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A GENERALIZATION ON THE NTH COMMUTATIVITY DEGREE OF ALTERNATING GROUPS OF DEGREE 4 AND 5

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Graphical abstract

$$P_n(G) = \frac{\left|\left\{(x,y) \in G \times G \middle| x^n y = yx^n\right\}\right|}{|G|^2}.$$

Abstract

The theory of commutativity degree is important in determining the abelianness of a group. The commutativity degree of a finite group G is the probability that a pair of elements chosen randomly from a group G, commute. The concept of commutativity degree can be generalized to the n^{th} commutativity degree of a group which is defined as the probability of commuting the n^{th} power of a randomly chosen element with another random element from the same group. In this research, the n^{th} commutativity degree of alternating groups of degree 4 and 5 are presented.

Keywords: Abelianness; commutativity degree; alternating group

Abstrak

Teori darjah kekalisan tukar tertib adalah sangat penting dalam menentukan keabelanan satu kumpulan. Darjah kekalisan tukar tertib untuk kumpulan terhingga G ialah kebarangkalian dua unsur terpilih secara rawak dalam kumpulan G, kalis tukar tertib. Konsep darjah kekalisan tukar tertib boleh teritlak kepada darjah kekalisan tukar tertib kuasa ke-n suatu kumpulan yang ditakrifkan sebagai kebarangkalian bahawa kuasa ke-n bagi suatu unsur yang dipilih secara rawak berkalis tukar tertib dengan unsur yang lain daripada kumpulan yang sama. Dalam kajian ini, kebarangkalian kekalisan tukar tertib kuasa ke-n bagi kumpulan selang-seli darjah 4 dan 5 dipersembahkan.

Kata kunci: Keabelanan; darjah kekalisan tukar tertib; kumpulan selang-seli

1.0 INTRODUCTION

All groups mentioned in this paper are considered finite. The commutativity degree of a group G is the probability that a selected chosen pair of elements of G commute. It is denoted by $P(G)$. The definition of the commutativity degree is given as follows.

Definition 1.1 [1] The commutativity degree of a group G, denoted as $P(G)$, can be written as

$$P(G) = \frac{\left|\left\{(x,y) \in G \times G \middle| xy = yx\right\}\right|}{|G|^2}.$$

The concept of commutativity degree was first introduced by Miller [2] in 1944. He provided a list of open problems related to the commutativity degree and its generalization. In 1968, Erdos and Turan [3] investigate some problems of statistical group theory and commutativity degree in nonabelian group and introduced the concept of commutativity degree for symmetric groups, S_m. Later, Gustafson [4] and Machale [1] showed that the commutativity degree of all nonabelian groups is less than or equal to $\frac{5}{8}$.

In 2006, Mohd Ali and Sarmin [5] extended the concept of commutativity degree of a group G to the n^{th} commutativity degree of G, denoted as $P_n(G)$, which is the probability that the n^{th} power of a selected element commute with another element of G.

The formal definition of n^{th} commutativity degree is given in the following.

Definition 1.2 [5] The n^{th} commutativity degree of a group G, denoted as $P_n(G)$, is defined as

$$P_n(G) = \frac{\left|\left\{(x,y) \in G \times G \middle| x^n y = y x^n\right\}\right|}{|G|^2}.$$

Note that for $n = 1$, $P_1(G) = P(G)$. In finding $P_n(G)$, the power of each element in G is gradually raised until the power n is achieved.

There are two approaches on finding the probability that a pair of elements commute. First by using the Cayley Table (or symmetrical 0-1 Table) and second by using the number of conjugacy classes. MacHale [1] used the 0-1 Table to find the probability that two elements commute in a group. In this research, the 0-1 Table is used to determine the n^{th} commutativity degree of a group G.

In this research the n^{th} commutativity degree of alternating groups of degree 4 of order 12 and alternating groups of degree 5 of order 60 are found.

2.0 PRELIMINARIES

In this section, we provide some preliminaries and basic definitions that are needed in this research.

Definition 2.1 [6] Symmetric Group of Degree m

Let A be the finite set $\{1,2,\ldots,m\}$. The group of all permutations of A is the symmetric group on m letters, and is denoted by S_m. The order of S_m is $m!$.

Definition 2.2 [7] Alternating Group of Degree m

The set of all even permutation in S_m forms a subgroup of S_m for $m \geq 2$. This subgroup is called the alternating group of degree m, and denoted by A_m. The order of A_m is $\frac{m!}{2}$.

Definition 2.3 [1] The 0-1 Table for a Group G

If $xy = yx$ for all x, y in G, each of the boxes corresponding to xy and yx will be assigned the number 1. In other side, if $xy \neq yx$, the number 0 will be placed in each of these boxes.

3.0 RESULTS AND DISCUSSION

In this section, the results of $P_n(A_m)$, which is the n^{th} commutativity degree of alternating groups of degree m where $m = 4$ and 5 are determined using the 0-1 Table.

Clearly, A_4 is the alternating group of degree 4. The elements of A_4 are (1), (123), (124), (134), (132), (142), (143), (234), (243), (12)(34), (14)(23) and (13)(24). To compute the multiplication table for A_4, we let

$\beta_1 = (1)$ $\beta_7 = (143)$
$\beta_2 = (123)$ $\beta_8 = (234)$
$\beta_3 = (124)$ $\beta_9 = (243)$
$\beta_4 = (134)$ $\beta_{10} = (12)(34)$
$\beta_5 = (132)$ $\beta_{11} = (13)(24)$
$\beta_6 = (142)$ $\beta_{12} = (14)(23)$.

The Cayley table of A_4 is given in the following:

Table 1 The Cayley Table of A_4

$\cdot$	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
β_1	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
β_2	β_2	β_5	β_{11}	β_8	β_1	β_7	β_{12}	β_{10}	β_3	β_4	β_9	β_6
β_3	β_3	β_{12}	β_6	β_{11}	β_4	β_1	β_9	β_2	β_{10}	β_7	β_5	β_8
β_4	β_4	β_3	β_{10}	β_7	β_{12}	β_8	β_1	β_{11}	β_5	β_2	β_6	β_9
β_5	β_5	β_1	β_9	β_{10}	β_2	β_{12}	β_6	β_4	β_{11}	β_8	β_3	β_7
β_6	β_6	β_8	β_1	β_5	β_{11}	β_3	β_{10}	β_{12}	β_7	β_9	β_4	β_2
β_7	β_7	β_{10}	β_2	β_1	β_9	β_{11}	β_4	β_6	β_{12}	β_3	β_8	β_5
β_8	β_8	β_{11}	β_4	β_{12}	β_6	β_{10}	β_2	β_9	β_1	β_5	β_7	β_3
β_9	β_9	β_7	β_{12}	β_3	β_{10}	β_5	β_{11}	β_1	β_8	β_6	β_2	β_4
β_{10}	β_{10}	β_9	β_8	β_6	β_7	β_4	β_5	β_3	β_2	β_1	β_{12}	β_{11}
β_{11}	β_{11}	β_6	β_7	β_9	β_8	β_2	β_3	β_5	β_4	β_{12}	β_1	β_{10}
β_{12}	β_{12}	β_4	β_5	β_2	β_3	β_9	β_8	β_7	β_6	β_{11}	β_{10}	β_1

From Table 1, we can produce the 0-1 Table for A_4 as shown in the following.

Table 2 The 0-1 Table for A_4

•	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
β_1	1	1	1	1	1	1	1	1	1	1	1	1
β_2	1	1	0	0	1	0	0	0	0	0	0	0
β_3	1	0	1	0	0	1	0	0	0	0	0	0
β_4	1	1	0	1	0	0	1	0	0	0	0	0
β_5	1	0	1	0	1	0	0	0	0	0	0	0
β_6	1	0	0	1	0	1	0	0	0	0	0	0
β_7	1	0	0	0	0	0	1	0	0	0	0	0
β_8	1	0	0	0	0	0	0	1	1	0	0	0
β_9	1	0	0	0	0	0	0	1	1	0	0	0
β_{10}	1	0	0	0	0	0	0	0	0	1	1	1
β_{11}	1	0	0	0	0	0	0	0	0	1	1	1
β_{12}	1	0	0	0	0	0	0	0	0	1	1	1

From Table 2, 48 pairs of elements commute with each other. Therefore, $P(A_4) = \frac{48}{144} = \frac{1}{3}$.

In Table 3 and Table 4, the powers of each element in A_4 are computed up to a certain value (until it can be generalized) and the value of $P_n(A_4)$ is computed for $n = 1, 2, 3, \ldots, 12$.

Table 3 $P_n(A_4)$ for $n = 2, 3, 4, 5$ and 6

$x \in A_4$	x^2	x^3	x^4	x^5	x^6
β_1	$(\beta_1)^2 = \beta_1$	$(\beta_1)^3 = \beta_1$	$(\beta_1)^4 = \beta_1$	$(\beta_1)^5 = \beta_1$	$(\beta_1)^6 = \beta_1$
β_2	$(\beta_2)^2 = \beta_5$	$(\beta_2)^3 = \beta_1$	$(\beta_2)^4 = \beta_2$	$(\beta_2)^5 = \beta_5$	$(\beta_2)^6 = \beta_1$
β_3	$(\beta_3)^2 = \beta_6$	$(\beta_3)^3 = \beta_1$	$(\beta_3)^4 = \beta_3$	$(\beta_3)^5 = \beta_6$	$(\beta_3)^6 = \beta_1$
β_4	$(\beta_4)^2 = \beta_7$	$(\beta_4)^3 = \beta_1$	$(\beta_4)^4 = \beta_4$	$(\beta_4)^5 = \beta_7$	$(\beta_{44})^6 = \beta_1$
β_5	$(\beta_5)^2 = \beta_2$	$(\beta_5)^3 = \beta_1$	$(\beta_5)^4 = \beta_5$	$(\beta_5)^5 = \beta_2$	$(\beta_5)^6 = \beta_1$
β_6	$(\beta_6)^2 = \beta_3$	$(\beta_6)^3 = \beta_1$	$(\beta_6)^4 = \beta_6$	$(\beta_6)^5 = \beta_3$	$(\beta_6)^6 = \beta_1$
β_7	$(\beta_7)^2 = \beta_4$	$(\beta_7)^3 = \beta_1$	$(\beta_7)^4 = \beta_7$	$(\beta_7)^5 = \beta_4$	$(\beta_7)^6 = \beta_1$
β_8	$(\beta_8)^2 = \beta_9$	$(\beta_8)^3 = \beta_1$	$(\beta_8)^4 = \beta_8$	$(\beta_8)^5 = \beta_9$	$(\beta_8)^6 = \beta_1$
β_9	$(\beta_9)^2 = \beta_8$	$(\beta_9)^3 = \beta_1$	$(\beta_9)^4 = \beta_9$	$(\beta_9)^5 = \beta_8$	$(\beta_9)^6 = \beta_1$
β_{10}	$(\beta_{10})^2 = \beta_1$	$(\beta_{10})^3 = \beta_{10}$	$(\beta_{10})^4 = \beta_1$	$(\beta_{10})^5 = \beta_{10}$	$(\beta_{10})^6 = \beta_1$
β_{11}	$(\beta_{11})^2 = \beta_1$	$(\beta_{11})^3 = \beta_{11}$	$(\beta_{11})^4 = \beta_1$	$(\beta_{11})^5 = \beta_{11}$	$(\beta_{11})^6 = \beta_1$
β_{12}	$(\beta_{12})^2 = \beta_1$	$(\beta_{12})^3 = \beta_{12}$	$(\beta_{12})^4 = \beta_1$	$(\beta_{12})^5 = \beta_{12}$	$(\beta_{12})^6 = \beta_1$
	$P_2(A_4) = \frac{1}{2}$	$P_3(A_4) = \frac{5}{6}$	$P_4(A_4) = \frac{1}{2}$	$P_5(A_4) = \frac{1}{3}$	$P_6(A_4) = 1$

Table 4 $P_n(A_4)$ for $n = 7, 8, 9, 10, 11$ and 12

x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
$(\beta_1)^7 = \beta_1$	$(\beta_1)^8 = \beta_1$	$(\beta_1)^9 = \beta_1$	$(\beta_1)^{10} = \beta_1$	$(\beta_1)^{11} = \beta_1$	$(\beta_1)^{12} = \beta_1$
$(\beta_2)^7 = \beta_2$	$(\beta_2)^8 = \beta_5$	$(\beta_2)^9 = \beta_1$	$(\beta_2)^{10} = \beta_2$	$(\beta_2)^{11} = \beta_5$	$(\beta_2)^{12} = \beta_1$
$(\beta_3{}^7 = \beta_3$	$(\beta_3)^8 = \beta_6$	$(\beta_3)^9 = \beta_1$	$(\beta_3)^{10} = \beta_3$	$(\beta_3)^{11} = \beta_6$	$(\beta_3)^{12} = \beta_1$
$(\beta_4)^7 = \beta_4$	$(\beta_4)^8 = \beta_7$	$(\beta_4)^9 = \beta_1$	$(\beta_4)^{10} = \beta_4$	$(\beta_4)^{11} = \beta_7$	$(\beta_4)^{12} = \beta_1$
$(\beta_5)^7 = \beta_5$	$(\beta_5)^8 = \beta_2$	$(\beta_5)^9 = \beta_1$	$(\beta_5)^{10} = \beta_5$	$(\beta_5)^{11} = \beta_2$	$(\beta_5)^{12} = \beta_1$
$(\beta_6)^7 = \beta_6$	$(\beta_6)^8 = \beta_3$	$(\beta_6)^9 = \beta_1$	$(\beta_6)^{10} = \beta_6$	$(\beta_6)^{11} = \beta_3$	$(\beta_6)^{12} = \beta_1$
$(\beta_7)^7 = \beta_7$	$(\beta_7)^8 = \beta_4$	$(\beta_7)^9 = \beta_1$	$(\beta_7)^{10} = \beta_7$	$(\beta_7)^{11} = \beta_4$	$(\beta_7)^{12} = \beta_1$
$(\beta_8)^7 = \beta_8$	$(\beta_8)^8 = \beta_9$	$(\beta_8)^9 = \beta_1$	$(\beta_8)^{10} = \beta_8$	$(\beta_8)^{11} = \beta_9$	$(\beta_8)^{12} = \beta_1$
$(\beta_9)^7 = \beta_9$	$(\beta_9)^8 = \beta_8$	$(\beta_9)^9 = \beta_1$	$(\beta_9)^{10} = \beta_9$	$(\beta_9)^{11} = \beta_8$	$(\beta_9)^{12} = \beta_1$
$(\beta_{10})^7 = \beta_{10}$	$(\beta_{10})^8 = \beta_1$	$(\beta_{10})^9 = \beta_{10}$	$(\beta_{10})^{10} = \beta_1$	$(\beta_{10})^{11} = \beta_{10}$	$(\beta_{10})^{12} = \beta_1$
$(\beta_{11})^7 = \beta_{11}$	$(\beta_{11})^8 = \beta_1$	$(\beta_{11})^9 = \beta_{11}$	$(\beta_{11})^{10} = \beta_1$	$(\beta_{11})^{11} = \beta_{11}$	$(\beta_{11})^{12} = \beta_1$
$(\beta_{12})^7 = \beta_{12}$	$(\beta_{12})^8 = \beta_1$	$(\beta_{12})^9 = \beta_{12}$	$(\beta_{12})^{10} = \beta_1$	$(\beta_{12})^{11} = \beta_{12}$	$(\beta_{12})^{12} = \beta_1$
$P_7(A_4) = \frac{1}{3}$	$P_8(A_4) = \frac{1}{2}$	$P_9(A_4) = \frac{5}{6}$	$P_{10}(A_4) = \frac{1}{2}$	$P_{11}(A_4) = \frac{1}{3}$	$P_{12}(A_4) = 1$

From Table 3 and Table 4, we can generalize the n^{th} commutativity degree of alternating group of degree 4, $P_n(A_4)$ as in the following theorem.

Theorem 3.1 Let A_4 be an alternating group of degree 4. Then for $n, k \in \mathbb{Z}^+$ where $k = 0, 1, 2, \ldots$, $P_n(A_4)$ is given as follows:

$$P_n(A_4) = \begin{cases} \frac{1}{3}, & n = 1 + 6k,\ n = 5 + 6k \\ \frac{1}{2}, & n = 2 + 6k,\ n = 4 + 6k \\ \frac{5}{6}, & n = 3 + 6k \\ 1, & n = 6k \end{cases}$$

Proof For all elements x in A_4, the order of x is 1, 2 or 3. Furthermore, for any $x \in A_4$, $x^6 = e$ and $x^n = e$ for $n = 6k$ where $k \in \mathbb{Z}^+$.

The number of (x, y) where $x \cdot y = y \cdot x$ also equal to the number of (x, y) when $x^5 \cdot y = y \cdot x^5$, $x^7 \cdot y = y \cdot x^7$ and $x^{11} \cdot y = y \cdot x^{11}$.

Now we need to prove that $x^5 \cdot y = y \cdot x^5, x^7 \cdot y = y \cdot x^7$ and $x^{11} \cdot y = y \cdot x^{11}$ can be reduced to $x \cdot y = y \cdot x$.

Suppose $x^6 = e$. This implies $x^{-1} = x^5$. Therefore $x^5 \cdot y = y \cdot x^5$ is the same as $x^{-1} \cdot y = y \cdot x^{-1}$. By cancellation we have $x \cdot y = y \cdot x$.

Next $x^7 \cdot y = y \cdot x^7$ can be written as
$$x \cdot x^6 \cdot y = y \cdot x \cdot x^6$$
$$x \cdot e \cdot y = y \cdot x \cdot e$$
$$x \cdot y = y \cdot x.$$

By the same calculations and argument, it can be shown that $x^{11} \cdot y = y \cdot x^{11}$ can be reduced to $x \cdot y = y \cdot x$.

Next $x^4 \cdot y = y \cdot x^4, x^8 \cdot y = y \cdot x^8$ and $x^{10} \cdot y = y \cdot x^{10}$ are equal to $x^2 \cdot y = y \cdot x^2$ and $x^9 \cdot y = y \cdot x^9$ is equal to $x^3 \cdot y = y \cdot x^3$.

Suppose $x^6 = e$. This implies $(x^2)^{-1} = x^4$. Therefore $x^4 \cdot y = y \cdot x^4$ is the same as $(x^2)^{-1} \cdot y = y \cdot (x^2)^{-1}$. By cancellation we have $x^2 \cdot y = y \cdot x^2$.

Next $x^8 \cdot y = y \cdot x^8$ can be written as
$$x^2 \cdot x^2 \cdot x^4 \cdot y = y \cdot x^2 \cdot x^2 \cdot x^4$$
$$x^2 \cdot e \cdot y = y \cdot x^2 \cdot e$$
$$x^2 \cdot y = y \cdot x^2.$$

By the same calculations and argument, it can be shown that $x^{10} \cdot y = y \cdot x^{10}$ can be reduced to $x^2 \cdot y = y \cdot x^2$.

Next $x^9 \cdot y = y \cdot x^9$ can be written as
$$x^3 \cdot x^3 \cdot x^3 \cdot y = y \cdot x^3 \cdot x^3 \cdot x^3$$
$$x^3 \cdot e \cdot y = y \cdot x^3 \cdot e$$
$$x^3 \cdot y = y \cdot x^3.$$

Clearly x^6 is an identity in A_4 then $x^{12} \cdot y = y \cdot x^{12}$ can also be reduced to $x^6 \cdot y = y \cdot x^6$.

By some calculations,

$x^{1+6k} \cdot y = y \cdot x^{1+6k}$ is equal to $x \cdot y = y \cdot x$.
Suppose $x^{6k} = e$,
then,
$$x^{1+6k} \cdot y = y \cdot x^{1+6k}$$
$$x \cdot x^{6k} \cdot y = y \cdot x \cdot x^{6k}$$
$$x \cdot e \cdot y = y \cdot x \cdot e$$
$$x \cdot y = y \cdot x$$

$x^{5+6k} \cdot y = y \cdot x^{5+6k}$ is equal to $x^5 \cdot y = y \cdot x^5$.
Suppose $x^{6k} = e$,
then,
$$x^{5+6k} \cdot y = y \cdot x^{5+6k}$$
$$x^5 \cdot x^{6k} \cdot y = y \cdot x^5 \cdot x^{6k}$$
$$x^5 \cdot e \cdot y = y \cdot x^5 \cdot e$$
$$x^5 \cdot y = y \cdot x^5$$

$x^{2+6k} \cdot y = y \cdot x^{2+6k}$ is equal to $x^2 \cdot y = y \cdot x^2$.
Suppose $x^{6k} = e$,
then,
$$x^{3+6k} \cdot y = y \cdot x^{3+6k}$$
$$x^3 \cdot x^{6k} \cdot y = y \cdot x^3 \cdot x^{6k}$$
$$x^3 \cdot e \cdot y = y \cdot x^3 \cdot e$$
$$x^3 \cdot y = y \cdot x^3$$

$x^{3+6k} \cdot y = y \cdot x^{3+6k}$ is equal to $x^3 \cdot y = y \cdot x^3$.
Suppose $x^{6k} = e$,
then,
$$x^{2+6k} \cdot y = y \cdot x^{2+6k}$$
$$x^2 \cdot x^{6k} \cdot y = y \cdot x^2 \cdot x^{6k}$$
$$x^2 \cdot e \cdot y = y \cdot x^2 \cdot e$$
$$x^2 \cdot y = y \cdot x^2$$

$x^{4+6k} \cdot y = y \cdot x^{4+6k}$ is equal to $x^4 \cdot y = y \cdot x^4$.
Suppose $x^{6k} = e$,
then,
$$x^{4+6k} \cdot y = y \cdot x^{4+6k}$$
$$x^4 \cdot x^{6k} \cdot y = y \cdot x^4 \cdot x^{6k}$$
$$x^4 \cdot e \cdot y = y \cdot x^4 \cdot e$$
$$x^4 \cdot y = y \cdot x^4$$

Suppose x^{6k} is the identity in A_4 then, clearly $x^{6k} \cdot y = y \cdot x^{6k}$.

Using similar method, we found the generalization of the n^{th} commutativity degree of alternating group of degree 5, $P_n(A_5)$ given as follows.

Theorem 3.2 Let A_5 be an alternating group of degree 5. Then for $n, k \in \mathbb{Z}^+$ where $k = 0, 1, 2, \ldots$, $P_n(A_5)$ is given as follows:

$$P_n(A_5) = \begin{cases} \frac{1}{12}, & n = 1+30k, n = 7+30k, n = 11+30k, n = 13+30k, \\ & n = 17+30k, n = 19+30k, n = 23+30k, n = 29+30k \\ \frac{1441}{3600}, & n = 3+30k, n = 9+30k, n = 21+30k, n = 27+30k \\ \frac{19}{60}, & n = 2+30k, n = 4+30k, n = 8+30k, n = 14+30k, \\ & n = 16+30k, n = 22+30k, n = 26+30k, n = 28+30k \\ \frac{1619}{3600}, & n = 5+30k, n = 25+30k \\ \frac{2281}{3600}, & n = 6+30k, n = 12+30k, n = 18+30k, n = 24+30k \\ \frac{2399}{3660}, & n = 10+30k, n = 20+30k \\ \frac{23}{30}, & n = 15+30k \\ 1, & n = 30+30k \end{cases}$$

4.0 CONCLUSION

As a conclusion, the n^{th} commutativity degree of alternating groups of degree 4 and alternating groups of degree 5 are determined. The 0-1 Table was used in finding $P_n(A_4)$ and $P_n(A_5)$.

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