A BOUNDARY INTEGRAL METHOD FOR THE PLANAR EXTERNAL POTENTIAL FLOW AROUND AIRFOILS

ALI H. M. MURID[1], MOHAMED M. S. NASSER[2] & NORSARAHAIDA AMIN[3]

Abstract: This paper presents a boundary integral equation for the external potential flow problem around airfoils without cusped trailing edge angle. The derivation of the integral equation is based upon reducing the external potential flow problem to an exterior Riemann problem. The solution technique is different from the known techniques in the literature since it involves an application of the Riemann problem, instead of the usual Dirichlet or Neumann problems. The solution of the integral equation contains an arbitrary real constant, which may be determined by imposing the Kutta-Joukowski condition. The integral equation is solved numerically using the Nyström method with Kress quadrature rule. Comparisons between the calculated and analytical values of the pressure coefficient for the van de Vooren airfoil and the Karman-Trefftz airfoil with 15% thickness ratio and different angles of attack show very good agreement. Numerical results of the pressure coefficient for NACA0012 airfoil with different angles of attack are also presented.

Keywords: Boundary integral equation, planar potential flow, Riemann problem, Kutta-Joukowski condition

Abstrak: Kertas ini membincangkan persamaan kamiran sempadan bagi masalah aliran potensi luaran di sekitar kerajang udara tanpa sudut pinggir mengekor yang terjuring. Penerbitan persamaan kamiran ini adalah berdasarkan penurunan masalah aliran potensi luaran kepada masalah Riemann luaran. Teknik penyelesaian ini berbeza dari pada teknik-teknik lain yang terdapat dalam literatur kerana ia melibatkan aplikasi masalah Riemann, berbanding dengan masalah Dirichlet atau Neumann seperti lazimnya. Penyelesaian persamaan kamiran mengandungi satu pemalar nyata sebarangan yang boleh ditentukan dengan mengenakan syarat Kutta-Joukowski. Persamaan kamiran tersebut kemudian diselesaikan mengunakan kaedah berangka Nystrom dengan petua kuadratur Kress. Perbandingan antara kiraan berangka dan kiraan analisis bagi pekali tekanan untuk kerajang udara van de Vooren dan kerajang udara Karman-Trefftz dengan nisbah ketebalan 15% dan pelbagai sudut serang menunjukkan penghampiran yang amat baik. Keputusan berangka bagi pekali tekanan untuk kerangka NACA0012 dengan pelbagai sudut serangan juga diberikan.

Kata kunci: Persamaan kamiran sempadan, aliran potensi satah, masalah Riemann, syarat Kutta-Joukowski

1.0 INTRODUCTION

The boundary integral equation method (also called boundary element method, panel method) is a very economical method from the computational point of view for investigating the potential flow past airfoils. According to [1], the starting point of this

---

[1,2&3]Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia, 81310 Skudai, Johor, Malaysia.

method may be due to Hess and Smith [2]. The method becomes one of the most frequently used numerical methods for calculating 2D and 3D potential flow (see e.g. [3]). Various integral equations for studying the external potential flow problem have been discussed in [4], and recently in [1] and [5].

To derive an integral equation for the determination of the complex velocity $W(z)$, the external potential flow problem is first formulated as a boundary value problem and then integral equations for the boundary value problem can be derived. The boundary condition on the boundary C is derived from the requirement that on a stationary impervious curve C, the normal component of fluid velocity must vanish. The boundary condition can be modified if the curve C is moving or if a non-zero normal velocity is prescribed. In this paper, we shall assume that the curve C is stationary and impervious.

It is known that the external potential flow problem can be formulated as a Neumann problem or a Dirichlet problem and then several integral equations can be derived for the external potential flow problem [4]. Recently, the external potential flow problem has been formulated by Mokry [6,7] as an exterior Riemann problem. If the curve C is smooth, then based on the integral equation which has been recently formulated by Murid and Nasser [8] for the exterior Riemann problem, a boundary integral equation has been formulated by the authors in [9] for the exterior potential flow problem around obstacles with smooth boundaries.

The problem considered in this paper is that of flow of an incompressible irrotational inviscid fluid in a region Ω exterior to a given airfoil C which is a simple, counterclockwise oriented, closed curve. The formulated integral equation for external potential flow around obstacles with smooth boundaries will be extended to the external potential flow around airfoils.

2.0 EXTERNAL POTENTIAL FLOW AROUND OBSTACLES WITH SMOOTH BOUNDARIES

Suppose that C is an obstacle with smooth boundary, the complex analytic function $W(z)$ is the complex velocity and $w(z)$ is the complex disturbance velocity due to the obstacle C. If the free stream velocity is of unit magnitude and angle α to the real axis, then $W(z)$ can be decomposed into the free stream part $e^{-i\alpha}$ and the complex disturbance velocity $w(z)$ as [6,7]:

$$W(z) = e^{-i\alpha} + w(z), \quad z \in \Omega \bigcup C. \tag{1}$$

Since in unconfined flow, the velocity disturbance is required to vanish far away from the airfoil, we have $w(\infty) = 0$. Assuming that one can solve the complex velocity $W(z)$ or the complex disturbance velocity $w(z)$, by the Bernoulli theorem, the pressure coefficient $C_p(z)$ is then given by:

$$C_p(z) = 1 - W(z)\overline{W(z)}, \quad z \in \Omega \bigcup C. \tag{2}$$

Let $T(\eta)$ be the unit tangent vector to C at the point $\eta \in C$ in the direction of C. It has been shown in [6,7] that the function $w(z)$ is a solution of the exterior Riemann problem.

$$\text{Re}\left[c(\eta)w(\eta)\right] = \gamma(\eta), \quad \eta \in C \tag{3}$$

where $c(\eta) = -iT(\eta)$ and $\gamma(\eta) = -\text{Im}[e^{-i\alpha}T(\eta)]$. It follows from [9] that the complex disturbance velocity $w(z)$ is given by:

$$w(z) = \frac{1}{2\pi i} \int_C \frac{\rho(\eta)}{T(\eta)(\eta - z)} \, d\eta, \quad z \in C, \tag{4}$$

where $\rho(\eta)$ is the general solution of the integral equation.

$$\rho(\eta) - \int_C \kappa(\eta, \zeta)\rho(\zeta)|d\zeta| = -2\,\text{Re}\left[e^{-i\alpha}T(\eta)\right], \quad \eta \in C, \tag{5}$$

and the kernel $\kappa(\eta, \zeta)$ is given on $C \times C$ by:

$$\kappa(\eta, \zeta) = \begin{cases} \dfrac{1}{\pi}\,\text{Im}\left[\dfrac{T(\eta)}{\eta - \zeta}\right] & \eta \neq \zeta, \\[3mm] \dfrac{1}{2\pi|\eta'(\tau)|}\,\text{Im}\left[\dfrac{\eta''(\tau)}{\eta'(\tau)}\right] & \eta = \zeta = \eta(\tau). \end{cases} \tag{6}$$

Furthermore, the boundary values of the function are given by:

$$w(\eta) = -e^{-i\alpha} - \frac{\rho(\eta)}{T(\eta)}, \quad \eta \in C, \tag{7}$$

and the circulation Γ of the fluid along the boundary C is given by:

$$\Gamma = -\int_C \rho(\eta)|d\eta|. \tag{8}$$

Note that the kernel $\kappa(\eta, \zeta)$ is known as the Neumann kernel [10].

3.0   EXTERNAL POTENTIAL FLOW AROUND AIRFOIL

Suppose that the obstacle C is an airfoil (Figure 1) with the parametric representation in τ

$$C: \quad \eta = \eta(\tau), \qquad 0 \leq \tau \leq 2\pi, \tag{9}$$

such that $\eta_0 = \eta(0) = \eta(2\pi)$ is the corner point of the airfoil. The vicinity points preceeding η_0 in describing C in the counterclockwise direction will be denoted by

η_L and the vicinity points following η_0 will be denoted by η_U, i.e., $\eta_U = \eta(0+)$ and $\eta_L = \eta(2\pi-)$. The function $\eta(\tau)$ will be assumed to be such that $\eta'(\tau) \neq 0$ and $\eta''(\tau)$ exists and is continuous for all $\tau \in (0, 2\pi)$. Suppose further that the interior angle θ of the corner point η_0 satisfies $0 < \theta < \pi$. Under the above assumptions, the function $T(\eta)$ is parameterized by:

$$T(\eta) = \frac{\eta'(\tau)}{|\eta'(\tau)|}, \quad 0 < \tau < 2\pi, \quad \eta = \eta(\tau) \in C \setminus \{\eta_0\}.$$

Hence $T(\eta)$ is a continuously differentiable function for all $\eta \in C$ except at the corner point η_0 where the tangent is undefined. However, at the points η_L and η_U the one side tangent vector are defined by:

$$T_U = T(\eta_U) = \frac{\eta'(0+)}{|\eta'(0+)|} \text{ and } T_L = T(\eta_L) = \frac{\eta'(2\pi-)}{|\eta'(2\pi-)|}.$$

It is then clear from Figure 1, that:

$$T_L = e^{i(\pi+\theta)}T_U = -e^{i\theta}T_U. \tag{10}$$

The property of smoothness of the boundary C was employed in [8,9] twice, the first time in applying the Sokhotski formulas, and again in proving the continuity of the kernel $\kappa(\eta, \zeta)$ of the integral Equation (5). However, if C is an airfoil (Figure 1), then from [10] the kernel $\kappa(\eta, \zeta)$ is continuous for all $(\eta, \zeta) \in C \setminus \{\eta_0\} \times C$. Using this fact and the fact that the Sokhotski formulas [11] remain valid for all $\eta \in C \setminus \{\eta_0\}$, we find from the above results for the case for which the obstacle C is smooth is still valid for the airfoil case, i.e., that the complex disturbance velocity $w(z)$ of the flow around the airfoil C is given by Equation (4) and the boundary values $w(\eta)$ of the function $w(z)$ are given by:

$$w(\eta) = -e^{-i\alpha} - \frac{\rho(\eta)}{T(\eta)}, \quad \eta \in C \setminus \{\eta_0\}, \tag{11}$$
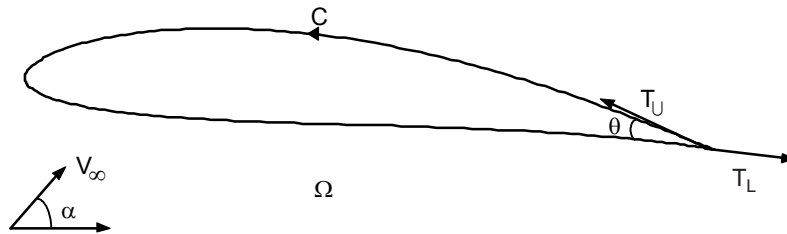

Figure 1   The airfoil

where $\rho(\eta)$ is the general solution of the integral equation:

$$\rho(\eta) - \int_C \kappa(\eta,\zeta)\rho(\zeta)|d\zeta| = -2\,\mathrm{Re}\left[e^{-i\alpha}T(\eta)\right], \quad \eta \in C \setminus \{\eta_0\} \tag{12}$$

By the Kutta-Joukowski condition, the function $w(z)$ must be bounded at the corner point η_0, then from [11] the function $\rho(\eta)/T(\eta)$ must satisfy the condition:

$$\frac{\rho(\eta_U)}{T(\eta_U)} = \frac{\rho(\eta_L)}{T(\eta_L)} \tag{13}$$

By Equation (10) and since $\rho(\eta)$ is a real-valued function, the condition Equation (13) implies that the function $\rho(\eta)$ must satisfy the conditions:

$$\rho(\eta_U) = 0 \text{ and } \rho(\eta_L) = 0. \tag{14}$$

Since the external potential flow problem is solvable and its solution is (finite) bounded at the corner point η_0, the integral Equation (12) with the added conditions Equation (14) is always solvable. However, from [10], $\lambda = 1$ is a simple eigenvalue of the kernel $\kappa(\eta,\zeta)$. This implies in accordance with the Fredholm alternative theorem that the general solution of the integral equation Equation (12) can be written as:

$$\rho(\eta) = \rho_p(\eta) + c_0\rho_h(\eta), \qquad t \in C \setminus \{\eta_0\}, \tag{15}$$

where $\rho_p(\eta)$ is a particular solution of the integral Equation (12), $\rho_h(\eta)$ is a solution of the homogenous integral equation:

$$\rho_h(\eta) - \int_C \kappa(\eta,\zeta)\rho_h(\zeta)|d\zeta| = 0, \quad \eta \in C \setminus \{\eta_0\}, \tag{16}$$

and c_0 is an arbitrary real constant. Moreover, from [10], the function $\rho_h(\eta(\tau))$ satisfies:

$$\lim_{\tau \to 0+} |\eta'(\tau)|\rho_h(\eta(\tau)) \neq 0, \quad \lim_{\tau \to 2\pi-} |\eta'(\tau)|\rho_h(\eta(\tau)) \neq 0. \tag{17}$$

Thus one of the conditions (Equation (14)) is enough to determine the arbitrary constant c_0 in Equation (15). Consequently, the integral Equation (12) with conditions (Equation (14)) is uniquely solvable.

From Equations (1), (2) and (11), the pressure coefficient $C_p(\eta)$ is then given for all $\eta \in C \setminus \{\eta_0\}$ by:

$$C_p(\eta) = 1 - \rho(\eta)^2. \tag{18}$$

It is clear that if $\rho(\eta)$ satisfies the conditions (Equation (14)), then the pressure coefficient given by Equation (18) satisfies:

$$C_p(\eta_L) = C_p(\eta_U). \tag{19}$$

The relation (Equation (19)) is known as the Kutta-Joukowski condition [3]. Furthermore, since the function $\rho(\eta)$ is continuous on $C \setminus \{\eta_0\}$, then by Equation (14) the function $\rho(\eta)$ can be made continuous at η_0 by defining it there by $\rho(\eta_0) = 0$. Then $\rho(\eta)$ is continuous for all $\eta \in C$ and then the pressure coefficient $C_p(\eta)$ is given by Equation (18) for all $\eta \in C$. It is clear then that:

$$C_p(\eta_0) = 1, \tag{20}$$

which implies that $W(\eta_0) = 0$, and hence the corner point η_0 is a stagnation point (point at which the velocity is zero). Hence, if $\rho(\eta)$ is the unique solution of the integral Equation (12) with conditions (Equation (14)), then the complex disturbance function $w(z)$ is given by Equation (4) and the Kutta-Joukowski condition is satisfied. Thus the conditions (Equation (14)) may be called the Kutta-Joukowski condition for the integral Equation (12).

4.0   NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

In this section, the boundary integral Equation (12) which has been derived in the physical plane will be solved numerically. Using the counterclockwise parametric representation (Equation (9)) of the curve C, the integral Equation (12) can be written as:

$$\tilde{\rho}(\tau) - \int_0^{2\pi} v(\tau, \sigma)\, \tilde{\rho}(\sigma)\, d\sigma = \psi(\tau), \quad 0 < \tau < 2\pi, \tag{21}$$

where for $0 < \sigma, \tau < 2\pi$,

$$\tilde{\rho}(\tau) = |\eta'(\tau)|\, \rho(\eta(\tau)),$$
$$v(\tau, \sigma) = \kappa(\eta(\tau), \eta(\sigma)) |\eta'(\tau)|,$$
$$\psi(\tau) = -2|\eta'(\tau)| \operatorname{Re}\left[e^{-i\alpha}\eta'(\tau) \right].$$

Due to the discontinuity of the kernel and the right hand side of Equation (21) at $\tau = 0$ and $\tau = 2\pi$, the use of the Nyström method with any quadrature method based on equidistant mesh points to discretize Equation (21) yields poor convergence. Kress [12] has introduced a graded quadrature formula to discretize such integral equations. Using the Nyström method with the Kress quadrature formula with $2n - 1$ node points to discretize the integral in Equation (21), we obtained:

$$\tilde{\rho}_n(\tau) - \sum_{j=1}^{2n-1} \omega_j K\left(\tau_j, \tau\right) \tilde{\rho}_n\left(\tau_j\right) = \psi(\tau), \quad 0 < \tau < 2\pi, \tag{22}$$

where the weights ω_j and the mesh points τ_j are given by:

$$\omega_j = \frac{\pi}{n} h'\left(\frac{j\pi}{n}\right), \quad \tau_j = h\left(\frac{j\pi}{n}\right), \quad j = 1, 2, \ldots, 2n - 1, \tag{23}$$

and the function h is given by:

$$h(s) = 2\pi \frac{(v(s))^p}{(v(s))^p + (v(2\pi - s))^p}, \quad 0 \le s \le 2\pi, \tag{24}$$

where

$$v(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p}\frac{s - \pi}{\pi} + \frac{1}{2}, \quad p \ge 2. \tag{25}$$

Choosing the grading parameter $p = 2$, collocating at the node points τ_j, $j = 1,2,...,$ $2n-1$, and defining the matrix and the vectors,

$$\mathbf{K}_n = \left(k_{ij}\right)_{(2n-1)\times(2n-1)}, \mathbf{x}_n = \left(x_i\right)_{(2n-1)\times 1} \text{ and } \mathbf{y}_n = \left(y_i\right)_{(2n-1)\times 1}, \tag{26}$$

for $i,j = 1,2,...,2n-1$, by:

$$k_{ij} = \omega_j K\left(\tau_j, \tau_i\right), \quad x_i = \tilde{\rho}_n\left(\tau_i\right) \text{ and } y_i = \psi\left(\tau_i\right), \tag{27}$$

we obtain from Equation (22) the $(2n - 1) \times (2n - 1)$ linear system

$$(\mathbf{I} - \mathbf{K}_n)\mathbf{x}_n = \mathbf{y}_n. \tag{28}$$

The kernel $\kappa(\eta, \zeta)$ has a simple eigenvalue $\lambda = 1$, thus for sufficiently large n the matrix $\mathbf{K}_n$ has also a simple eigenvalue $\lambda = 1$. Thus the linear system has infinitely many solutions. To remove the non-uniqueness in the solution of the linear system (Equation (28)), conditions (Equation (14)) will be imposed on the solution of the linear system (Equation (28)). This will be done by approximating $\tilde{\rho}_n(\tau)$ at the nodes τ_1 and τ_{2n-1} as follows:

$$\tilde{\rho}_n\left(\tau_1\right) \approx \frac{\tilde{\rho}_n(0) + \tilde{\rho}_n\left(\tau_2\right)}{2} = \frac{\tilde{\rho}_n\left(\tau_2\right)}{2} \quad \text{and}$$

$$\tilde{\rho}_n\left(\tau_{2n-1}\right) \approx \frac{\tilde{\rho}_n(2\pi) + \tilde{\rho}_n\left(\tau_{2n-2}\right)}{2} = \frac{\tilde{\rho}_n\left(\tau_{2n-2}\right)}{2}.$$

Consequently, we have two additional equations:

$$x_1 - \frac{1}{2}x_2 = 0, \quad x_{2n-1} - \frac{1}{2}x_{2n-2} = 0. \tag{29}$$

By adding the two equations (Equation (29)) to the linear system (Equation (28)), we arrive at an over-determined $(2n + 1) \times (2n - 1)$ linear system. According to [13], the best numerical method for solving such linear system is to use the QR factorization algorithm. In our numerical experiments we use the MATLAB's operator/that makes use of QR factorization with column pivoting [13]. Once the solution

$$\mathbf{x}_n = \big(\tilde{\rho}_n(\tau_1), \tilde{\rho}_n(\tau_2), \ldots, \tilde{\rho}_n(\tau_{2n-1})\big), \tag{30}$$

of the new $(2n+1) \times (2n-1)$ system has been computed, the Nyström interpolation formula (Equation (22)) can be used to obtain $\tilde{\rho}_n(\tau)$ for all $\tau \in (0, 2\pi)$. The approximate solution $w_n(z)$ to the complex disturbance velocity $w(z)$ is obtained by substituting:

$$\rho_n(\eta) = \frac{\tilde{\rho}_n(\tau)}{|\eta'(\tau)|}, \quad \eta = \eta(\tau) \in C \setminus \{\eta_0\}, \tag{31}$$

in Equation (4). The approximate pressure coefficient $C_{p,n}(\eta)$ on C can be obtained by substituting $\rho_n(\eta)$ in Equation (18).

5.0   NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the results obtained from the solution technique described above, numerical calculations have been performed for three airfoils, namely, the van de Vooren airfoil, the Karman-Trefftz airfoil and NACA0012 airfoil.

The profile of the van de Vooren airfoil is transformed from the circle $\xi = e^{is}$, $0 \le s \le 2\pi$, by means of the mapping:

$$\eta = 1 + a\frac{(\xi-1)^k}{(\xi-\varepsilon)^{k-1}}, \quad a = \left(\frac{1+\varepsilon}{2}\right)^{k-1}, \quad k = 2 - \frac{\theta}{\pi},$$

where ε is the thickness ratio, k is the trailing-edge angle parameter, and θ is the trailing-edge angle. In the following example, we set, as in [5], $\theta = \pi/12$, $\varepsilon = 0.06573$ which corresponds to a 15% thick airfoil, and an attack angle $\alpha = 5°$.

The second airfoil is the Karman-Trefftz airfoil obtained from the circle $\xi = (1-\xi_0)e^{is} + \xi_0, 0 \le s \le 2\pi$, by means of the mapping:

$$\eta = k\frac{1 + \left(\dfrac{\xi-1}{\xi+1}\right)^k}{1 - \left(\dfrac{\xi-1}{\xi+1}\right)^k}, \quad k = 2 - \frac{\theta}{\pi},$$

where k and θ are as above. The values of the parameters are chosen such that the airfoil has a 15% thickness ratio. We set $\theta = \pi/15$, and $\xi = 0.06573$. The angle of attack is assumed to be $\alpha = 5°$.

Table 1 shows the errors $\left|C_p(\eta_j) - C_{p,n}(\eta_j)\right|$, i.e., the difference between the analytic solution $C_p(\eta)$ and the numerical solution $C_{p,n}(\eta)$ at selected points on the airfoil C for the van de Vooren and the Karman-Trefftz airfoils. As we can see from Table 1, the convergence of the method is excellent except near the trailing edges

Table 1   The error for the two airfoils

| | $\left| C_p\left(\eta(\tau)\right) - C_{p,n}\left(\eta(\tau)\right) \right|$ | | | | | |
| τ | van de Vooren airfoil | | | Karman-Trefftz airfoil | | |
	n = 32	n = 64	n = 128	n =32	n = 64	n = 128
0.0016	0.0402	0.1073	0.2093	0.2657	0.2016	0.1042
0.0065	0.0418	0.1664	0.2807	0.2566	0.1374	0.0285
0.0152	0.1650	0.0111	0.0582	0.2182	0.0722	0.0067
0.0278	0.2879	0.1603	0.1207	0.1685	0.0382	0.0015
0.0448	0.0989	0.0401	0.0613	0.1368	0.0193	0.0003
0.0665	0.1246	0.0228	0.0118	0.0997	0.0094	0.0001
0.0934	0.0853	0.0093	0.0036	0.0718	0.0045	0.0000
0.1257	0.0601	0.0032	0.0003	0.0510	0.0022	0.0000
6.1575	0.0590	0.0027	0.0003	0.0488	0.0017	0.0000
6.1898	0.0825	0.0088	0.0036	0.0693	0.0041	0.0000
6.2167	0.1136	0.0222	0.0117	0.0953	0.0090	0.0000
6.2384	0.0611	0.0405	–0.0613	0.1264	0.0189	0.0003
6.2554	0.3160	0.1604	0.1204	0.1766	0.0384	0.0014
6.2680	0.1625	0.0148	0.0582	0.2172	0.0756	0.0067
6.2767	0.0415	0.1761	0.2804	0.2566	0.1280	0.0286
6.2816	0.0401	0.1073	0.2112	0.2657	0.2016	0.1023

where the kernel and the right hand side of the integral equation have singularity there. However, the convergence there is still acceptable. The comparisons between the analytic and numerical solutions for the pressure coefficients for the van de Vooren and the Karman-Trefftz airfoils are shown in Figures 2 and 3, respectively.

The third airfoil is the NACA0012 airfoil which has the parametric representation in τ [14]

$$t(\tau) = \tau \pm i0.6\left(0.2969\sqrt{\tau} - 0.126\tau - 0.3537\tau^2 + 0.2843\tau^3 - 0.1015\tau^4\right), \quad 0 \le \tau \le 1.$$

The numerical values for the pressure coefficients for the NACA0012 airfoil are shown in Figure 4. Numerical results of this airfoil have been calculated in [15] using three different methods. Let C_{pL} be the pressure coefficient on the lower surface of the airfoil and C_{pU} be the pressure coefficient on the upper surface of the airfoil. Then the lift coefficient is defined by:

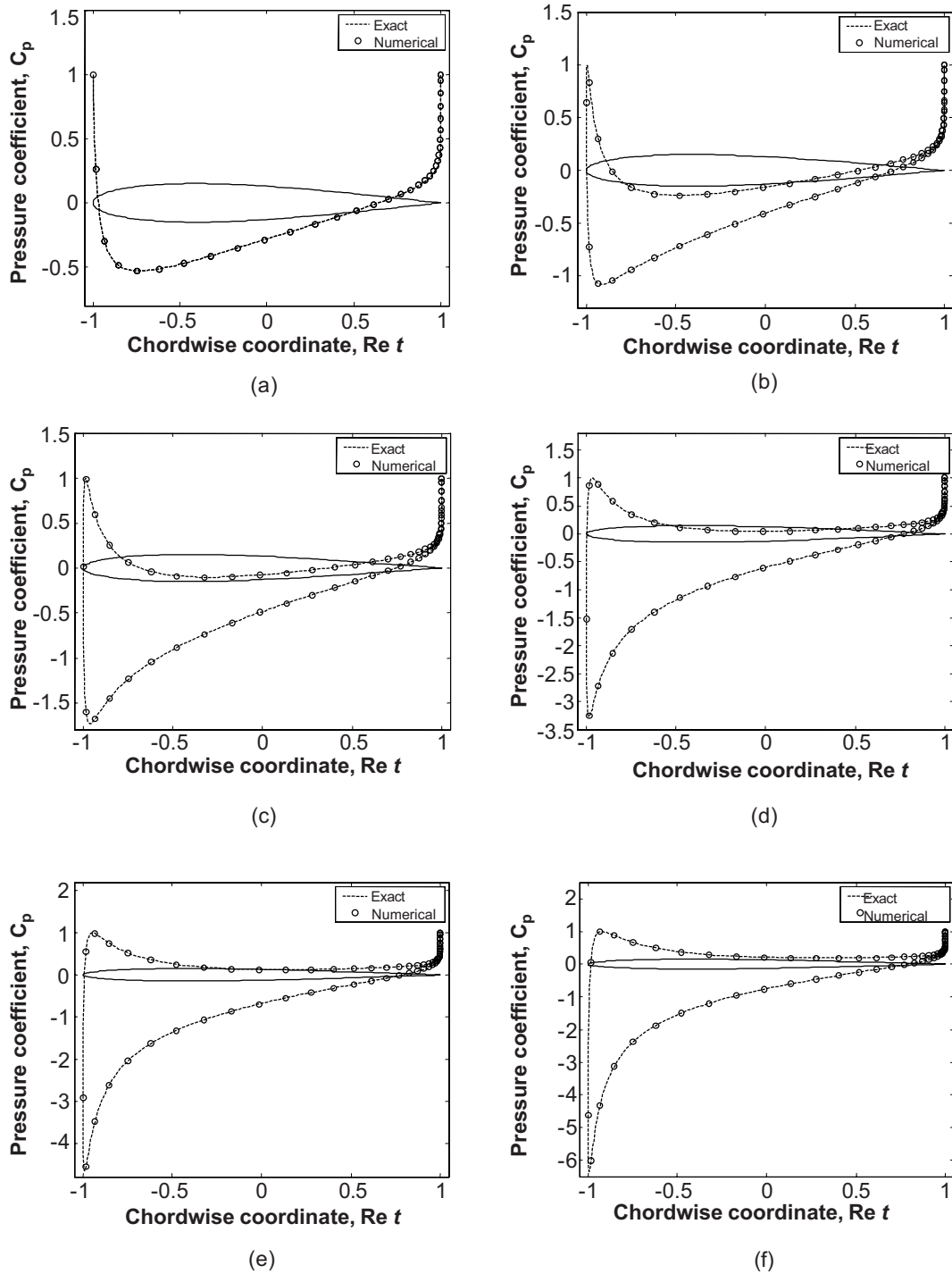$$C_L = \int_0^1 \left(C_{pL}(x) - C_{pU}(x)\right) dx. \tag{32}$$

Figure 2 van de Vooren airfoil: Comparison of the exact and numerical solutions (with 61 node points). (a) $\alpha = 0°$, (b) $\alpha = 3°$, (c) $\alpha = 5°$, (d) $\alpha = 8°$, (e) $\alpha = 10°$, (f) $\alpha = 12°$
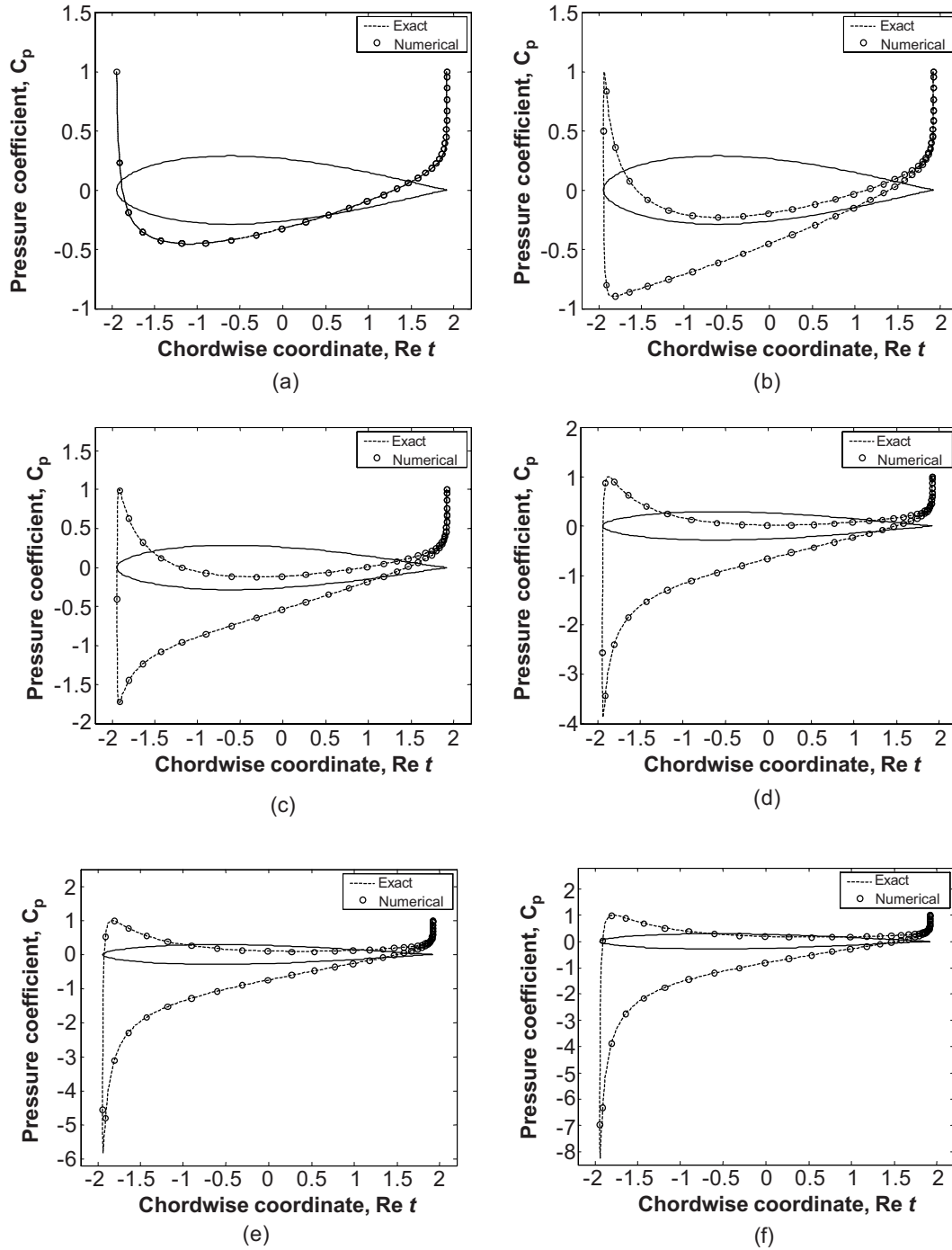
Figure 3   Karman-Trefftz airfoil: Comparison of the exact and numerical solutions (with 61 node points) (a) $\alpha = 0°$, (b) $\alpha = 3°$, (c) $\alpha = 5°$, (d) $\alpha = 8°$, (e) $\alpha = 10°$, (f) $\alpha = 12°$
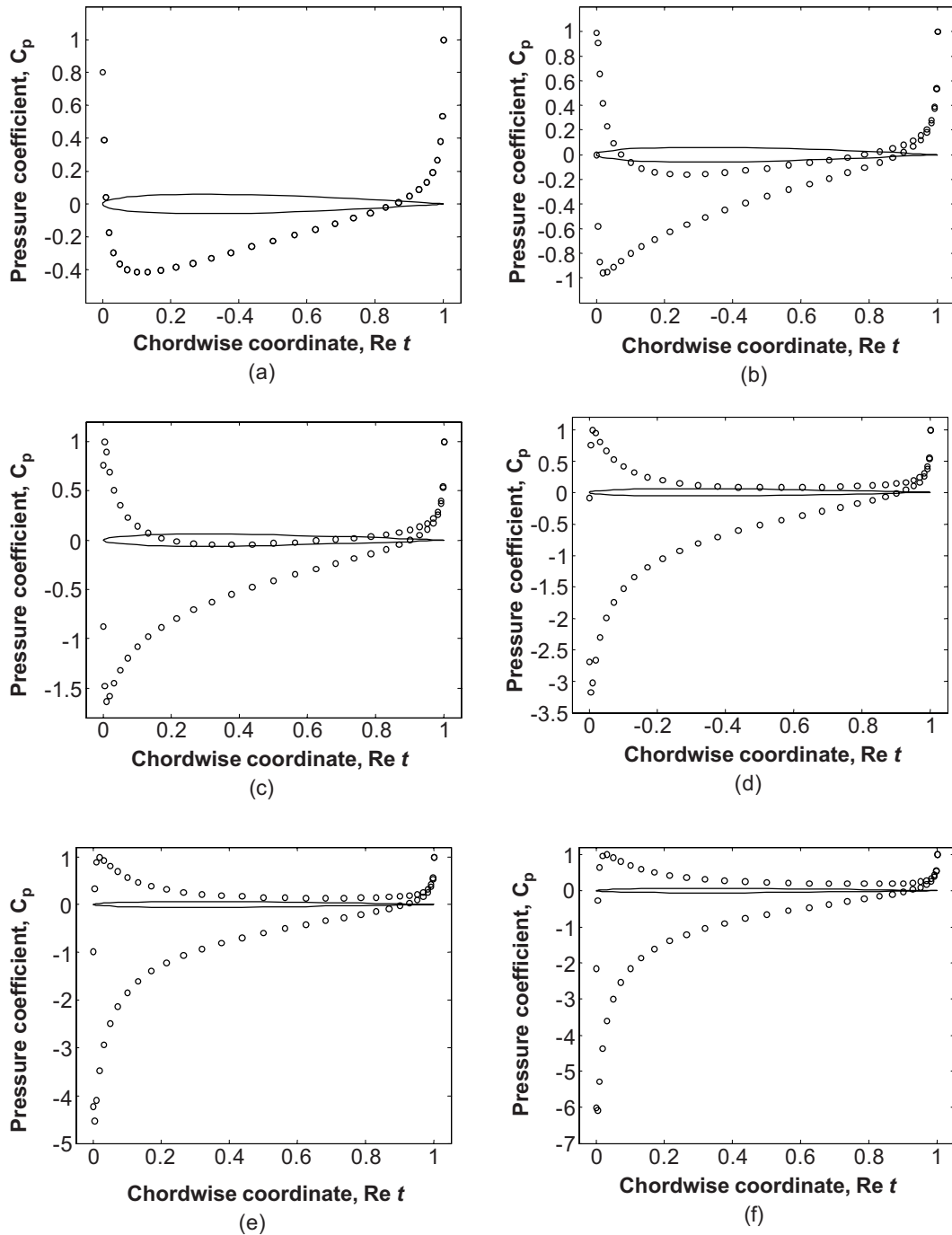
Figure 4   NACA0012 airfoil: The numerical solutions (with 61 node points) (a) $\alpha = 0°$, (b) $\alpha = 3°$, (c) $\alpha = 5°$, (d) $\alpha = 8°$, (e) $\alpha = 10°$, (f) $\alpha = 12°$

A comparison between the computing results (using the present methods), the methods presented in [15] and the experimental measurements [16] at various angles of attack is shown in Table 2.

Table 2 Comparison of lift coefficients at various angles of attack

α	Method I [15]	Method II [15]	Method III [15]	Present method	Experimental [16]
4 °	0.479	0.489	0.391	0.435	0.39
6 °	0.719	0.722	0.621	0.650	0.62
8 °	0.957	0.958	0.832	0.862	0.82
10 °	1.194	1.196	1.041	1.069	0.97
11 °	1.312	1.300	1.142	1.171	1.04
12 °	1.429	1.421	1.243	1.271	1.08

6.0   CONCLUSION

In this paper, a boundary integral equation has been presented for the external potential flow problem. The approach of deriving the integral equation is different from the known approaches in the literature since it depends on reducing the external potential flow problem to an exterior Riemann problem and using the new discovered integral equation for the exterior Riemann problem. Once the solution of the integral equation is computed, the complex disturbance velocity $w(z)$ is then given by a Cauchy type integral which can be calculated sufficiently and accurately. The integral equation has been solved numerically in this paper using the Nyström method with Kress quadrature rule. The numerical solution may be improved if a more efficient method, like the Galerkin method, for solving boundary integral equation in domains with corners is used. The extension of the results of this paper to the multi-element airfoils is straightforward.

REFERENCES

[1]   Carabineanu, A. 1996. A Boundary Element Approach to the 2D Potential Flow Problem Around Airfoils with Cusped Trailing Edge. *Comput. Methods Appl. Mech. Engrg*. 129: 213-219.

[2]   Hess, J. I., and A. M. O. Smith. 1967. Calculation of Potential Flow About Arbitrary Bodies. In Kuchemann, D. (ed.), *Progress in Aeronautical Sciences*. London: Pergamon. 8: 1-138.

[3]   Katz, J., and A. Plotkin. 2001. *Low-Speed Aerodynamics*. 2nd ed. Cambridge: Cambridge University Press.

[4]   Hess, J. L. 1975. Review of Integral-equation Techniques for Solving Potential-flow Problems with Emphasis on the Surface-source Method. *Comput. Methods Appl. Mech. Engrg*. 5: 145-196.

[5]   Hwang, W. S. 2000. A Boundary Node Method for Airfoils Based on the Dirichlet Condition. *Comput. Methods Appl. Mech. Engrg*. 190: 1679-1688.

[6]   Mokry, M. 1990. Complex Variable Boundary Element Method for External Potential Flows. 28th Aerospace Science Meeting. Reno, Nevada. AIAA 90-0127.

[7]   Mokry, M. 1996. Potential Flow Past Airfoils as a Riemann-Hilbert Problem. 1st AIAA Theoretical Fluid Mechanics Meeting. New Orleans, LA. AIAA 96-2161.

[8]   Murid, A. H. M., and M. M. S. Nasser. 2003. Eigenproblem of the Generalized Neumann Kernel. Bull. Malays. Math. Sc. Soc. (Second Series). 26: 13-33.

[9]   Murid, A. H. M., M. M. S. Nasser, and N. Amin. 2004. A Boundary Integral Equation for the External Potential Flow Around Obstacles with Smooth Boundaries. In Proc. Seminar Mengenang Jasa Prof. Dr. Shaharir, UKM, Selangor, Malaysia, 199-207.

[10]   Henrici, P. 1986. *Applied and Computational Complex Analysis*. 3. John Wiley, New York.

[11]   Gakhov, F. D. 1966. *Boundary Value Problem, English Translation of Russian edition 1963*. Oxford: Pergamon Press.

[12]   Kress, R. 1990. A Nyström Method for Boundary Integral Equations in Domains with Corners. *Numer. Math*. 58: 145-161.

[13]   Trefethen, L. N., and D. Bau III. 1997. Numerical Linear Algebra, SIAM Philadelphia.

[14]   Moran, J. 1984. *An Introduction to Theoretical and Computational Aerodynamics*. New York: John Wiley and Sons.

[15]   Xu, C. 1998. Surface Vorticity Modeling of Flow Around Airfoils. *Comput. Mech*. 21: 526-532.

[16]   Michos, A., G. Bergeles, and N. Athanassiadis. 1983. Aerodynamic Characteristics of NACA0012 Airfoil in Relation to Wind Generators. *Wind Eng*. 4: 1-8.