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POLYCYCLIC PRESENTATIONS OF THE TORSION FREE SPACE GROUP WITH QUATERNION POINT GROUP OF ORDER EIGHT

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Abstract

A space group of a crystal describes its symmetrical properties. Many mathematical approaches have been explored to study these properties. One of the properties is on exploration of the nonabelian tensor square of the group. Determining the polycyclic presentation of the group before computing the nonabelian tensor square is the method used in this research. Therefore, this research focuses on computing the polycyclic presentations of the torsion free space group named Bieberbach group with a quaternion point group of order eight.

Keywords: Polycyclic Presentations, Bieberbach group, Quaternion point group of Order Eight

Abstrak

Kumpulan ruang bagi sesuatu kristal menghuraikan ciri-ciri simetrinya. Banyak pendekatan matematik telah diterokai untuk mengkaji ciri-ciri ini. Salah satu ciri yang dikaji ialah kuasa dua tensor tak abelan bagi sesuatu kumpulan. Mencari persembahan polikitaran sesuatu kumpulan terlebih dahulu sebelum mengira kuasa dua tensor tak abelan merupakan cara yang digunakan di dalam penyelidikan ini. Justeru itu, penyelidikan ini memfokus pada pengiraan persembahan polikitaran kumpulan ruang bebas kilasan yang dinamakan kumpulan Bieberbach dengan kumpulan titik kuaternion berperingkat lapan.

Kata kunci: Persembahan Polikitaran, Kumpulan Bieberbach, Kumpulan Titik Kuaternion Berperingkat Lapan

1.0 INTRODUCTION

Mathematical approach has been widely used in many problems including the pattern of crystals such as in X-ray diffraction crystallography for powder samples [1]. Being one of the torsion free space group or known as the crystallographic group, Bieberbach group has the symmetry structure which can be applied in this application. One of the symmetry structures that have been explored is the nonabelian tensor square of a group. Therefore the research on the nonabelian tensor square of a torsion free space group named Bieberbach group has been carried out over the years. Masri [2] in 2009 has focused on the Bieberbach groups with cyclic point group of order two and elementary abelian point group of 2-group. The method of converting the matrix representation of a group into polycyclic presentation before computing its nonabelian tensor square has been used by Masri in [2] and Mohd Idrus in [3]. Mohd Idrus has considered the Bieberbach groups with dihedral point group of order eight. Recently in 2014, Tan *et al.* [4] also used the same method and found the polycyclic presentation for the Bieberbach group with symmetric point group of order six.

The Crystallographic, Algorithms and Table (CARAT) package [5] provides the lists of torsion free space groups with certain point groups up to dimension six. In this paper, our focus is on the torsion free space group with quaternion point group of order eight. Based on the CARAT package, there are four torsion free space groups with quaternion point group of order eight. All of them are of dimension six. Using the matrix presentation of the groups, the polycyclic presentations of the torsion free space groups with quaternion point group, denoted as $Q_n(6)$, where $n = 1,2,3$ and 4 are found with the assistance of Groups, Algorithms and Programming (GAP) software [6]. We need these polycyclic presentations to show the group is polycyclic in order to find its nonabelian tensor squares.

2.0 PRELIMINARY

This section provides the definition of polycyclic presentation which is used throughout this paper.

Definition 1 [7] Polycyclic Presentation

Let F_n be a free group on generators $g_i, \ldots, g_n$ and R be a set of relations of group F_n. The relations of a polycyclic presentation have the form:

$$g_i^{e_i} = g_{i+1}^{x_{i,i+1}} \ldots g_n^{x_{i,n}} \quad \text{for } i \in I,$$

$$g_j^{-1} g_i g_j = g_{j+1}^{y_{i,j,j+1}} \ldots g_n^{y_{i,j,n}} \quad \text{for } j < i,$$

$$g_j g_i g_j^{-1} = g_{j+1}^{z_{i,j,j+1}} \ldots g_n^{z_{i,j,n}} \quad \text{for } j < i \text{ and } j \notin I$$

for some $I \subseteq \{1, \ldots, n\}, e_i \in \mathbb{Z}$ for $i \in I$ and $x_{i,j,} y_{i,j,k} z_{i,j,k} \in \mathbb{Z}$ for all i, j and k.

3.0 RESULTS AND DISCUSSION

In this section, the computations of the polycyclic presentations of $Q_n(6)$, where $n = 1,2,3$ and 4 are shown.

For n=1,

the following is the matrix representation of the group $Q_1(6)$, which is listed in the CARAT package:

$Q_1(6) = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ where

$$a_0 = \begin{bmatrix} \frac{0}{4} & \frac{-4}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{-1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix}, \quad a_1 = \begin{bmatrix} \frac{-4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{4}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{-2}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix},$$

$$l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (1)$$

$$l_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$l_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix presentation is used to compute the polycyclic presentation using GAP software. GAP will assist in giving the relators of the group. Then, we analyze it to get the polycyclic presentation. Below are the commands to construct the group $Q_1(6)$ by using GAP software:

```
gap> Q1 := Group(a0,a1,l1,l2,l3,l4,l5,l6 );

<matrix group with 8 generators>

gap>
Gpcp:=Image(IsomorphismPcpGroup(AffineCrystGro
```

upOnRight(GeneratorsOfGroup(TransposedMatrixGroup(Q1)))));

Pcp-group with orders [2, 2, 2, 0, 0, 0, 0, 0, 0]

gap> GFp:=Image(IsomorphismFpGroup(Gpcp));

<fp group on the generators [f1, f2, f3, f4, f5, f6, f7, f8, f9]>

gap> RelatorsOfFpGroup(GFp);

[f1^2*f9^-1*f3^-1, f2^2*f9*f8^-1*f3^-1, f1^-1*f2*f1*f9^-2*f8^2*f3^-1*f2^-1, f3^2*f9*f8^-1, f1^-1*f3*f1*f9^-1*f8*f3^-1, f2^-1*f3*f2*f3^-1, <identity ...>,

f1^-1*f4*f1*f7*f5^-1*f4, f2^-1*f4*f2*f6, f3^-1*f4*f3*f4, <identity ...>,

f1^-1*f5*f1*f6^-1*f5^-1*f4, f2^-1*f5*f2*f7*f6*f4, f3^-1*f5*f3*f5, f4^-1*f5*f4*f5^-1, f4*f5*f4^-1*f5^-1, <identity ...>,

f1^-1*f6*f1*f7*f6*f5, f2^-1*f6*f2*f4^-1, f3^-1*f6*f3*f6, f4^-1*f6*f4*f6^-1, f4*f6*f4^-1*f6^-1, f5^-1*f6*f5*f6^-1, f5*f6*f5^-1*f6^-1, <identity ...>,

f1^-1*f7*f1*f7^-1*f6^-1*f4^-1, f2^-1*f7*f2*f6^-1*f5^-1*f4, f3^-1*f7*f3*f7, f4^-1*f7*f4*f7^-1, f4*f7*f4^-1*f7^-1, f5^-1*f7*f5*f7^-1, f5*f7*f5^-1*f7^-1, f6^-1*f7*f6*f7^-1, f6*f7*f6^-1*f7^-1, <identity ...>,

f1^-1*f8*f1*f9^-1, f2^-1*f8*f2*f8^-1, f3^-1*f8*f3*f8^-1, f4^-1*f8*f4*f8^-1, f4*f8*f4^-1*f8^-1, f5^-1*f8*f5*f8^-1, f5*f8*f5^-1*f8^-1, f6^-1*f8*f6*f8^-1, f6*f8*f6^-1*f8^-1, f7^-1*f8*f7*f8^-1, f7*f8*f7^-1*f8^-1, <identity ...>,

f1^-1*f9*f1*f8^-1, f2^-1*f9*f2*f9^-1, f3^-1*f9*f3*f9^-1, f4^-1*f9*f4*f9^-1, f4*f9*f4^-1*f9^-1, f5^-1*f9*f5*f9^-1, f5*f9*f5^-1*f9^-1, f6^-1*f9*f6*f9^-1, f6*f9*f6^-1*f9^-1, f7^-1*f9*f7*f9^-1, f7*f9*f7^-1*f9^-1, f8^-1*f9*f8*f9^-1, f8*f9*f8^-1*f9^-1]

Then, based on the GAP computations above, the following relations are established.
Let f1=a, f2=b, f3=c, f4=l_1, f5=l_2, f6=l_3, f7=l_4, f8=l_5, f9=l_6;

f1^2*f9^-1*f3^-1 --> $a^2=cl_6$,
f2^2*f9*f8^-1*f3^-1 --> $b^2 = cl_5 l_6^{-1}$,
f1^-1*f2*f1*f9^-2*f8^2*f3^-1*f2^-1 --> $b^a = bcl_5^{-2} l_6^2$,
f3^2*f9*f8^-1 --> $c^2 = l_5 l_6^{-1}$,
f1^-1*f3*f1*f9^-1*f8*f3^-1 --> $c^a = cl_5 l_6^{-1}$,
f2^-1*f3*f2*f3^-1 --> $c^b = c$,
<identity ...>,
f1^-1*f4*f1*f7*f5^-1*f4 --> $l_1^a = l_1^{-1} l_2 l_4^{-1}$,
f2^-1*f4*f2*f6 --> $l_1^b = l_3^{-1}$,
f3^-1*f4*f3*f4 --> $l_1^c = l_1^{-1}$,
<identity ...>,
f1^-1*f5*f1*f6^-1*f5^-1*f4 --> $l_2^a = l_1^{-1} l_2 l_3$,
f2^-1*f5*f2*f7*f6*f4 --> $l_2^b = l_1^{-1} l_3^{-1} l_4^{-1}$,
f3^-1*f5*f3*f5 --> $l_2^c = l_2^{-1}$,
f4^-1*f5*f4*f5^-1 --> $l_2^{l_1} = l_2$,
f4*f5*f4^-1*f5^-1 --> ${}^{l_1}l_2 = l_2$,
<identity ...>,
f1^-1*f6*f1*f7*f6*f5 --> $l_3^a = l_2^{-1} l_3^{-1} l_4^{-1}$,
f2^-1*f6*f2*f4^-1 --> $l_3^b = l_1$,
f3^-1*f6*f3*f6 --> $l_3^c = l_3^{-1}$,
f4^-1*f6*f4*f6^-1 --> $l_3^{l_1} = l_3$,
f4*f6*f4^-1*f6^-1 --> ${}^{l_1}l_3 = l_3$,
f5^-1*f6*f5*f6^-1 --> $l_2^{l_3} = l_3$,
f5*f6*f5^-1*f6^-1 --> ${}^{l_2}l_3 = l_3$,
<identity ...>,
f1^-1*f7*f1*f7^-1*f6^-1*f4^-1 --> $l_4^a = l_1 l_3 l_4$,
f2^-1*f7*f2*f6^-1*f5^-1*f4 --> $l_4^b = l_1^{-1} l_2 l_3$,
f3^-1*f7*f3*f7 --> $l_4^c = l_4^{-1}$,
f4^-1*f7*f4*f7^-1 --> $l_4^{l_1} = l_4$,
f4*f7*f4^-1*f7^-1 --> ${}^{l_1}l_4 = l_4$,
f5^-1*f7*f5*f7^-1 --> $l_4^{l_2} = l_4$,
f5*f7*f5^-1*f7^-1 --> ${}^{l_2}l_4 = l_4$,
f6^-1*f7*f6*f7^-1 --> $l_4^{l_3} = l_4$,
f6*f7*f6^-1*f7^-1 --> ${}^{l_3}l_4 = l_4$,
<identity ...>,
f1^-1*f8*f1*f9^-1 --> $l_5^a = l_6$,
f2^-1*f8*f2*f8^-1 --> $l_5^b = l_5$,
f3^-1*f8*f3*f8^-1 --> $l_5^c = l_5$,
f4^-1*f8*f4*f8^-1 --> $l_5^{l_1} = l_5$,
f4*f8*f4^-1*f8^-1 --> ${}^{l_1}l_5 = l_5$,
f5^-1*f8*f5*f8^-1 --> $l_5^{l_2} = l_5$,
f5*f8*f5^-1*f8^-1 --> ${}^{l_2}l_5 = l_5$,
f6^-1*f8*f6*f8^-1 --> $l_5^{l_3} = l_5$,
f6*f8*f6^-1*f8^-1 --> ${}^{l_3}l_5 = l_5$,
f7^-1*f8*f7*f8^-1 --> $l_5^{l_4} = l_5$,
f7*f8*f7^-1*f8^-1 --> ${}^{l_4}l_5 = l_5$,
<identity ...>,
f1^-1*f9*f1*f8^-1 --> $l_6^a = l_5$,
f2^-1*f9*f2*f9^-1 --> $l_6^b = l_6$,
f3^-1*f9*f3*f9^-1 --> $l_6^c = l_6$,
f4^-1*f9*f4*f9^-1 --> $l_6^{l_1} = l_6$,
f4*f9*f4^-1*f9^-1 --> ${}^{l_1}l_6 = l_6$,
f5^-1*f9*f5*f9^-1 --> $l_6^{l_2} = l_6$,

f5*f9*f5^-1*f9^-1 --> ${}^{l_2}l_6 = l_6$,

f6^-1*f9*f6*f9^-1 --> $l_6^{l_3} = l_6$,

f6*f9*f6^-1*f9^-1 --> ${}^{l_3}l_6 = l_6$,

f7^-1*f9*f7*f9^-1 --> $l_6^{l_4} = l_6$,

f7*f9*f7^-1*f9^-1--> ${}^{l_4}l_6 = l_6$,

f8^-1*f9*f8*f9^-1 --> $l_6^{l_5} = l_6$,

f8*f9*f8^-1*f9^-1--> ${}^{l_5}l_6 = l_6$.

Therefore, based on Definition 1, the polycyclic presentation of a torsion free space group in (1) is established as in (2):

$$Q_1(6) = \left\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \;\middle|\; \begin{array}{l} a^2 = cl_6,\ b^2 = cl_5 l_6^{-1}, \\ b^a = bcl_5^{-2}l_6^2, c^2 = l_5 l_6^{-1}, \\ c^a = cl_5^{-1}l_6, c^b = c, \\ l_1^a = l_1^{-1}l_2 l_4^{-1},\ l_1^b = l_3^{-1}, \\ l_1^c = l_1^{-1},\ l_2^a = l_1^{-1}l_2 l_3, \\ l_2^b = l_1^{-1}l_3^{-1}l_4^{-1},\ l_2^c = l_2^{-1}, \\ l_3^a = l_2^{-1}l_3^{-1}l_4^{-1},\ l_3^b = l_1,\ l_3^c = l_3^{-1}, \\ l_4^a = l_1 l_3 l_4,\ l_4^b = l_1^{-1}l_2 l_3, l_4^c = l_4^{-1}, \\ l_5^a = l_6, l_5^b = l_5, l_5^c = l_5, l_6^a = l_5, \\ l_6^b = l_6, l_6^c = l_6,\ l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j \\ \text{for } j > i,\ 1 \le i,j \le 6 \end{array} \right\rangle . \quad (2)$$

For n=2,

$Q_2(6) = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ where

$$a_0 = \begin{bmatrix} \frac{0}{4} & \frac{-4}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{-2}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{2}{4} \\ \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{-1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix}, \quad a_1 = \begin{bmatrix} \frac{-4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{4}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{-2}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix},$$

$$l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3)$$

$$l_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$l_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the same method, the polycyclic presentation of (3) is established as follows:

$$Q_2(6) = \left\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \;\middle|\; \begin{array}{l} a^2 = cl_2 l_3 l_4 l_6,\ b^2 = cl_5 l_6^{-1}, \\ b^a = bcl_1^{-1}l_2 l_5^{-2}l_6^2,\ c^2 = l_5 l_6^{-1}, \\ c^a = cl_2 l_4 l_5^{-1}l_6,\ c^b = c, \\ l_1^a = l_1^{-1}l_2 l_4^{-1},\ l_1^b = l_3^{-1}, l_1^c = l_1^{-1}, \\ l_2^a = l_1^{-1}l_2 l_3,\ l_2^b = l_1^{-1}l_3^{-1}l_4^{-1}, \\ l_2^c = l_2^{-1},\ l_3^a = l_2^{-1}l_3^{-1}l_4^{-1},\ l_3^b = l_1, \\ l_3^c = l_3^{-1},\ l_4^a = l_1 l_3 l_4, \\ l_4^b = l_1^{-1}l_2 l_3, l_4^c = l_4^{-1},\ l_5^a = l_6, \\ l_5^b = l_5, l_5^c = l_5, l_6^a = l_5, \\ l_6^b = l_6, l_6^c = l_6, l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j \\ \text{for } j > i,\ 1 \le i,j \le 6 \end{array} \right\rangle . \quad (4)$$

The following is the matrix representation of the group where n=3,

$Q_3(6)=\langle a_0,a_1,l_1,l_2,l_3,l_4,l_5,l_6\rangle$ where

$$a_0=\begin{bmatrix} \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{-1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{2}{4} \\ \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{1}{4} \\ \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix},\ a_1=\begin{bmatrix} \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{4}{4} & \frac{-4}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{-4}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{-2}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix},$$

$$l_1=\begin{bmatrix} 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ l_2=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&1\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ (5)$$

$$l_3=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ l_4=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$l_5=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ l_6=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{bmatrix}.$$

Therefore, by using the same method, the polycyclic presentation of (5) is established as in (6):

$$Q_3(6)=\left\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \;\middle|\; \begin{array}{l} a^2=cl_3^{-1}l_5,\ b^2=cl_2^{-1}l_4^{-1}l_5^{-1}l_6^{-1}, \\ b^a=bcl_2l_4^2l_5^2l_6^2,\ c^2=l_2^{-1}l_4^{-1}l_5^{-1}l_6^{-1}, \\ c^a=cl_3^{-1}l_4l_5l_6, c^b=c,\ l_1^a=l_3^{-1}, \\ l_1^b=l_4^{-1}, l_1^c=l_1^{-1},\ l_2^a=l_4, \\ l_2^b=l_3^{-1},\ l_2^c=l_2^{-1},\ l_3^a=l_1,\ l_3^b=l_2, \\ l_3^c=l_3^{-1},\ l_4^a=l_2^{-1},\ l_4^b=l_1, l_4^c=l_4^{-1}, \\ l_5^a=l_1l_4^{-1}l_6^{-1}, l_5^b=l_2l_4l_5, \\ l_5^c=l_1l_2l_3^{-1}l_4l_5,\ l_6^a=l_1^{-1}l_4^{-1}l_5^{-1}, \\ l_6^b=l_1^{-1}l_3l_6,\ l_6^c=l_1^{-1}l_2l_3l_4l_6, \\ l_j^{l_i}=l_j,\ l_j^{l_i^{-1}}=l_j \\ \text{for } j>i,\ 1\le i,j\le 6 \end{array}\right\rangle.\ (6)$$

Lastly, the matrix representation of the group for n=4 as listed in the CARAT package,

$Q_4(6)=\langle a_0,a_1,l_1,l_2,l_3,l_4,l_5,l_6\rangle$ where

$$a_0=\begin{bmatrix} \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{-1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{1}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix},\ a_1=\begin{bmatrix} \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{-4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{-2}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix},$$

$$l_1=\begin{bmatrix} 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ l_2=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&1\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ (7)$$

$$l_3=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ l_4=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$l_5=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},\ l_6=\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{bmatrix}.$$

Hence, by using the same method, the polycyclic presentation of (7) is established as in the following:

$$Q_4(6)=\left\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \;\middle|\; \begin{array}{l} a^2=cl_6,\ b^2=cl_5l_6^{-1}, \\ b^a=bcl_5^{-2}l_6^2,\ c^2=l_5l_6^{-1}, \\ c^a=cl_5^{-1}l_6,\ c^b=c, \\ l_1^a=l_4^{-1},\ l_1^b=l_3^{-1}, l_1^c=l_1^{-1}, \\ l_2^a=l_3,\ l_2^b=l_4^{-1},\ l_2^c=l_2^{-1}, \\ l_3^a=l_2^{-1},\ l_3^b=l_1,\ l_3^c=l_3^{-1}, \\ l_4^a=l_1,\ l_4^b=l_2, l_4^c=l_4^{-1}, \\ l_5^a=l_6, l_5^b=l_5, l_5^c=l_5, l_6^a=l_5, \\ l_6^b=l_6, l_6^c=l_6,\ l_j^{l_i}=l_j,\ l_j^{l_i^{-1}}=l_j \\ \text{for } j>i,\ 1\le i,j\le 6 \end{array}\right\rangle.\ (8)$$

All polycyclic presentations of the torsion free space group with quaternion point group have been computed. These presentations must be shown to be consistent before the nonabelian tensor square of the group can be computed.

4.0 CONCLUSION

In this paper, the polycyclic presentations of all four torsion free space groups of dimension six with quaternion point group of order eight, denoted as $Q_n(6)$, where $n = 1, 2, 3$ and 4 have been computed with the assistance of GAP software. These polycyclic presentations are needed in finding the nonabelian tensor squares of these groups.

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