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DEVELOPMENT OF NEW HARMONIC EULER USING NONSTANDARD FINITE DIFFERENCE TECHNIQUE FOR SOLVING STIFF PROBLEMS

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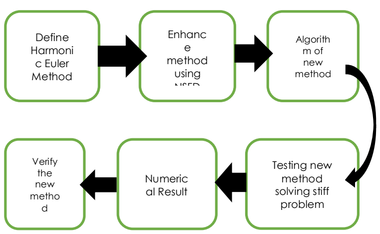
Graphical abstract



Abstract

Solving stiff problem always required very tiny size of meshes if it is solved via traditional numerical algorithm. Using insufficient of mesh size, will triggered instabilities. In this paper, we develop an algorithm applying Harmonic Mean on Euler method to solve the stiff problems. The main purpose of this paper is to discuss the improvement of Harmonic Euler using Nonstandard Finite Difference (NSFD). The combination of these methods can provide new advantages that Euler method could offer. Four set of stiff problems are solved via three schemes, i.e. Harmonic Euler, Nonstandard Harmonic Euler and Nonstandard EO with Harmonic Euler. Findings show that both nonstandard schemes produce high accuracy results.

Keywords: Harmonic Euler, nonstandard, stiff

1.0 INTRODUCTION

A finite difference scheme for numerical solution of ordinary and partial differential equations is one of the most frequently used. One of the disadvantages is that the standard finite difference method of the qualitative characteristics of the exact solution is usually not transferred to the numerical solution. This could have an impact on the stability characteristics of the standard approach[1]. Besides, in fact, the limit of step size also do not reach the standard methods.

In reality, the nonstandard finite difference method is an extension of the standard finite difference method. Nonstandard finite difference scheme (NSFD) was introduced as an alternative method for solving numerous problems in mathematical models engage of algebra, biology and differentiation, and the chaotic system[1, 2]. Mickens introduced Nonstandard schemes in 1988 to resolve some of the issues related to numerical instabilities. In general, nonstandard finite difference schemes (NSFD) is deemed general finite-difference schemes by including some of the 'exact' forms[3]. As a result, there are many advantages by using NSFD compared classical techniques. For example NSFD provide an efficient numerical solution from an aspect of the higher efficiency and better accuracy. In this study, many of ideas are motivated by the works of Mickens [4-6], Ibijola and Obayomi [7-10], K.F. Gurski [11] and Erdogan and Ozis [12].

In this paper, stiff problems will be solved via some modification to Euler method with NSFD. Euler's method is also called a Tangent Line method or one step method and is the simplest numerical method to solve the problem. This method was developed by Leonhard Euler in 1768 and it suitable for quick programming and simple implementation[13]. Because of that factors, Euler method offer a low cost computational application[13]. However, the factor of accuracy and instabilities persuades the authors to modified Euler method. We previously modify the traditional Euler scheme by adapting Harmonic Mean[14, 15]. The primary aim of this investigation is to avoid inaccuracies given by a standard finite difference method. As we know more complex method and sophisticated solvers have been chosen by scholars to replace Euler Method[16, 17]. However, a very small step sizes need to be used for solving the stiff problem[17-19]. In this paper, we apply two nonstandard strategies to avoid numerical instabilities and to overcome the need of tiny step sizes.

For this study we proposed a new algorithm that gave better solution in solving the stiff problem. Algorithm is a step by step procedure to solve a problem[20, 21]. In computer science, algorithm is an instruction model that giving solution to problems logically in simple language[21]. The algorithm contains a set of instruction clearly that describe the flow of process. Two popular way to convey the algorithm are pseudo code and flow chart. The authors choose the pseudo code as the algorithm in solving the stiff problem. After the algorithm is constructed, it will be convert into the code program. Then, the code was executed to test the effectiveness of the algorithm method. The researchers used Scilab 5.5.1 to write the code program in this research.

Figure 1 shows the graphical process of this study. At the final stage, we proposed two new algorithms via combination of Nonstandard Finite Difference (NSFD) scheme with Harmonic Euler in this research. The proposed algorithm called as Nonstandard Harmonic Euler and Nonstandard EO Harmonic Euler.

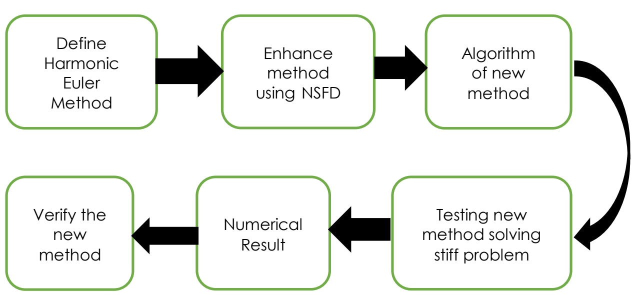


Figure 1 The flow of construction the new Harmonic Euler algorithm

2.0 MATERIALS AND METHOD

Currently, NSFD plays an imperative role in the invention of reliable numerical methods in various areas in Sciences and Engineering. Since, the standard finite difference method does not accurately solve some sensitive mathematical problems. NSFD offers better solution for the problem. NSFD is an extension of the standard finite difference method to resolve issues that related to numerical instabilities[1, 11]. Research on nonstandard scheme was pioneered by Mickens [2, 4-6]. We consider at least one of the nonstandard modeling proposed by Mickens. For this investigation, rule#2 [2]of was implemented.

"**Rule 2:** Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step-sizes than those conventionally used. (These denominator functions, generally, are functions, that are related to particular solutions or properties of the general solution to the differential equation)."

Definition 2.1

A standard Euler method defined by

$$y_{n+1} - y_n = hf(x_n, y_n) \qquad (1)$$

According to rule #2, a new step size as a part denominator function is replaced as a new one. In this research and discussed about new step size.

Definition 2.2

Define the new step size according to Mickens rule, gives the following new finite difference equation for the equation 1.

$$\frac{y_{n+1} - y_n}{\left(\frac{1-e^{-\lambda h}}{\lambda}\right)} = -\lambda y_n \qquad (2)$$

Equation 2, will use when a function has fixed λ. The Harmonic method will enhance equation 2 in solving the stiff problem.

$$\frac{y_{n+1} - y_n}{h} = \frac{\lambda}{2}\left(y_{n+1} + y_n\right) \qquad (3)$$

Equation (3) is the NSFD scheme by Erdogan and Ozis[11, 12]. Erdogan and Ozis also used Mickens rule to develop equation 3 that called as NSFDEO in this paper.

3.0 DEVELOPMENT USING NONSTANDARD IN HARMONIC EULER METHOD

This section will be discussed about implementation of Nonstandard in Harmonic Euler method. The objective of that implementation, to give a better solution. Harmonic Euler is one method of the authors [14, 15] using modified Euler and concept of average. The Harmonic Euler method tries to find an average value of a slope for y, between $x_n + h$, by using harmonic averaging the slopes at x_n and x_{n+1} [14]. The Harmonic Euler method is given in equation (4)

$$y_{n+1} = y_n + \Delta t\, f(t_0 + \tfrac{\Delta t}{2}, y_0 + \left(\tfrac{\Delta t}{2}\right)\left(\frac{2\left(f(t_0,y_0)*f(t_1,y_1)\right)}{f(t_0,y_0)+f(t_1,y_1)}\right)) \qquad (4)$$

Next, we apply the nonstandard rule to equation (4), yield a new scheme called Nonstandard Harmonic Euler, as given in equation (5).

$$y_{n+1} = y_n + \Delta t\, f(t_0 + \tfrac{\Delta t}{2}, y_0 + \left(\tfrac{\Delta t}{2}\right)\left(\frac{2\left(f(t_0,y_0)*(y_0-\lambda f(t_0,y_0)*\left(\frac{1-e^{-\lambda\Delta t}}{\lambda}\right))\right)}{f(t_0,y_0)+(y_0-\lambda f(t_0,y_0)*\left(\frac{1-e^{-\lambda\Delta t}}{\lambda}\right))}\right)) \qquad (5)$$

Next the authors implement equation (3) into the equation (4), resulted in a new method called Nonstandard EO Harmonic Euler. The Nonstandard EO Harmonic Euler equation can be written as an equation (6).

$$y_{n+1} = y_n + \Delta t\, f(t_0 + \frac{\Delta t}{2}, y_0 + \left(\frac{\Delta t}{2}\right)\left(\frac{2\left(f(t_0, y_0) * \left(\frac{y_0 + (1 - \frac{\lambda * \Delta t}{2})}{(1 + \frac{\lambda * \Delta t}{2})}\right)\right)}{f(t_0, y_0) + \left(\frac{y_0 + (1 - \frac{\lambda * \Delta t}{2})}{(1 + \frac{\lambda * \Delta t}{2})}\right)}\right)) \tag{6}$$

4.0 NONSTANDARD HARMONIC EULER ALGORITHM

In this section, algorithms for three type of Harmonic Euler Methods are displayed. The comparison refer on maximum error value. The purpose of this research to solve four stiff problem using step size of *h*=0.001, 0.01 and 0.1. In Tables 2-5, Harmonic Euler Method (equation (4)) will be refer as Scheme 1, Non-standard with Harmonic Euler Method (equation (5)) will be refer as Scheme 2. While Non-standard EO with Harmonic Euler Method (equation (6)) will be refer as Scheme 3.

4.1 Harmonic Euler Algorithm

1 Start
2. Problem Equation
3. Set x, y, h, y(n) , k and a value.
4. Start processing time
5. Condition loop $(n \leq k)$ for
  a. Calculate exact solution
  b. Set A $\leftarrow f\,(x_{n+1}, y_{n+1})$
  c. Set B $\leftarrow f\,(x_n, y_n)$
  d. Set C $\leftarrow\; y_n + h/2 * [2 * (A * B)/\; A + B]$
  e. Set D $\leftarrow f\,(x_n + h/2,\;\; C)$
  f. $y_{n+1} \leftarrow y_n + h * D$
  g. calculate maximum error, |exact -y_n|
  End for
6. Print value, location, exact solution and y at maximum error.
7. End processing time.
8. Print processing time.
9. End

4.2 Nonstandard with Harmonic Euler Algorithm

1 Start
2. Problem Equation
3. Set x, y, h, y(n) , k and a value.
4. Start processing time
5. Condition loop $(n \leq k)$ for
  a. Calculate exact solution
  b. Calculate nonstandard scheme,
    (1- exponent(λ h))/ λ*
  c. Set A $\leftarrow f\,(x_{n+1}, y_{n+1})$
  d. Set B $\leftarrow f\,(x_n, y_n)$
  e. Set C $\leftarrow\; y_n + h/2 * [2 * (A * B)/\; A + B]$
  f. Set D $\leftarrow f\,(x_n + h/2,\;\; C)$
  g. $y_{n+1} \leftarrow y_n + h * D$
  h. calculate maximum error, |exact -y_n|
  End for
6. Print value, location, exact solution and y at maximum error.
7. End processing time.
8. Print processing time.
9. End

4.3 Nonstandard EO with Harmonic Euler Algorithm

1 Start
2. Problem Equation
3. Set x, y, h, y(n) , k and a value.
4. Start processing time
5. Condition loop $(n \leq k)$ for
  a. Calculate exact solution
  b. Calculate nonstandard EO scheme,
    $\{y_n * [1 - (\lambda/2) * h]\} \,/ * [1 + (\lambda/2) * h]$
  c. Set A $\leftarrow f\,(x_{n+1}, y_{n+1})$
  d. Set B $\leftarrow f\,(x_n, y_n)$
  e. Set C $\leftarrow\; y_n + h/2 * [2 * (A * B)/\; A + B]$
  f. Set D $\leftarrow f\,(x_n + h/2,\;\; C)$
  g. $y_{n+1} \leftarrow y_n + h * D$
  h. calculate maximum error, |exact -y_n|
  End for
6. Print value, location, exact solution and y at maximum error.
7. End processing time.
8. Print processing time.
9. End

5.0 NUMERICAL RESULT

This section discusses the results of the four stiff problems with three different step sizes. Table 1 refers to a set of problem with exact solution in ordinary differential equations.

Table 1 Set of Stiff Problem Ordinary Differential Equation

	Equation	Initial Values	Interval of Integration	Source
1.	$y' = -0.5y$ Exact solution : $y(x) = e^{-0.5x}$	$y(0) = 1$	$0 \leq x \leq 20$	[22]
2	$y' = -y$ Exact solution : $y(x) = e^{-x}$	$y(0) = 1$	$0 \leq x \leq 20$	[22]
3	$y' = -30y$ Exact solution : $y(x) = e^{-30x}$	$y(0) = 1$	$0 \leq x \leq 20$	[22]
4	$y' = -10(y-1)^2$ Exact solution : $y(x) = 1 + \frac{1}{1+10x}$	$y(0) = 2$	$0 \leq x \leq 0.1$	[19]

Four stiff problems are tested by using three type of Harmonic Euler methods discussed in section 6.0. The following Table 2 to Table 5 displays the results of the stiff problems. The result shown in the tables are comparisons of maximum error with Scheme 1, Scheme 2 and Scheme 3.

Table 2 Results for Problem 1

Method	*h*=0.001	*h*=0.01	*h*=0.1
Scheme 1	0.00E+00	1.00E-06	8.90E-05
Scheme 2	0.00E+00	1.00E-06	8.30E-05
Scheme 3	0.00E+00	1.00E-06	8.30E-05

Table 3 Results for Problem 2

Method	*h*=0.001	*h*=0.01	*h*=0.1
Scheme 1	0.00E+00	3.00E-06	4.09E-04
Scheme 2	0.00E+00	3.00E-06	3.54E-04
Scheme 3	0.00E+00	3.00E-06	3.55E-04

Table 4 Results for Problem 3

Method	*h*=0.001	*h*=0.01	*h*=0.1
Scheme 1	3.00E-05	6.16E-03	inf
Scheme 2	2.90E-05	4.13E-03	2.26E+39
Scheme 3	2.90E-05	4.21E-03	2.90E-05

Table 5 Results for Problem 4

Method	*h*=0.001	*h*=0.01	*h*=0.1
Scheme 1	4.00E-06	6.27E-04	5.00E-01
Scheme 2	4.00E-06	5.31E-04	2.76E-01
Scheme 3	3.40E-05	3.98E-03	3.10E-01

6.0 DISCUSSION

In this study, a solution of each method is compared with the exact solution using maximum error. About three different step sizes are used for testing in each problem. Those values are 0.001, 0.01 and 0.1. The different step size will give impact in the solution stiff problems using Harmonic Euler Method and NSFD schemes.

Table 2 shows maximum error for stiff problem 1. All schemes produced "exact result" when using small step size which is 0.001. Solution for step size 0.01, all schemes gave the same maximum error which is 1.06E-06. When using larger step size, 0.1 increase all maximum error to 8.30E-05.

Table 3 shows maximum error for stiff problem 2. In this table, all scheme gave zero for the value of maximum error when used step size 0.001. After took step size 0.01, all scheme gave the small value of maximum error which is 3.00E-06. Only scheme 2 gave the smallest value of maximum error when used larger step size which is 3.54E-04.

Table 4 illustrated for stiff Problem 3. Value of small maximum error at step size 0.001 gave the answer 2.90E-05 when used Scheme 2 and Scheme 3. Scheme 3 also gave the small value of maximum error which is 4.13E-03 when used step size 0.01. When used a larger step size 0.1, scheme 1 scheme 2 produces unstable result for maximum error. Only scheme 3 gave the small value of maximum error which is 2.90E-05.

For the last table as refer to Table 5 presented for stiff Problem 4. In this table Scheme 2 gave the smallest value of maximum error when used different step sizes. For h=0.001, the maximum error value was 4.00E-06. When used step size 0.01, the answer of maximum error value was 5.31E-04 and for largest step size gave value which is 2.76E-01.

We can conclude that (all discussed stiff problems) that using Nonstandard scheme is better than standard scheme for all step sizes. Thus the stiff problem can used NSFD scheme for avoid numerical instabilities and overly small step sizes.

7.0 CONCLUSION

The nonstandard schemes have been tested numerically in terms of their consistency with known behavior of an analytic solution. NSFD scheme is applied to Harmonic Euler to give better solutions. Otherwise NSFD method are used to avoid numerical instabilities and slightly small step sizes in stiff problems. The results are presented in the tables which provide comparison of the exact solution using maximum error with Scheme 1, Scheme 2 and Scheme 3. The tables definitely show that NSFD schemes approaching the exact solution, although using huge step sizes. The best NSFD scheme is the Nonstandard EO with Harmonic Euler Method since it can give an accurate result while other methods are unstable as shown in Table 4.

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