

# Effect of Axial Compressibility on the Snap-Through Buckling of Prestressed Arches.

by: JOHN V. HUDDLESTON, Ph.D. (Columbia)

## ABSTRACT

A generalized theory is used to investigate the effect of axial compressibility on the stability of a prestressed arch obtained by buckling a strut into a deformed shape and then attaching it to its supports. For various amounts of compressibility, geometric and mechanical quantities caused by a uniform load that produces large displacement are computed. Symmetric and, in some cases, unsymmetric modes of buckling are found.

## LIST OF PRINCIPAL SYMBOLS

A	Area of cross section
C	Axial compressibility
E	Young's modulus
H	Thrust
h	Height of prestressed arch
I	Moment of inertia of cross section
$I_0$	Reference moment of inertia
L	Original length of strut
l	span of arch
M	Bending couple
N	Normal force
$q_x$	Intensity of distributed force in x-direction
$q_y$	Intensity of distributed force in y-direction
$q_0$	Reference intensity of distributed force
$R_A$	Reaction at A
$R_B$	Reaction at B
S	Shear force
s	Distance along deformed centerline

Professor John Huddleston is a Visiting Professor in Civil Engineering at Universiti Teknologi Malaysia, Gurney Road, Kuala Lumpur, Malaysia.

$u_y$	Net displacement in y-direction
x	Coordinate of cross section in unstressed state
$\xi$	Coordinate of final cross section
$\theta$	Inclination of final centerline

## INTRODUCTION

In a previous paper [1]<sup>1</sup>, the author investigated the buckling and snap-through behavior of a prestressed arch made by first buckling a horizontal strut into its natural curved shape and then attaching it to its supports. Symmetric and unsymmetric modes of buckling under two types of vertical load were discerned, and bifurcation points located. The theory used, however, contained the assumption that the arch was axially incompressible.

In a later paper [2], the author generalized the theory in two ways. First, the effect of centerline compressibility was introduced. This modified the governing nonlinear first-order differential equations and increased their number from four to five. Second, three nonlinear differential equations of equilibrium were added to the boundary-value problem to give a total system of eight differential equations. This opened the way for analyzing a prestressed arch under any type of load, including the weight of the arch itself. Although the theory was generalized to include axial compressibility in that paper, the problem that was solved contained the assumption that the compressibility was zero. The work is continued in the present paper by using the generalized theory to study the effect of axial compressibility on the snap-through buckling of prestressed arches.

## THEORY

Figure 1 shows a strut which has been buckled into an arch and then attached to its supports (dashed line). Subsequent loads applied to the arch cause it to deform further to its final position (solid line).

The differential equations describing the final position are taken from [2] and are the following:

$$\frac{d\theta}{dx} = \frac{M}{EI} \quad (1)$$

$$\frac{du_y}{dx} = \left(1 + \frac{N}{EA}\right) \sin \theta \quad (2)$$

$$\frac{d\xi}{dx} = \left(1 + \frac{N}{EA}\right) \cos \theta \quad (3)$$

$$\frac{ds}{dx} = 1 + \frac{N}{EA} \quad (4)$$

<sup>1</sup>Number in brackets designate References at end of paper.

$$\frac{dN}{dx} = -\frac{SM}{EI} - q_x \left(1 + \frac{N}{EA}\right) \cos \theta - q_y \left(1 + \frac{N}{EA}\right) \sin \theta \quad (5)$$

$$\frac{dS}{dx} = \frac{NM}{EI} - q_x \left(1 + \frac{N}{EA}\right) \sin \theta + q_y \left(1 + \frac{N}{EA}\right) \cos \theta \quad (6)$$

$$\frac{dM}{dx} = S \left(1 + \frac{N}{EA}\right) \quad (7)$$

$$\frac{du_y}{dx} = \frac{du_y}{dx} \text{ for initial configuration} \quad (8a)$$

$(q_x = q_y = 0)$

$$\frac{du_x}{dx} = \frac{d\xi}{dx} - 1 \text{ for load acting, symmetric or unsymmetric mode} \quad (8b)$$

In these equations, the independent variable  $x$  is the distance from the origin to the generic cross section as measured *when the strut is unstressed*,  $\theta$  is the inclination of the centerline in the final position,  $u_y$  is the net displacement in the  $y$ -direction,  $\xi$  is the  $x$ -coordinate of the final position of the cross section,  $s$  is final centerline distance,  $N$  is normal force (positive if tensile),  $S$  is shear force (positive if toward the "outside" of the arch on a negative face), and  $M$  is the bending couple (positive if producing compression on the "outside" of the arch). The primed variables are introduced for use as control variables during the integration of the differential equations. On the right side of the equations,  $E$  is Young's modulus,  $I$  is moment of inertia of cross section (allowed to vary with  $x$  but assumed to remain constant during all deformations),  $A$  is area of cross section (same assumptions as for  $I$ ), and  $q_x$  and  $q_y$  are intensities of distributed forces per unit length of deformed centerline.

The initial conditions corresponding to differential equations (1)–(8) are as follows:

$$\theta(0) = \theta_A \quad (1)$$

$$u_y(0) = 0 \quad (2)$$

$$\xi(0) = 0 \quad (3)$$

$$s(0) = 0 \quad (4)$$

$$N(0) = -H \cos \theta_A - R_A \sin \theta_A \quad (5)$$

$$S(0) = -H \sin \theta_A + R_A \cos \theta_A \quad (6)$$

$$M(0) = 0 \quad (7)$$

$$u_y'(0) = -h \text{ for initial configuration} \quad (8a)$$

$(q_x = q_y = 0)$

$$u_x'(0) = -u_{xC0} \text{ for load acting, symmetric mode} \quad (8b)$$

$$u_x'(0) = -u_{xB0} \text{ for load acting, unsymmetric mode} \quad (8c)$$

where

$$u_{xC0} = \frac{\ell}{2} - \frac{L}{2}$$

( $L$  original length of strut), and

$$u_{xB0} = 2u_{xC0}$$

Since the quantities  $\theta_A$  and  $H$  are not known *a priori*, the set of initial conditions is not complete, and some terminal conditions must be adopted to create a two-point boundary-value problem that admits of solution. Such terminal conditions are the following:

$$\theta\left(\frac{L}{2}\right) = \theta_C = 0 \text{ for initial configuration or for symmetric mode} \quad (1)$$

$$u_y(L) = \Delta_{BV} = 0 \text{ for unsymmetric mode} \quad (2)$$

$$u_y'\left(\frac{L}{2}\right) = e = 0 \text{ for initial configuration} \quad (8a)$$

$$u_x'\left(\frac{L}{2}\right) = f = 0 \text{ for symmetric mode} \quad (8b)$$

$$u_x'(L) = \Delta_{BH} = 0 \text{ for unsymmetric mode} \quad (8c)$$

Equations for  $q_x$  and  $q_y$  for various types of load are as follows:

(I) Dead load ( $w$  is specific weight of material):

$$q_x = 0, \quad q_y = -\frac{Aw}{1 + \frac{N}{EA}} \quad (9)$$

(II) Uniform load  $q_0$  per length of horizontal projection

$$q_x = 0, \quad q_y = -q_0 \cos \theta \quad (10)$$

(III) Concentrated vertical load  $Q$  at  $x = x_Q$ :

$$q_x = 0, \quad q_y = -Q \langle x - x_Q \rangle^{-1} \quad (11)$$

(IV) Uniform load  $q_1$  per length of vertical projection:

$$q_x = q_1 \sin \theta, \quad q_y = 0 \quad (12)$$

(V) External pressure  $p$ :

$$q_x = p \sin \theta, \quad q_y = -p \cos \theta \quad (13)$$

For combinations of load, one obtains the total  $q_x$  or  $q_y$  by adding the contributions of the various loads acting.

#### METHOD OF SOLUTION

The nonlinear two-point boundary-value problem formulated in the previous section can be solved by the shooting method used previously by the author. To that end, consider a numerical example in which  $I$  and  $A$  are assumed to be constant with  $x$  and to have values  $I_0$  and  $A_0$ , respectively. In the process of non-dimensionalizing the boundary-value problem, the measure of axial compressibility of the strut that emerges in a natural way is the parameter  $C = I_0/A_0 L^2$ . In the previous paper [2], an arch with  $C = 0$  under load type (I) was analyzed. In the present paper, a set of arches with  $h/\ell = 0.25$  under load type (II) with varying amounts of axial compressibility (from 0 to 0.0100) is considered.

The shooting method used necessitates assuming values for two input quantities in order to create a complete set of initial conditions. The quantities required here are  $\theta_A$  and  $H$ , since  $R_A$  can be computed in every case from the equation  $R_A = q_0 \ell / 2$ . Once values are assigned, the differential equations can be integrated over half or all of the arch (for symmetric and unsymmetric modes, respectively) by a standard technique such as the Runge-Kutta method. Values for two output quantities corresponding to  $\theta_A$  and  $H$ , respectively, are then computed. Those used in this work were, for the initial configuration,  $\theta_C$  and  $e$ ; for a symmetric mode  $\theta_C$  and  $f$ ; and, for an unsymmetric mode,  $\Delta_{BV}$  and  $\Delta_{BH}$ . After a search is conducted to find two values of an input quantity that produce opposite signs in the corresponding output quantity, *regula falsi* is used to systematically adjust the input until the output is zero. This operation is carried out at two levels to determine the two inputs that make both outputs zero simultaneously.

#### RESULTS AND CONCLUSIONS

Figures 2, 3, and 4 show graphs of dimensionless load vs. vertical position of midpoint, angle of inclination at support, and horizontal thrust, respectively, for various values of  $C$ . The curves for  $C = 0$  are those obtained in [1] and are repeated here for comparison. By shifting the origin in Figure 2 to the point at the left where each curve crosses the horizontal axis, these curves become load-deflection curves.

Starting from zero load, all the arches display a symmetric mode of deformation as the load is increased. These "symmetric" curves eventually reach peak ordinates (of approximately 117.0, 80.3, 33.6, 15.7, and 6.4 for  $C = 0, 0.0010, 0.0025, 0.0050$ , and  $0.0100$ , respectively), and then decrease. Before that happens, however, the arches for which  $C = 0, 0.0010$ , and  $0.0025$  experience a bifurcation point (with ordinates of approximately 48.3, 45.4, and 33.6, respectively), after which

the load decreases and the arch deforms in an unsymmetric mode. As yet, no unsymmetric modes have been found for the cases  $C = 0.0050$  and  $0.0100$ .

The three arches for which  $C = 0.0025, 0.0050$ , and  $0.0100$  return to a straight configuration at the point where the symmetric curves cross the horizontal axis (as manifested by the value  $\theta_A = 0$  in Figure 3).

The computations have shown in a general way that for  $C < 0.0001$ , the effect of axial compressibility is minor, and the approximation  $C = 0$  is normally permissible.

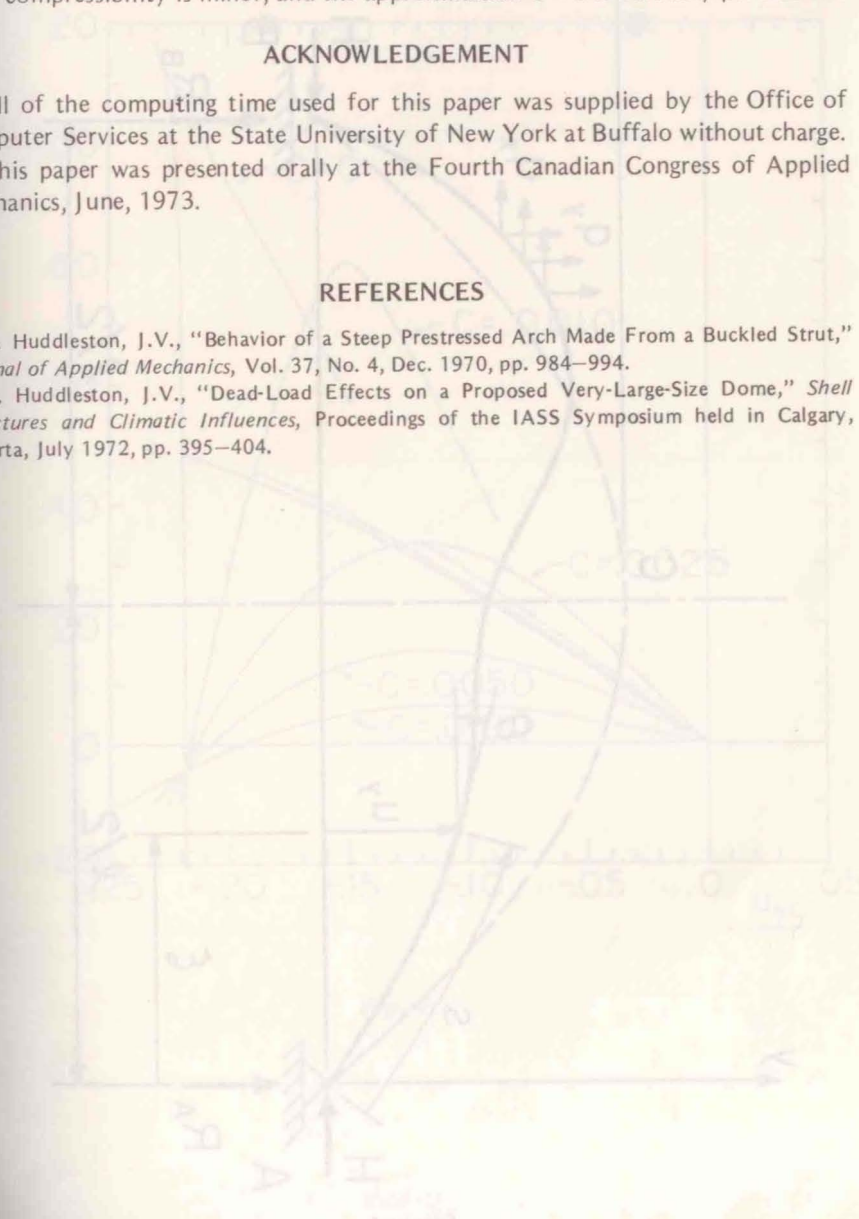
#### ACKNOWLEDGEMENT

All of the computing time used for this paper was supplied by the Office of Computer Services at the State University of New York at Buffalo without charge.

This paper was presented orally at the Fourth Canadian Congress of Applied Mechanics, June, 1973.

#### REFERENCES

1. Huddleston, J.V., "Behavior of a Steep Prestressed Arch Made From a Buckled Strut," *Journal of Applied Mechanics*, Vol. 37, No. 4, Dec. 1970, pp. 984-994.
2. Huddleston, J.V., "Dead-Load Effects on a Proposed Very-Large-Size Dome," *Shell Structures and Climatic Influences*, Proceedings of the IASS Symposium held in Calgary, Alberta, July 1972, pp. 395-404.



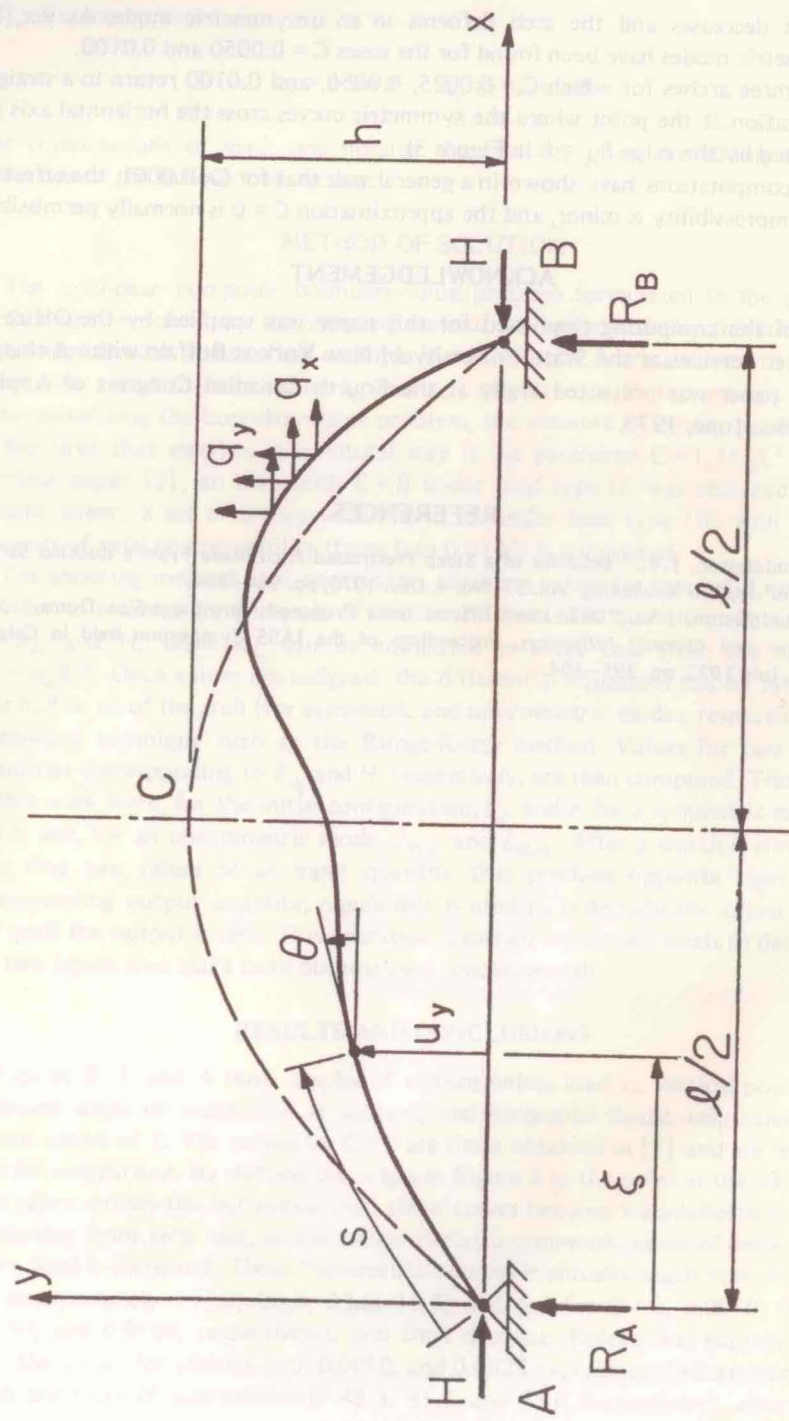


Figure 1.

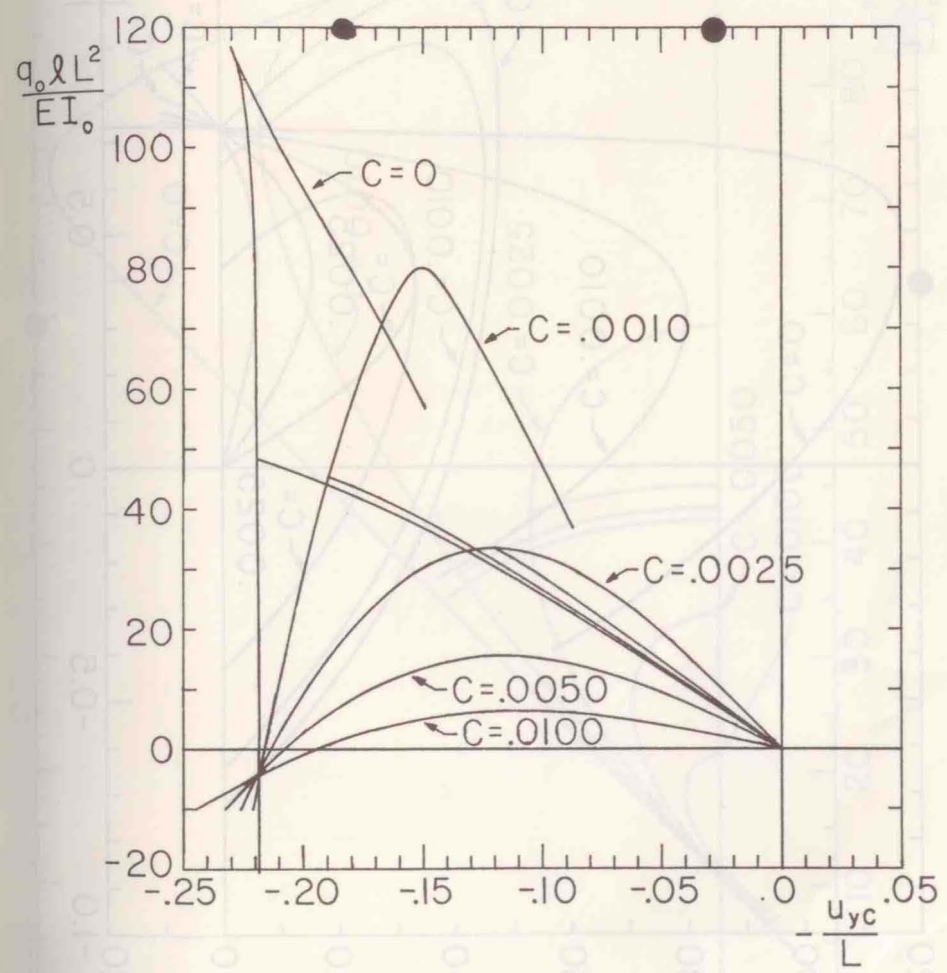


Figure 2.

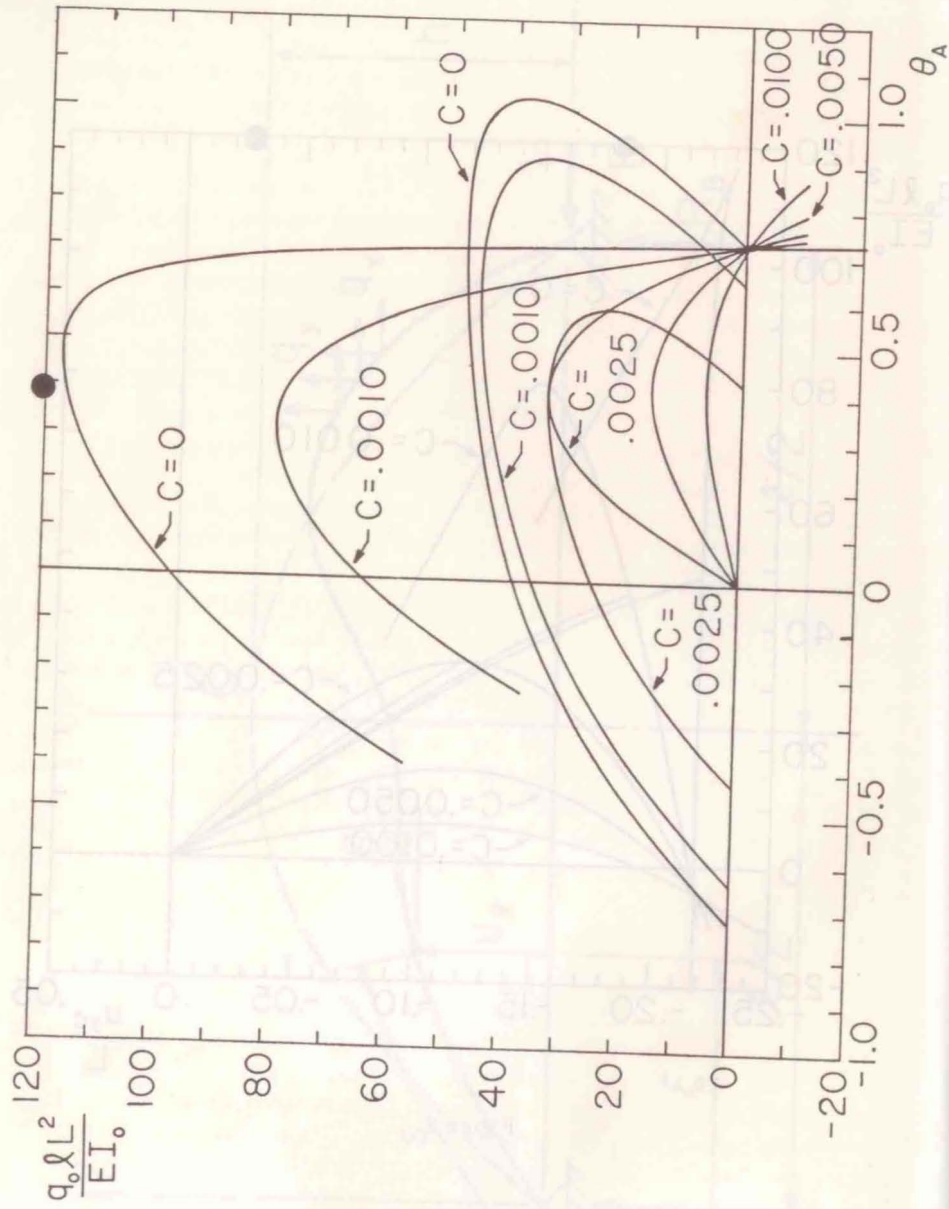


Figure 3.

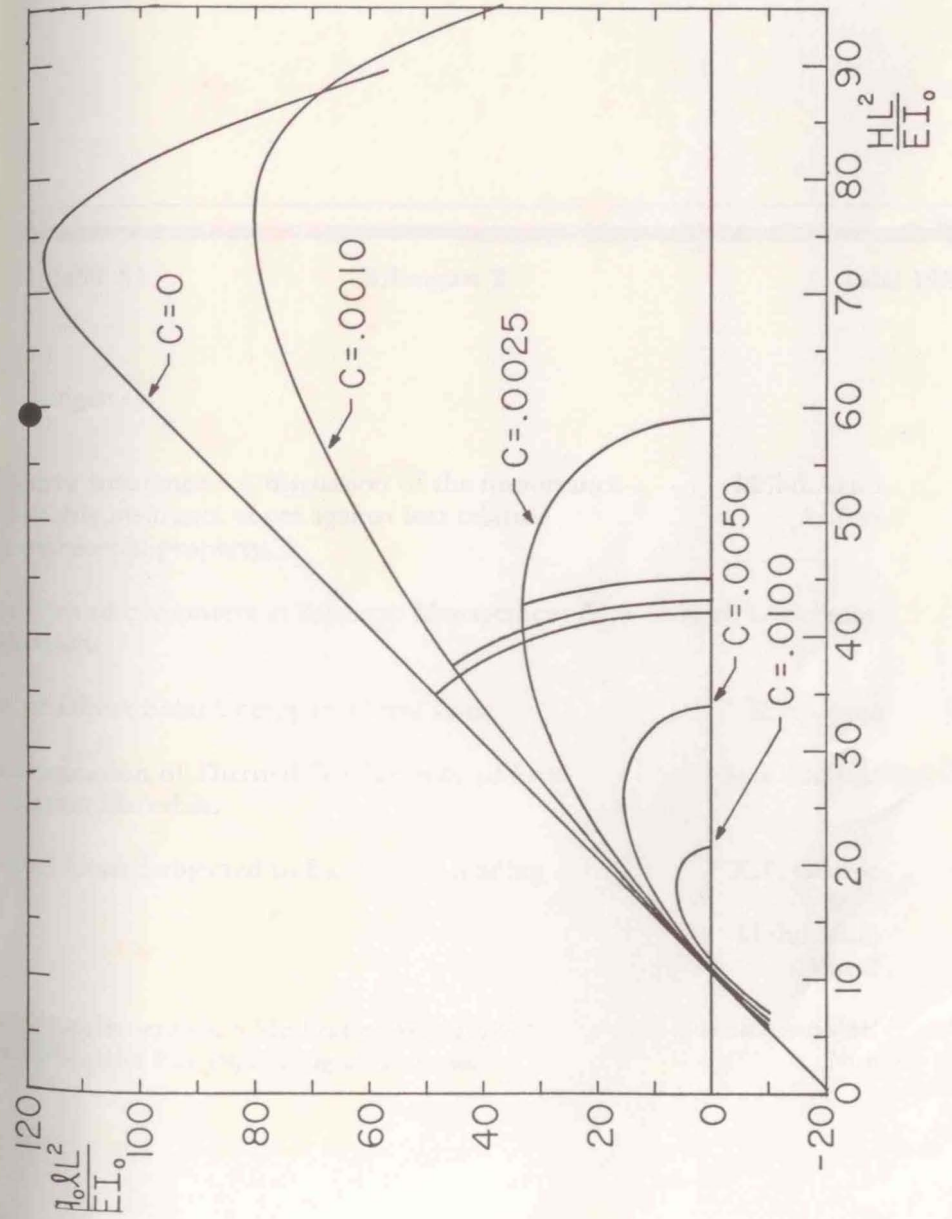


Figure 4.