the delay term

7.0 CONCLUSIONS

When analyzing a stability region, we are usually concerned with the non-positive half plane of the region. From Figures 1-3, looking only at the left half plane, it was observed that Hermite Interpolation gives region of stability which is slightly bigger than stability regions for CERK method and Lagrange interpolation. Based on the above arguments, it can be concluded that in general, Hermite Interpolation is a better choice of interpolation since it has bigger region of stability and hence larger step size can be used in solving DDEs. This is consistent with the numerical results obtained in Lwin [9].

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