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FREE CONVECTION BOUNDARY LAYER OVER A NONISOTHERMAL VERTICAL FLAT PLATE

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Abstract. Numerical solutions are presented for the free convection boundary layer problems on a vertical flat plate with prescribed temperature or heat flux distributions, namely the sinusoidal wall temperature and the exponential heat flux variations. The numerical computation is carried out using a very efficient implicit finite difference scheme known as the Keller-box method. Illustrative computational examples are carried out and the present results are compared with previously available theoretical results obtained using other methods of solution, and they are found to be in good agreement. Comparisons of nondimensional temperature gradient for sinusoidal wall temperature variation and of nondimensional wall temperature with exponential variation in wall heat flux are made between the present and previous results. New results for the variation of the surface shear stress with various Prandtl numbers are also presented. In addition, for the case of sinusoidal wall temperature variation, representative velocity and temperature profiles are presented for Prandtl numbers 0.7, 1, 10 and 100, while for the case of exponential heat flux distribution, the velocity and temperature profiles for various transformed streamwise coordinate $\xi = 0, 1, 10$ and 100 are illustrated.

Keywords: Free convection, boundary layer, nonisothermal vertical plate, numerical method

Abstrak. Penyelesaian berangka bagi masalah lapisan sempadan olakan bebas terhadap plat menegak dengan taburan suhu atau fluks haba yang ditetapkan akan dibincangkan. Perbincangan tertumpu kepada dua jenis syarat sempadan iaitu, variasi sinusoid suhu dinding dan variasi eksponen fluks haba. Dalam kajian ini, pengiraan berangka dilakukan menggunakan suatu skema beza terhingga tersirat yang efisien, yang dinamakan kaedah kotak Keller. Ilustrasi pengiraan dijalankan dan keputusan baru yang diperolehi dibandingkan dengan keputusan-keputusan lepas yang diperolehi menggunakan kaedah-kaedah penyelesaian yang lain, dan didapati bahawa hasil perbandingan adalah baik. Perbandingan melibatkan kecerunan suhu tak berdimensi bagi variasi sinusoid suhu dinding dan suhu tak berdimensi bagi variasi eksponen fluks haba. Keputusan baru bagi variasi tegasan ricih permukaan dengan pelbagai nombor Prandtl juga dibincangkan. Di samping itu, bagi kes variasi sinusoid suhu dinding, profil halaju dan suhu diberikan bagi beberapa nombor Prandtl iaitu 0.7, 1, 10 dan 100, manakala bagi kes variasi eksponen fluks haba, profil halaju dan suhu juga diketengahkan untuk beberapa nilai koordinat mengikut-strim yang terjelmakan ξ iaitu 0, 1, 10 and 100.

Kata kunci: Olakan bebas, lapisan sempadan, plat menegak tak isoterma, kaedah berangka

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1.0 INTRODUCTION

Since the Schmidt and Beckmann experiments in 1930 [1], the study of free convection along a vertical flat plate has been intensively studied due to its extensive applications in engineering like electronic cooling equipment, building applications or crystal growth processes. It remains a subject of interest either theoretically or experimentally due to the number of possible variations in the boundary conditions.

The free convection problem on a nonisothermal vertical plate under boundary layer approximations has been studied by several authors [2–5] using approximate methods such as the integral and series expansion methods. Similarity solutions for free convection over nonisothermal vertical plates have been provided by Sparrow and Gregg [6] and Finston [7]. Yang [8] has verified that the similarity possibilities have essentially been covered by these authors, namely the cases of power law and exponential wall temperature distributions.

However, many problems of interest in boundary layer flow and heat transfer do not admit similarity solutions. A number of solution methods have been proposed to deal with such problems. Similarity transformations are limited to some specific cases, hence research has been undertaken in order to expand available solutions to include problems with non-similar surface conditions. The non-similarity of boundary layer may stem from a variety of causes. The most common is perhaps the nonsimilarity of the velocity boundary layer [9]. A number of solution methods involving different degrees of approximations and various levels of numerical effort have been proposed to deal with such problems.

Sparrow and co-workers [9,10] presented an approximate method known as the local nonsimilarity method for solving nonsimilar boundary layer problems and to improve the local similarity method. This method gives more accurate results than the series and integral approximate methods. For the case of free convection boundary layer over a vertical flat plate, the local non-similarity method was employed by Kao [11] in which only the case of sinusoidal wall temperature variation is considered. Kao and Elrod [12] have developed another approximate method known as the strained coordinate method, for the solution of nonsimilar boundary layer problems. The results are found to be slightly more accurate than those of the local nonsimilarity method. Kao *et al.* [13] have applied this technique to study the free convection problem on a vertical flat plate with sinusoidal and exponential wall temperature variations and linearly varying and exponentially increasing heat flux. The general theory of this technique is developed with local similarity as a first approximation, and universal functions for improvement. Yang *et al*. [14] applied appropriate coordinate transformation and the Merk-type series to solve a similar type of free convection problems with variable surface temperature and heat flux as considered in [13]. Previous works [11,13,14] have demonstrated the need for a study of this problem by an efficient numerical method. Numerical computations, using finite difference methods, have been reported for the cases of uniform, step jump discontinuity, power law, sinusoidal and exponential variations of wall temperature [15–17].

In this paper, a numerical study is considered for the problem of free convection boundary layer over a nonisothermal vertical plate for the cases of sinusoidal wall temperature and exponential heat flux variations using a very efficient finite difference scheme known as the Keller-box method [18]. The present numerical method yields accurate results, and this method has been successfully used recently by the present authors [19]–[22]. Graphs and table are provided for the cases of Prandtl number equal to 0.7 to facilitate such computation.

Comparisons of nondimensional temperature gradient for sinusoidal wall temperature variation and of nondimensional wall temperature with exponential variation in wall heat flux are made between the present method and the corresponding similarity [6], local similarity [13,14], local nonsimilarity [11], strained coordinate [13], and the Merk-type series [14] solutions. The results of the present numerical method are found to be in good agreement particularly with [13] and [14]. New results for the variation of the surface shear stress for various Prandtl numbers are presented. Representative results of velocity and temperature profiles for different values of Prandtl number (Pr = 0.7, 1, 10 and 100) and transformed streamwise coordinate (ξ = 0, 1, 10 and 100) are also presented. The different values of Prandtl number, i.e. Pr = 0.7, 1, 10 and 100 are chosen because, Pr = 0.7 is well known for convection in air, while significant differences may exist in the convection styles of fluids due to changes in Prandtl number in the intermediate range from Pr = 1 to Pr = 100. Fluids in this range have important industrial applications: gases, with Pr ~ 1; water depending on temperature, with Pr ~ 10; and oils or water solutions, with Pr ~ 100. On the other hand, different values of the streamwise coordinate, i.e. ξ = 0, 1, 10 and 100 are presented in order to see how the velocity and temperature profiles change as the streamwise coordinate increases from 0 to 100.

2.0 BASIC EQUATIONS AND TRANSFORMATIONS

The governing equations for the steady state laminar boundary layer free convection over a vertical flat plate are given as follows:

Continuity
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = g\beta\left(T - T_\infty\right) + v\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Energy
$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The boundary conditions are

$$u(x,0) = v(x,0) = 0, \quad u(x,\infty) = 0, \quad T(x,\infty) = T_\infty \tag{4}$$

$$T(x,0) = T_w(x) \quad \text{or} \quad \frac{\partial T(x,0)}{\partial y} = -\frac{q_w(x)}{k} \tag{5a,b}$$

The fluid density is assumed to be constant (Boussinesq approximation), and u, v, T, T_w, T_∞, ν, g, β, α, k and q_w are velocity components in x- and y-direction, respectively, local temperature, surface temperature, ambient temperature, kinematic viscosity, gravitational acceleration, coefficient of thermal expansion, thermal diffusivity, thermal conductivity and heat flux, respectively. The coordinate system is shown in Figure 1 below.

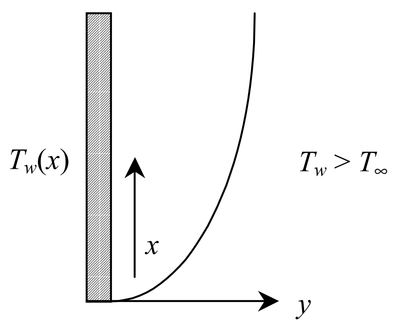

Figure 1 Physical model and coordinate system

A stream function ψ is defined as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{6}$$

The continuity equation (1) is automatically satisfied by the introduction of the stream function. The (x,y) coordinate system is transformed into the (ξ,η) system by introducing [13,14]

$$\xi = \int_0^x F(x)\,dx, \tag{7}$$

and
$$\eta = C_1 F(x)^{1/2} \frac{y}{\xi^{1/4}} \tag{8}$$

where
$$F(x) = T_w(x) - T_\infty \tag{9}$$

for the case of prescribed variable wall temperature,

and
$$F(x) = Q^{2/3}\left[\frac{5}{6}\int_0^x Q^{2/3}dx\right]^{1/5}, \tag{10}$$

where
$$Q(x) = \frac{q_w(x)}{C_1 k}, \text{ and } C_1 = \left(\frac{g\beta}{4\nu^2}\right)^{1/4}, \tag{11}$$

for the case of prescribed variable wall heat flux.

The following non-dimensional stream function and temperature function are introduced:

$$f(\xi,\eta) = F(x)^{1/2}\psi 4C_1\nu\xi^{3/4}, \tag{12}$$

$$\theta(\xi,\eta) = \frac{T - T_\infty}{F(x)}. \tag{13}$$

With these substitutions, the set of the conservation equations (2) and (3), governing laminar free convection adjacent to a vertical wall are transformed into the following ordinary differential equations:

$$f''' + (3-2B)ff'' - 2f'^2 + \theta = 4\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right), \tag{14}$$

$$\theta'' + (3-2\text{B})\Pr f\theta' - 4\text{B}\Pr f'\theta = 4\Pr\xi\left(f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right), \tag{15}$$

where ξ, η, and Pr are the transformed streamwise coordinate, transformed normal coordinate and Prandtl number, respectively. Primes denote derivatives with respect to η, and B is defined as

$$\text{B} = \frac{\xi}{F(x)^2}\frac{dF}{dx}, \tag{16}$$

where F is as defined previously in (9) and (10).

The transformed boundary conditions associated with the transformed equations are:

$$F(\xi,0) = f'(\xi,0) = 0, \quad f'(\xi,\infty) = 0, \quad \theta(\xi,\infty) = 0, \tag{17}$$

$$\theta(\xi,0) = 1 \quad \text{or} \quad \theta'(\xi,0) = -1 \tag{18a,b}$$

3.0 NUMERICAL METHOD

Equations (14) and (15) consisting of nonlinear, coupled equations are solved simultaneously by numerical integration subject to boundary conditions (17) and (18). They were solved numerically using a very efficient finite difference scheme. The scheme employed is the box method developed by Keller [18]. This method has been shown to be particularly accurate for parabolic problems. It is much faster, easier to program and it is chosen because it seems to be the most flexible of the common methods, being easily adaptable to solving equations of any order. The Keller-box method is essentially an implicit finite difference scheme, which has been found to be very suitable in dealing with nonlinear problems. Details of the method may be found in many recent publications, and here we have used the procedure outlined in [23].

One of the basic ideas of the box method is to write the governing system of equations in the form of a first order system. First derivatives of u and other quantities with respect to η must therefore be introduced as new unknown functions. With the resulting first order equations, the "centered-difference" derivatives and averages at the midpoints of net rectangles and net segments are used, as they are required to get accurate finite difference equations.

The resulting finite difference equations are implicit and nonlinear. Newton's method is first introduced to linearize the nonlinear system of equations before a block-tridiagonal factorization scheme is employed on the coefficient matrix of the finite difference equations for all η at a given ξ. The solution of the linearized difference equations can be obtained in a very efficient manner by using the block-elimination method [23]. All the results quoted here were obtained using uniform grids in both the ξ and η directions.

4.0 RESULTS AND DISCUSSIONS

A numerical example involving a sinusoidal surface temperature distribution will be considered. The distribution is given by

$$T_w(x) - T_\infty = \sin x.$$

With this surface condition, the parameters required for the heat transfer calculation can be expressed as

$$\xi = 1 - \cos x, \qquad \text{and} \qquad \mathrm{B} = \frac{\cos x}{1 + \cos x}.$$

The resulting dimensionless wall temperature gradients are compared with the results obtained by other methods in Figure 2. The numerical data for comparison are given in Table 1. The present numerical method shows excellent agreement with the Merk-type series method [14] and good agreement with the strained coordinate [13] and local non-similarity [11] methods.

Further, Figure 3 shows the variation of the surface shear stress, $f''(x,0)$, with x for sinusoidal variation in wall temperature, when Pr = 0.1, 0.7, 1, 7 and 10. Since these are new results, there are no previous results available for comparison. From Figure 3, we notice that as the Prandtl number increases, the surface shear stress decreases. The

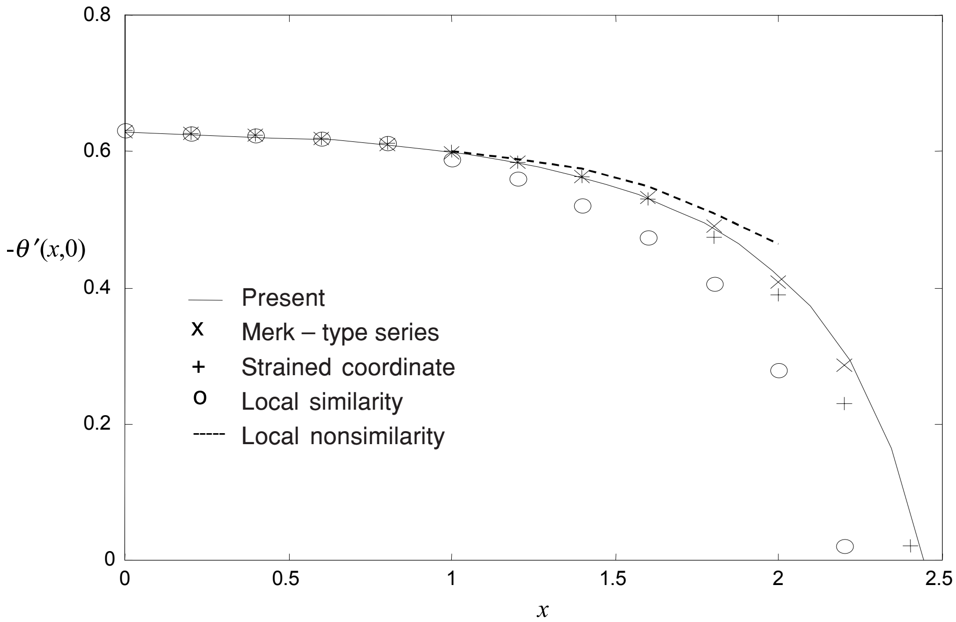

Figure 2 Comparison of dimensionless temperature gradient for sinusoidal variation in wall temperature (Pr = 0.7)

Table 1 Comparison of dimensionless wall temperature gradients: sinusoidal wall temperature variation for Pr = 0.7

x \ θ'	Present	Merk-type Series [14]	Strained Coordinate [13]	Local Non-similarity [11]
0.0	–0.628	–0.628	–0.630	–0.630
0.2	–0.627	–0.627	–0.627	–0.627
0.4	–0.624	–0.624	–0.625	–0.625
0.6	–0.619	–0.619	–0.620	–0.620
0.8	–0.612	–0.611	–0.612	–0.612
1.0	–0.600	–0.600	–0.600	–0.600
1.2	–0.584	–0.584	–0.585	–0.585
1.4	–0.564	–0.563	–0.564	–0.570
1.6	–0.530	–0.532	–0.530	–0.545
1.8	–0.485	–0.489	–0.475	–0.515
2.0	–0.425	–0.410	–0.390	–0.480
2.2	–0.320	–0.290	–0.230	–

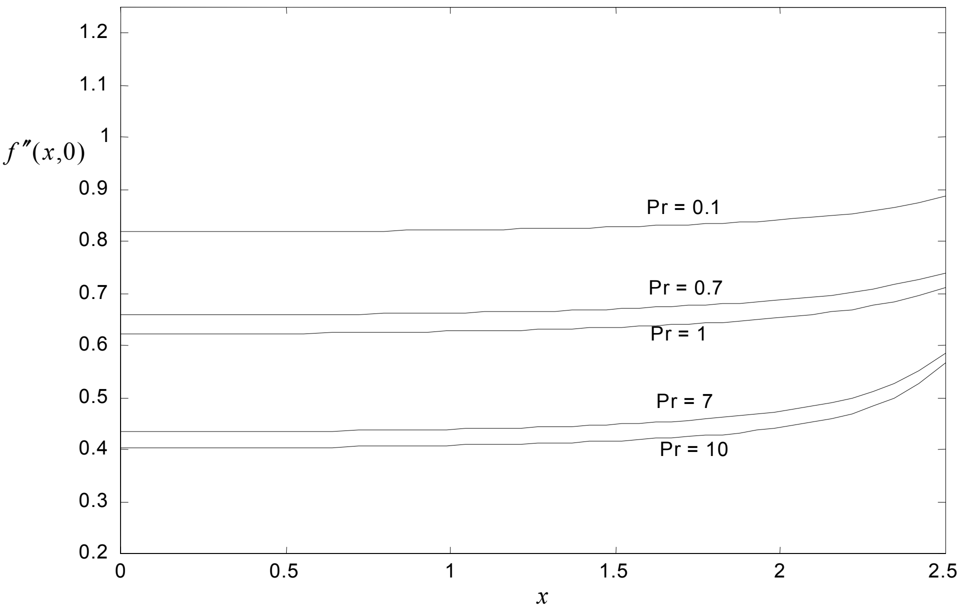

Figure 3 Variation of the surface shear stress ($f''(x,0)$) with x for sinusoidal variation in wall temperature and various Prandtl number, Pr

velocity and temperature profiles for various Prandtl numbers (Pr = 0.7, 1, 10 and 100) at $\xi = 0$ are as given in Figures 4 and 5 below. As the value of Prandtl number increases, the value of velocity and temperature decreases.

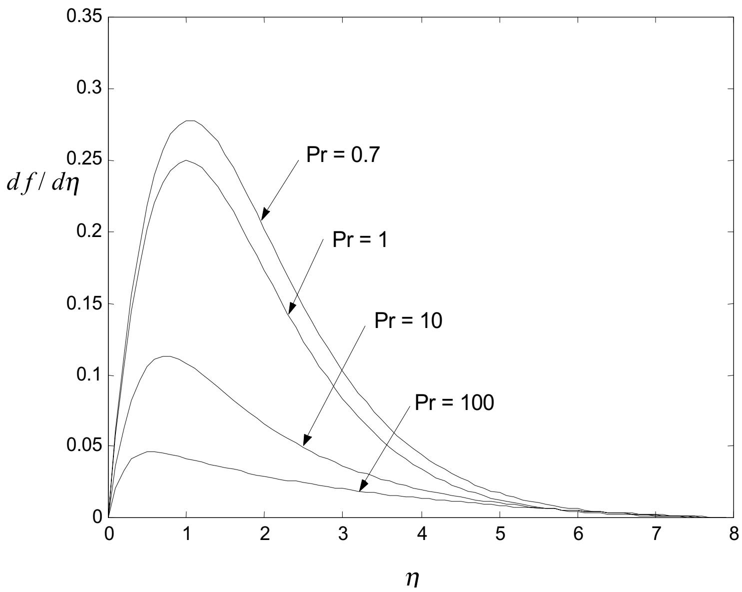

Figure 4 Velocity profiles ($x = 0$)

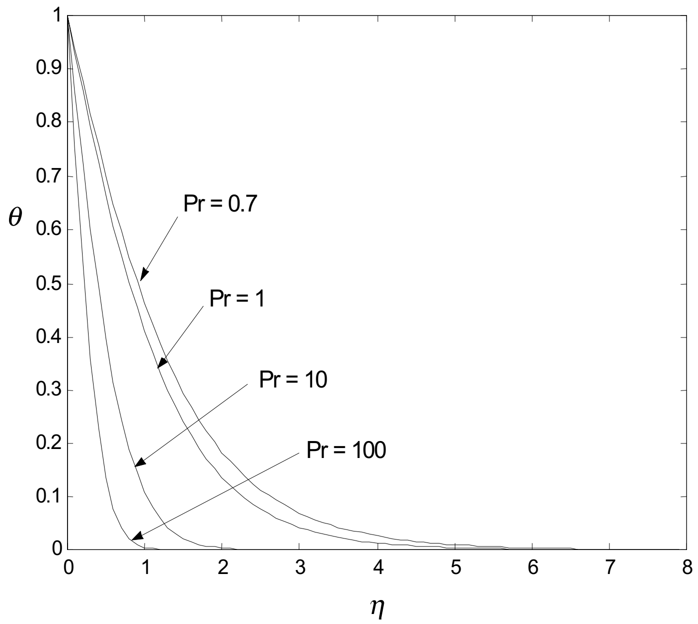

Figure 5 Temperature profiles $(x = 0)$

Another numerical example for specified surface heat flux is also considered. In the case of exponentially increasing surface heat flux, the variable surface heat flux is written as

$$\frac{q_w(x)}{k} = C_1 Q(x) = Ae^{mx},$$

where A has units of temperature divided by length. For comparison, A, m and C_1 are given the value unity. With this surface condition, the parameters required for the heat transfer calculation can be expressed as

$$\xi = \left(\frac{5}{4}\left(e^{2x/3} - 1\right)\right)^{6/5} \quad \text{and} \quad B = 1 - \frac{5}{6}e^{-2x/3}$$

The results for the dimensionless surface temperature for Pr = 0.7 are presented in Figure 6 at several streamwise locations, along with the results obtained by other solution methods. The present numerical method shows good agreement with the strained coordinate method [13] as well as the Merk-type series method [14].

The variation of the surface shear stress, $f''(x,0)$, with x is shown in Figure 7 for exponential variation in surface heat flux, and Pr = 0.1, 0.7, 1, 7, 10 and 100. We notice from this figure that as the Prandtl number increases, the surface shear stress decreases. The velocity and temperature profiles for various ξ (ξ = 0, 1, 10 and 100) with Pr = 0.7 are as given in Figures 8 and 9. As the value of ξ increases, the value of velocity and temperature decreases, but as ξ gets larger, the decrease is small.

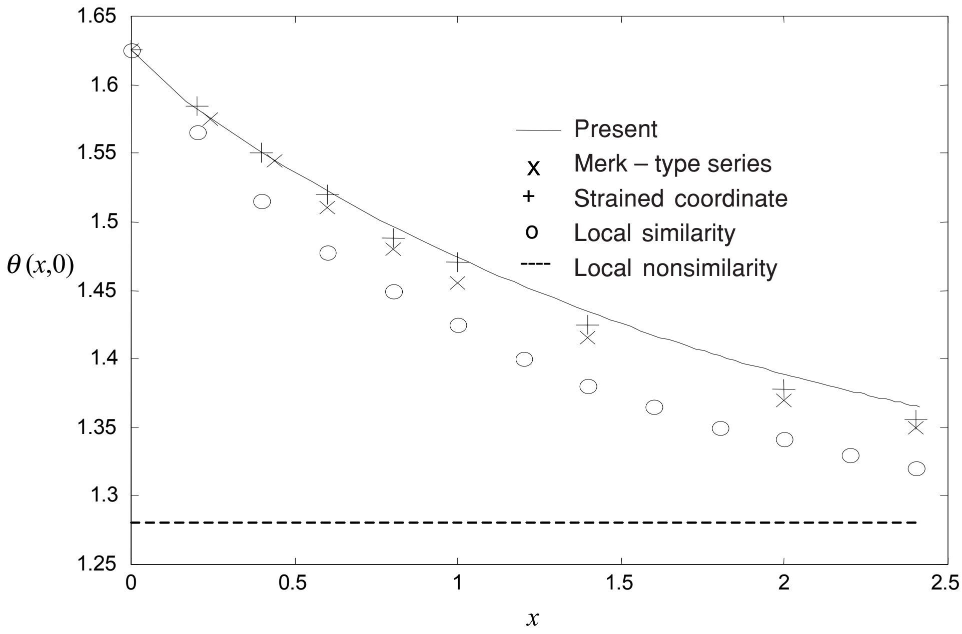

Figure 6 Comparison of dimensionless temperature for exponential variation in surface heat flux (Pr = 0.7)

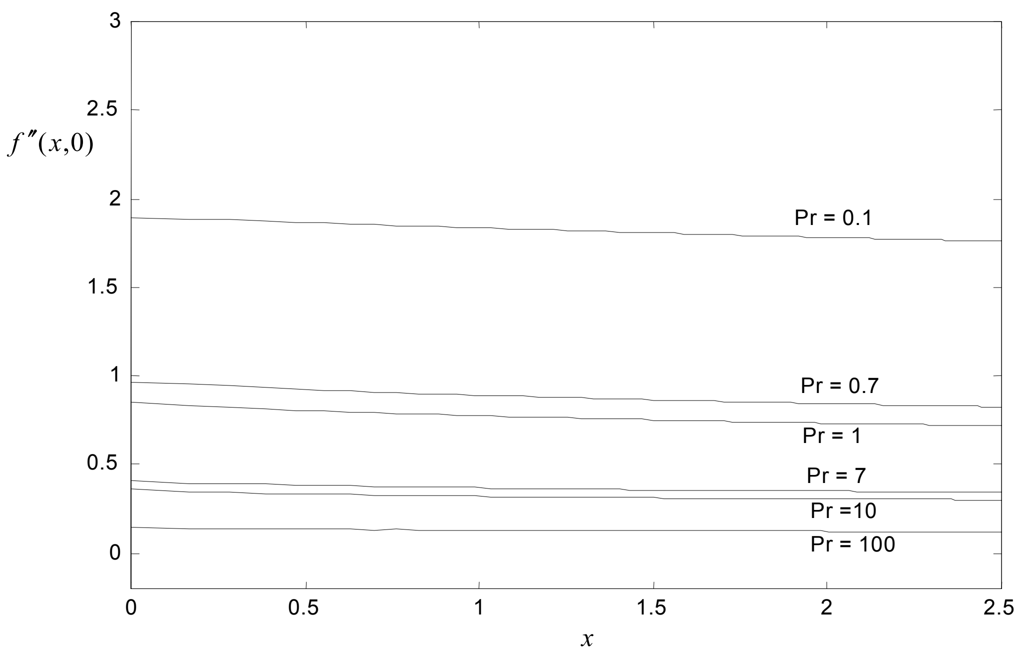

Figure 7 Variation of the surface shear stress ($f''(x,0)$) with x for exponential variation in surface heat flux and various Prandtl number, Pr

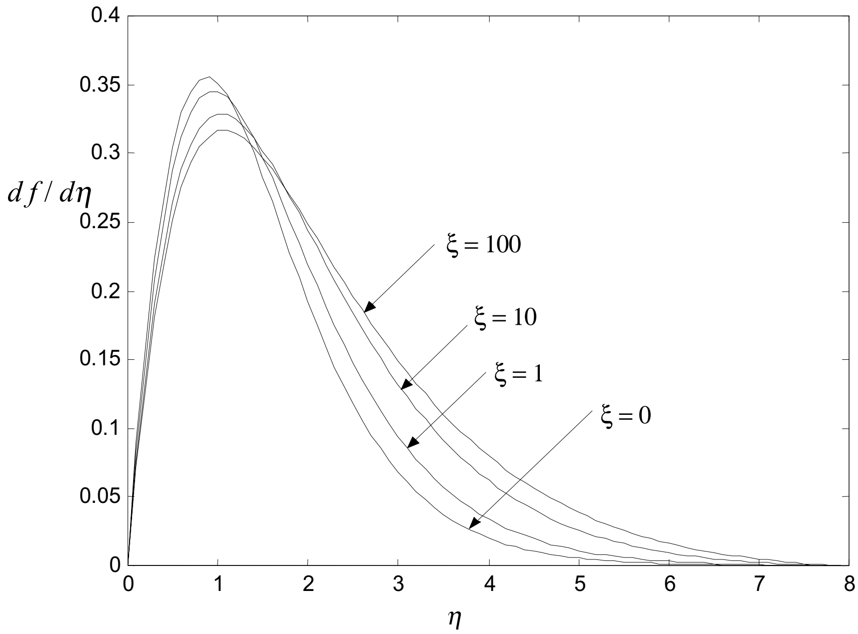

Figure 8 Velocity profiles (Pr = 0.7)

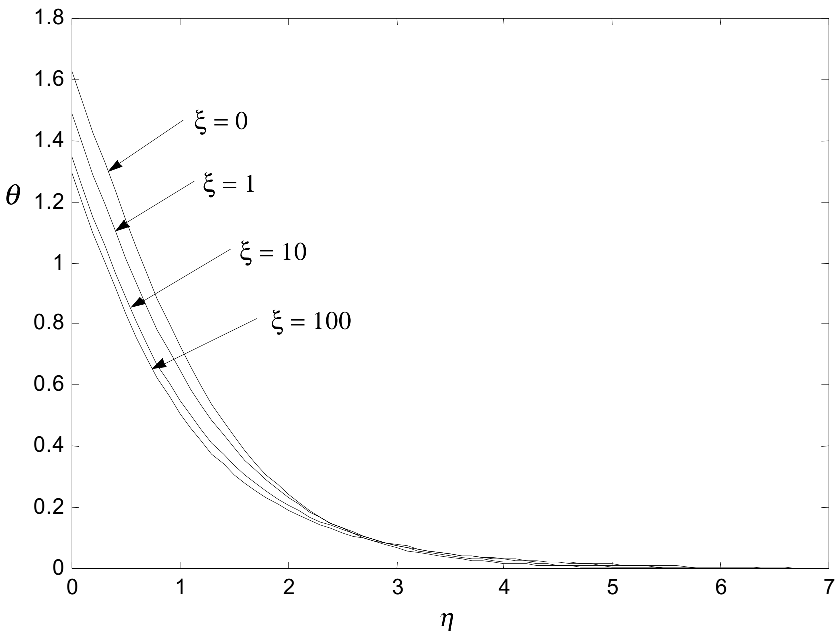

Figure 9 Temperature profiles (Pr = 0.7)

5.0 CONCLUSIONS

A numerical method of determining the temperature and heat flux relations in laminar free convection along a nonisothermal vertical plate has been described. The problems have been solved numerically using the Keller-box scheme for the cases of sinusoidal variation in wall temperature and exponentially increasing surface heat flux. The results for the dimensionless surface temperature $\theta(x,0)$ and for the dimensionless temperature gradient, $-\theta'(x,0)$ for Pr = 0.7 are obtained and graphed at several streamwise locations x, along with the results obtained by other solution methods, namely the local similarity, local nonsimilarity, Merk-type series and the strained coordinate method. The numerical solutions show good agreement with other methods particularly methods by [13] and [14]. This numerical method is found to be simpler and accurate. Further, the variation of the surface shear stress for various values of Prandtl number, as well as the representative temperature and velocity profiles for different values of Prandtl number and transformed streamwise coordinate are also presented for both cases.

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