Jurnal Teknologi

Full paper

A Homological Invariant of a Bieberbach Group with Dihedral Extension

Siti Afiqah Mohammad, Nor Haniza Sarmin,* Nor Muhainiah Mohd Ali, Hazzirah Izzati Mat Hassim

Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor Malaysia

*Corresponding author: nhs@utm.my

Article history

Received : 31 July 2014
Received in revised form :
23 November 2014
Accepted : 1 December 2014

Graphical abstract

$$G \wedge G = {(G \otimes G)}\big/{\nabla(G)}$$

Abstract

A Bieberbach group is a crystallographic group which is an extension of a free abelian group of finite rank by a finite extension. Meanwhile, research on homological invariants has been on interest of many authors since it is related to the study of the properties of the crystal using mathematical approach. One of the homological invariants is the exterior square. In this paper, the exterior square of a Bieberbach group of dimension four with dihedral extension is computed theoretically.

Keywords: Bieberbach group; homological invariant; dihedral extension

Abstrak

Kumpulan Bieberbach merupakan kumpulan kristal yang merupakan perluasan kepada kumpulan abelan bebas melalui perluasan terhingga. Sementara itu, kajian terhadap homologi tak varian telah menjadi tumpuan ramai penyelidik kerana ia berkait dengan kajian terhadap ciri-ciri suatu kristal menggunakan pendekatan matematik. Salah satu daripada homologi tak varian ini ialah kuasa dua peluaran. Dalam kertas kerja ini, kuasa dua peluaran kumpulan Bieberbach berdimensi empat dengan perluasan dwihedron telah dikira secara teori.

Kata kunci: Kumpulan Bieberbach; homologi tak varian; perluasan dwihedron

1.0 INTRODUCTION

The nonabelian tensor square, denoted as $G \otimes G$, is defined as the product of two groups where the two groups act on each other in a compatible way and their actions are taken to be conjugation. A factor group of the nonabelian tensor square with a central subgroup of the nonabelian tensor square of the group, known as the exterior square, denoted as $G \wedge G$, is one of the homological invariants that we are focusing in this paper. Research on this topic has been growing over the years. In 2008, the exterior squares for symmetric groups of order six have been computed.[1] While in 2011, the exterior square for infinite nonabelian 2-generator groups of nilpotency class two has been found.[2] Furthermore, in 2012, Mat Hassim *et al.* computed the exterior squares for some Bieberbach groups with cyclic point group of order two.[3] In this paper we are motivated to compute the exterior square for a Bieberbach group of dimension four with dihedral extension.

2.0 PRELIMINARIES

The definitions, theorems and propositions that are used in this paper are given in the following.

Definition 1[4]

Let G be a group with a presentation $\langle \varsigma \mid \mathfrak{R} \rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \left\langle \begin{array}{l} \varsigma, \varsigma^{\varphi} \mid \mathfrak{R}, \mathfrak{R}^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x\varphi}[g, h^{\varphi}], \\ \forall x, g, h \in G \end{array} \right\rangle.$$

Theorem 1[5]

Let G be a group. The map $\sigma : G \otimes G \to \left[G, G^{\varphi}\right] \triangleright \nu(G)$ defined by $\sigma(g \otimes h) = \left[g, h^{\varphi}\right]$ for all g, h in G is an isomorphism.

Definition 2[6]

Let G be any group. Then $\tau(G)$ is defined to be quotient group ${\nu(G)}\big/{\sigma(\nabla(G))}$, where $\sigma : G \otimes G \to \left[G, G^{\varphi}\right]$ is as defined in Theorem 1.

Definition 2[7] **Exterior Square**

The exterior square of G is defined as $G \wedge G = {}^{(G \otimes G)}\!\big/_{\nabla(G)}.$

Proposition 1[6]

Let G be any group. The map $\hat{\sigma} : G \wedge G \to \left[G, G^{\varphi}\right]_{\tau(G)} \rhd \tau(G)$ defined by $\hat{\sigma}(g \wedge h) = \left[g, h^{\varphi}\right]_{\tau(G)}$ is an isomorphism.

Since $\tau(G)$ is a subgroup of $\nu(G)$, $\left[g, h^{\varphi}\right]_{\tau(G)}$ coincides with $\left[g, h^{\varphi}\right]$. Therefore, for simplification, we use $\left[g, h^{\varphi}\right]$ instead of $\left[g, h^{\varphi}\right]_{\tau(G)}$.

Proposition 2[6]

Let G be a polycyclic group with a polycyclic generating sequence $g_1, g_2, \ldots\ldots, g_k$. Then $\left[G, G^{\varphi}\right]$ a subgroup of $\nu(G)$ is given by

$$\left[G, G^{\varphi}\right] = \left\langle \left[g_i, g_i^{\varphi}\right], \left[g_i^{\varepsilon}, (g_j^{\varphi})^{\partial}\right], \left[g_i^{\varepsilon}, (g_j^{\varphi})^{\partial}\right]\left[g_j^{\partial}, (g_i^{\varphi})^{\varepsilon}\right] \right\rangle$$

and

$$\left[G, G^{\varphi}\right]_{\tau(G)} = \left\langle \left[g_i^{\varepsilon}, (g_j^{\varphi})^{\partial}\right], \left[g_j^{\varepsilon}, (g_i^{\varphi})^{\partial}\right] \right\rangle \text{ for } 1 \le i < j \le k, \text{ where}$$

$$\varepsilon = \begin{cases} 1 & \text{if } |g_i| < \infty \\ \pm 1 & \text{if } |g_i| = \infty \end{cases} \quad \text{and} \quad \partial = \begin{cases} 1 & \text{if } |g_i| < \infty \\ \pm 1 & \text{if } |g_i| = \infty \end{cases}$$

By Theorem 1, the subgroup $\left[G, G^{\varphi}\right]$ of $\nu(G)$ is isomorphic to $G \otimes G$.

The commutator calculus that are used in this paper are as follows:

$$y^x = y[y, x]; \tag{1.1}$$

$$[xy, z] = [x, z]^y . [y, z] = [x, z].[[x, z], y].[y, z]; \tag{1.2}$$

$$[x, yz] = [x, z].[x, y]^z = [x, z].[x, y].[[x, y], z]; \tag{1.3}$$

$$\left[x, y^{-1}\right] = [x, y]^{-y^{-1}} = \left[y^{-1}, [x, y]\right].[x, y]^{-1}. \tag{1.4}$$

The following lemma is used in the computation.

Lemma 1[5-6]

Let G be a group. The following relations hold in $\nu(G)$:

i) $\left[g_1, g_2^{\varphi}\right]^{\left[g_3, g_4^{\varphi}\right]} = \left[g_1, g_2{}^{\varphi}\right]^{\left[g_3, g_4\right]}$ for all g_1, g_2, g_3, g_4 in G;

ii) $\left[g_1, g_2^{\varphi}, g_3\right] = \left[g_1, g_2, g_3^{\varphi}\right] = \left[g_1, g_2^{\varphi}, g_3^{\varphi}\right]$ and

$\left[g_1^{\varphi}, g_2, g_3\right] = \left[g_1^{\varphi}, g_2, g_3^{\varphi}\right] = \left[g_1^{\varphi}, g_2^{\varphi}, g_3\right]$

for all g_1, g_2, g_3 in G.

Theorem 2[8]

The polycyclic presentation of a Bieberbach group of dimension four with dihedral point group of order eight denoted as $B_1(4)$ has been shown to be:

$$B_1(4) = \left\langle a, b, c, l_1, l_2, l_3, l_4 \;\middle|\; \begin{array}{l} a^2 = l_3^{-1},\ b^2 = l_2,\ b^a = cl_3^{-1},\ c^2 = l_1^{-1}, \\ c^a = bl_3^{-1},\ c^b = cl_1 l_2 l_4^{-1},\ l_1^a = l_2^{-1}, \\ l_1^b = l_1^{-1}, l_1^c = l_1,\ l_2^a = l_1^{-1},\ l_2^b = l_2, \\ l_2^c = l_2^{-1},\ l_3^a = l_3,\ l_3^b = l_3^{-1},\ l_3^c = l_3^{-1}, \\ l_4^a = l_4^{-1},\ l_4^b = l_4^{-1}, l_4^c = l_4^{-1},\ l_j^{l_i} = l_j, \\ l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \le i, j \le 4 \end{array} \right\rangle \tag{1.5}$$

Theorem 3[8]

The nonabelian tensor square of a Bieberbach group of dimension four with dihedral point group of order eight denoted as $B_1(4)$ has been established as in the following:

$$B_1(4) \otimes B_1(4) = \left\langle \begin{array}{l} \left[a, a^{\varphi}\right], \left[c, c^{\varphi}\right], \left[a, b^{\varphi}\right], \left[a, c^{\varphi}\right], \left[a, l_1^{\varphi}\right], \\ \left[b, l_3^{\varphi}\right], \left[c, l_2^{\varphi}\right], \left[a, c^{\varphi}\right]\left[c, a^{\varphi}\right] \end{array} \right\rangle$$

3.0 MAIN RESULT

By using Theorem 2 and Theorem 3, our main result in this research is given in the following.

Main Theorem

Let $B_1(4)$ be a Bieberbach group of dimension four with dihedral point group of order eight. Then, the exterior square of $B_1(4)$ is given as:

$$B_1(4) \wedge B_1(4) = \left\langle \begin{array}{l} a \wedge b,\ c \wedge b,\ a \wedge l_1, \\ b \wedge l_3,\ c \wedge l_2 \end{array} \right\rangle.$$

Proof:

Proposition 1 shows that

$$B_1(4) \wedge B_1(4) \cong \left[B_1(4), B_1(4)^{\varphi}\right]_{\tau(B_1(4))}.$$

Then, based on Proposition 2 , since $B_1(4)$ is a polycyclic group generated by polycyclic generating sequence $a, b, c, l_1, l_2, l_3, l_4$, then

$$\left[B_1(4), B_1(4)^{\varphi}\right]_{\tau(B_1(4))} = \left\langle \begin{array}{l} \left[a^{\pm 1}, b^{\pm\varphi}\right], \left[b^{\pm 1}, a^{\pm\varphi}\right], \left[a^{\pm 1}, c^{\pm\varphi}\right], \left[c^{\pm 1}, a^{\pm\varphi}\right], \\ \left[b^{\pm 1}, c^{\pm\varphi}\right], \left[c^{\pm 1}, b^{\pm\varphi}\right], \left[a^{\pm 1}, l_1^{\pm\varphi}\right], \left[l_1^{\pm 1}, a^{\pm\varphi}\right], \\ \left[a^{\pm 1}, l_2^{\pm\varphi}\right], \left[l_2^{\pm 1}, a^{\pm\varphi}\right], \left[a^{\pm 1}, l_3^{\pm\varphi}\right], \left[l_3^{\pm 1}, a^{\pm\varphi}\right], \\ \left[a^{\pm 1}, l_4^{\pm\varphi}\right], \left[l_4^{\pm 1}, a^{\pm\varphi}\right], \left[b^{\pm 1}, l_1^{\pm\varphi}\right], \left[l_1^{\pm 1}, b^{\pm\varphi}\right], \\ \left[b^{\pm 1}, l_2^{\pm\varphi}\right], \left[l_2^{\pm 1}, b^{\pm\varphi}\right], \left[b^{\pm 1}, l_3^{\pm\varphi}\right], \left[l_3^{\pm 1}, b^{\pm\varphi}\right], \\ \left[b^{\pm 1}, l_4^{\pm\varphi}\right], \left[l_4^{\pm 1}, b^{\pm\varphi}\right], \left[c^{\pm 1}, l_1^{\pm\varphi}\right], \left[l_1^{\pm 1}, c^{\pm\varphi}\right], \\ \left[c^{\pm 1}, l_2^{\pm\varphi}\right], \left[l_2^{\pm 1}, c^{\pm\varphi}\right], \left[c^{\pm 1}, l_3^{\pm\varphi}\right], \left[l_3^{\pm 1}, c^{\pm\varphi}\right], \\ \left[c^{\pm 1}, l_4^{\pm\varphi}\right], \left[l_4^{\pm 1}, c^{\pm\varphi}\right], \left[l_1^{\pm 1}, l_2^{\pm\varphi}\right], \left[l_2^{\pm 1}, l_1^{\pm\varphi}\right], \\ \left[l_1^{\pm 1}, l_3^{\pm\varphi}\right], \left[l_3^{\pm 1}, l_1^{\pm\varphi}\right], \left[l_1^{\pm 1}, l_4^{\pm\varphi}\right], \left[l_4^{\pm 1}, l_1^{\pm\varphi}\right], \\ \left[l_2^{\pm 1}, l_3^{\pm\varphi}\right], \left[l_3^{\pm 1}, l_2^{\pm\varphi}\right], \left[l_2^{\pm 1}, l_4^{\pm\varphi}\right], \left[l_4^{\pm 1}, l_2^{\pm\varphi}\right], \\ \left[l_3^{\pm 1}, l_4^{\pm\varphi}\right], \left[l_4^{\pm 1}, l_3^{\pm\varphi}\right] \end{array} \right\rangle . \quad (1.6)$$

The generators of $\left[B_1(4), B_1(4)^{\varphi}\right]_{\tau(B_1(4))}$ can be reduced to the minimum independent generators as they might be written as products of other generators. All commutators in that set that have negative power can actually be eliminated since they can be written as integer powers of its positive commutators or some other positive commutators. For examples,

$$\begin{aligned} \left[b, l_1^{-\varphi}\right] &= \left[l_1^{-1}, \left[b, l_1^{\varphi}\right]\right]\left[b, l_1^{\varphi}\right]^{-1} && \text{by (1.4)} \\ &= \left[l_1^{-1}, \left[b, l_1\right]^{\varphi}\right]\left[b, l_1^{\varphi}\right]^{-1} && \text{by Lemma 1 (ii)} \\ &= \left[l_1^{-1}, (l_1)^{2\varphi}\right]\left[b, l_1^{\varphi}\right]^{-1} && \text{by the relation of } B_1(4) \\ &= \left[c^2, c^{-\varphi}\right]\left[b, l_1^{\varphi}\right]^{-1} && \text{by the relation of } B_1(4) \\ &= \left[c, c^{\varphi}\right]^{-2}\left[b, l_1^{\varphi}\right]^{-1} && \text{by Lemma 1 (i)} \\ &= \left[b, l_1^{\varphi}\right]^{-1} && \text{since } \left[c, c^{\varphi}\right] \text{ is trivial in } \left[B_1(4), B_1(4)^{\varphi}\right]_{\tau(B_1(4))}, \end{aligned}$$

$$\begin{aligned} \left[a, c^{\varphi}\right] &= \left[l_3^{-2}, l_1^{-2\varphi}\right] && \text{by the relation of } B_1(4) \\ &= \left[l_3^{-2}, l_1^{-\varphi}\right]\left[l_3^{-2}, l_1^{-\varphi}\right] \\ &\quad \left[\left[l_3^{-2}, l_1^{-1}\right], l_1^{-\varphi}\right] && \text{by (1.3)} \\ &= \left[a, l_1^{-\varphi}\right]\left[a, l_1^{-\varphi}\right]\left[\left[a, l_1^{-1}\right], l_1^{-\varphi}\right] && \text{by the relation of } B_1(4) \\ &= \left[a, l_1^{-\varphi}\right]\left[a, l_1^{-\varphi}\right]\left[l_2^{-1} l_1^{-1}, l_1^{-\varphi}\right] && \text{by the relation of } B_1(4) \\ &= \left[a, l_1^{-\varphi}\right]^2\left[l_2^{-1}, l_1^{-\varphi}\right] \\ &\quad \left[\left[l_2^{-1}, l_1^{-1}\right], l_1^{-\varphi}\right]\left[l_1^{-1}, l_1^{-\varphi}\right] && \text{by (1.2)} \\ &= \left[a, l_1^{-\varphi}\right]^2. \end{aligned}$$

Hence (1.6) can be written as shown in the following.

$$\left[B_1(4), B_1(4)^{\varphi}\right]_{\tau(B_1(4))} = \left\langle \begin{array}{l} \left[a, b^{\varphi}\right], \left[c, b^{\varphi}\right], \left[a, l_1^{\varphi}\right], \\ \left[b, l_3^{\varphi}\right], \left[c, l_2^{\varphi}\right] \end{array} \right\rangle .$$

Therefore, by Proposition 2,

$$B_1(4) \wedge B_1(4) = \left\langle \begin{array}{l} a \wedge b,\ c \wedge b,\ a \wedge l_1, \\ b \wedge l_3,\ c \wedge l_2 \end{array} \right\rangle .$$

4.0 CONCLUSION

In this paper, one of the homological invariants, namely the exterior square of a torsion free space group with dihedral extension has been computed by transforming first the presentation into polycyclic form.

Acknowledgement

The authors would like to express their appreciation for the support of the sponsor; Ministry of Education (MOE) for the financial funding of this research through Research Acculturation Collaborative Effort (RACE) grant Vote No: 00M04 and Fundamental Research Grant Scheme (FRGS), Vote No: 4F545 from Research Management Centre (RMC) Universiti Teknologi Malaysia (UTM). The first author is also indebted to UTM for her Zamalah Scholarship.

References

[1] V.A.R. Ramachandran, N.H. Sarmin, N.M. Mohd Ali. 2008. *Proceeding of Regional Annual Fundamental Science Seminar*. 350–354.
[2] H.I. Mat Hassim, N.H. Sarmin, N.M. Mohd Ali and M.S. Mohamad. 2011. *Journal of Quality Measurement and Analysis*. 7(1): 153.
[3] H.I. Mat Hassim, N.H. Sarmin, N.M. Mohd Ali, R. Masri and N. Mohd Idrus. 2012. *Jurnal Teknologi*. 20: 55.
[4] N.R. Rocco. 1991. *Bol.Soc.Brasil.Mat (N.S)*. 22(1): 63.
[5] G. Ellis, F. Leonard. 1995. *Proc.Roy.Irish Acad, Sect. A*. 95(2): 137.
[6] R.D. Blyth, R.F. Morse. 2009. *Journal of Algebra*. 321: 2139.
[7] M. Bacon, L.C. Kappe. 2003. *Illinois Journal of Math*. 47: 49.
[8] N. Mohd Idrus. 2011. PhD. Thesis, Universiti Teknologi Malaysia.