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ANALYTICAL APPROXIMATE SOLUTION FOR THE FORCED KORTEWEG-DE VRIES (FKDV) ON CRITICAL FLOW OVER A HOLE USING HOMOTOPY ANALYSIS METHOD

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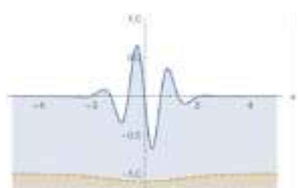
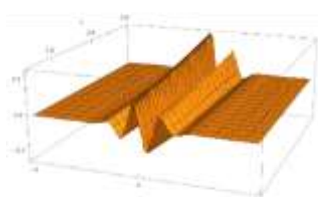
Graphical abstract





Abstract

Free surface flows in a two-dimensional channel past over a hole is studied using shallow water forced Korteweg-de Vries (fKdV) equation. The forcing term of fKdV equation represents the hole shaped bottom topography. Froude number (*Fr*), which represents the ratio of flow speed to the wave speed, will also be used in solving fKdV equation. The fKdV equation is solved using Homotopy Analysis Method (HAM). HAM is an approximate analytical technique used to obtain series of solutions for the nonlinear problems where HAM has an auxiliary parameter $c_o$to adjust and control the convergence region of the series solution. Solitary wave solutions are obtained from the series of solutions of HAM and wave flows are observed at particular time. The HAM solution shows the hole shaped bottom topography plays an important role in determining the evolution of solitary waves.

Keywords: fKdV equation; homotopy analysis method; approximate analytical solution; holed bottom topography

Abstrak

Aliran permukaan bebas dalam saluran dua dimensi melalui permukaan lubang dikaji menggunakan persamaan air cetek paksaan Korteweg-de Vries (fKdV). Sebutan paksaan dalam persamaan fKdV mewakili topografi bahagian bawah berbentuk lubang. Nombor Froude (*Fr*) yang mewakili nisbah kelajuan aliran kepada kelajuan gelombang juga akan digunakan. Persamaan fKdV akan diselesaikan dengan menggunakan kaedah analisis homotopi (HAM). HAM adalah teknik analisis beranggaran yang digunakan untuk mendapatkan siri penyelesaian untuk masalah tidak linear di mana ia mempunyai parameter tambahan c_o, untuk menyelaraskan dan mengawal rantau penumpuan penyelesaian siri itu. Penyelesaian gelombang solitari dicerap pada sesuatu masa tertentu menggunakan HAM. Penyelesaian HAM menunjukkan bahagian bawah topografi yang berbentuk lubang memainkan peranan penting dalam menentukan evolusi gelombang solitari.

Kata kunci: Persamaan fKdV; kaedah analisis homotopi; penyelesaian analisis anggaran; topografi bahagian bawah berlubang

1.0 INTRODUCTION

Nonlinear phenomena such as flow of water waves play a vital role in the field of fluid dynamics and wave propagation which is a branch of applied mathematics and physics. In the past decades, many researchers have worked on water waves flow over an obstacle. The initial work on water waves over flat bottom was started by John Scott Russell. The research then continued by Boussinesq, Korteweg and De-Vries who found a canonical equation that balance the nonlinear and dispersive effects[1].Zabusky and Kruskal confirmed Russell findings on solitary wave by using Korteweg de-Vries (KdV) equation [2]. The invention of the Inverse Spectral Transform (IST) to solve KdV equation creates great development in the modern nonlinear mathematical science [3].KdV equation can also be solved analytically using Hirota Method [4], Bäcklund transformation [5], Darboux transformation [6], and Adomian Decomposition Method [7].

KdV equation with a forcing term is established by recent studies to describe the physics of a shallow layer of fluid subject to external forcing. KdV equation with forcing term is known as "forced KdV" (fKdV) equation. It is noted that KdV equation is completely integrable but when a forcing term is added, the fKdV equation is difficult to be integrated [8]. Lee et al. found that the forcing term in fKdV can be related to a physical meaning and showed that the forcing term represents surface pressure and bottom topography [9]. Moreover, Camassa & Wu studied solitary waves generated by a negative forcing function and described the features [10]. Zhang and Zhu presented a weakly nonlinear theory for subcritical, transcritical and supercritical flows over step topography [11]. Zhang and Chwang explored generation of solitary waves by bottom topographies using numerical model [12]. Furthermore, Grimshaw et al. found flow over a localised obstacle resulting in upstream and downstream nonlinear wavetrains with unsteady undular bores [13].Ee et al. worked on transcritical flow over a hole and investigated the effects of the width and amplitude of the hole [14].

Homotopy Analysis Method (HAM) was introduced by Liao in 1992 is an analytical method to solve nonlinear partial differential problems [15]. HAM has greater flexibility in the selection of a proper set of base functions for the solution and a much simpler way in the control of the convergence rate and region compared to perturbation approach [16-17]. This analytical technique does not have restriction of non-perturbation methods, such as Lyapunov's artificial small parameter method and the δ-expansion method. The analytical technique also has been applied successfully to solve nonlinear differential equation for modelling in science, engineering and finance [18-19]. Nazari et al. used HAM to obtain the solitary wave solution of KdV equation and have shown an excellent agreement with the existing exact solution [20]. David et al. used HAM to solve fKdV model with a specific choice of forcing term [21].

In this article, the critical flow over a hole will be examined using nonlinear shallow water fKdV model. The series solution of HAM demonstrates the flow of waves over a hole using fKdV model. This investigation aims (a) to describe two types of hole using forcing term in the fKdV model and relate it with sea bed topography (b) to find an analytic approximate solution for fKdV model using HAM (c) to describe the physical flow over a hole, and (d) to explain new findings. HAM solution is found to be capable to demonstrate critical flows of water over a particular hole. The changes in the slope of seabed geometry are found to generate multi solitary waves.

2.0 FORCED KDV AND HAM

2.1 The Shallow Water Forced KdV

The fKdV equation given by Wu [22] is:

$$\frac{1}{c}\frac{\partial\varphi(x,t)}{\partial t}+[(Fr-1)-\frac{3}{2}\frac{\varphi(x,t)}{h}]\frac{\partial\varphi(x,t)}{\partial x} -\frac{1}{6}h^2\frac{\partial^3\varphi(x,t)}{\partial x^3}=\frac{1}{2}\frac{\partial f(x)}{\partial x} \quad (2.1)$$

with $\varphi(x,0)=\varphi_0(x)$ (2.2)

$$f(x)=\frac{p_a(x)}{\rho g}+b(x) \quad (2.3)$$

where $\varphi(x,t)$ refers to the free water surface elevation measured from undisturbed water level , $f(x)$ is the external forcing term given by surface air pressure, $\frac{p_a(x)}{\rho g}$ and topography of rigid bottom, $b(x)$.

The Froude number *Fr* is the ratio of flow speed *U* with linear long wave speed c. It is also known as critical parameter where the value of *Fr* determines the type of critical flows over localised obstacle. When the value of *Fr* is more than 1, then the flow is considered supercritical whereby the flow assumed to be subcritical when the value of *Fr* is less than 1. The flow is considered transcritical when *Fr* =1. In this work, the forcing term of equation (2.3) is simplified by eliminating surface air pressure. Bottom topography is modeled in our work here as a hole in the seabed geometry by using equation (2.4) which is

$$b(x)=-0.1\exp[-\frac{x^n}{4}]-1 \quad (2.4)$$

Where n is constant. Two different types of hole analysed where the first one is set at $n=2$ using equation (2.4) where the hole is likely an inverse of bell-shaped. The second case study of hole is set at $n=8$ so that the hole is wider and more flattened at the bottom. Figure 1 shows a sea bed topography with a hole based on equation 2.4 using different values of n. Figure 1(a) and (b) will be discussed respectively in the section 3.0.

a)

b)

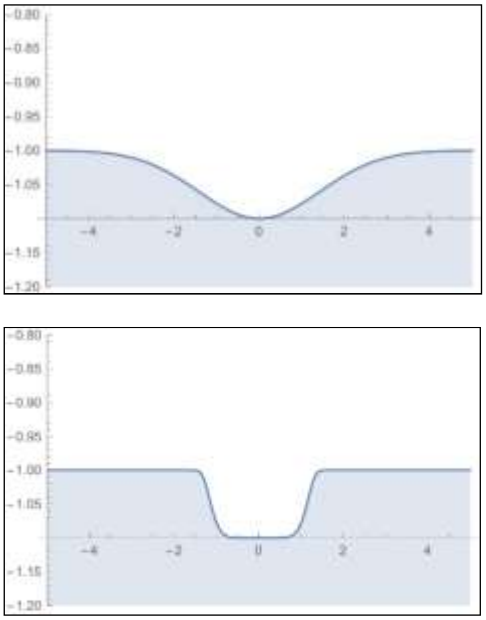


Figure 1 Sea bed geometry with a hole modelled using equation (2.4). (a) $n=2$ (b) $n=8$

2.2 Approximate Analytical Solution For fKdV Using HAM

Rewrite equation 2.3,

$$\frac{1}{c}\frac{\partial\varphi(x,t)}{\partial t}+[(Fr-1)-\frac{3}{2}\frac{\varphi(x,t)}{h}]\frac{\partial\varphi(x,t)}{\partial x} -\frac{1}{6}h^2\frac{\partial^3\varphi(x,t)}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x}=0 \quad (2.5)$$

Consider the constant mean water depth of sea is $h=1$ and wave speed, $c\approx\sqrt{gh}=\sqrt{9.81}$.

From HAM,

$$(1-q)\ell[\varphi(x,t;q)-\varphi_0(x,t)]=qc_o\mathrm{H}(x,t)N[\varphi(x,t;q)] \quad (2.6)$$

we use

$$\varphi_0(x,t)=\frac{-2e^x}{(1+e^x)^2} \quad (2.7)$$

as the initial guess and

$$\ell[\varphi(x,t;q)\,]=\frac{\partial\varphi(x,t;q)}{\partial t} \quad (2.8)$$

as the auxiliary linear operator satisfying

$$\ell[c_1]=0 \quad (2.9)$$

where c_1 is constant.

Considering

$$\mathrm{H}(x,t)=1, \quad (2.10)$$

$$N[\varphi(x,t;q)]=\frac{1}{c}\frac{\partial\varphi(x,t;q)}{\partial t}+(Fr-1)\frac{\partial\varphi(x,t;q)}{\partial x} -\frac{3}{2h}\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x}-\frac{1}{6}h^2\frac{\partial^3\varphi(x,t;q)}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x}, \quad (2.11)$$

and the *m*th-order deformation problem

$$\ell[\varphi_m(x,t)-\chi_m\varphi_{m-1}(x,t)]=qc_o[\frac{1}{c}\frac{\partial\varphi_{m-1}}{\partial t}+(Fr-1)\frac{\partial\varphi_{m-1-i}}{\partial x} -\frac{3}{2h}(\sum_{i=0}^{m-1}\varphi_i\frac{\partial\varphi_{m-1-i}}{\partial x})-\frac{1}{6}h^2\frac{\partial^3\varphi_{m-1}}{\partial x^3}-\frac{1}{2}\frac{\partial b_{m-1}}{\partial x}] \quad (2.12)$$

with

$$\varphi_m(x,0)=0 \text{ for } m>1 \quad (2.13)$$

3.0 RESULTS AND DISCUSSION

For both case 1 (*n*=2) and case 2 (*n*=8), the Froude number is fixed at 1 as the main focus of this work is on transcritical flow. MATHEMATICA Version 10 was used to solve the nonlinear equations.

3.1 Case 1 : *n* =2

HAM solution of equation (2.1) is obtained at 5th-order approximation. The auxiliary parameter c_o in HM solution must be determined by plotting the derivatives of φ for a fixed point of x and time, *t*. The sketching of derivatives of φ over fixed point of x is important in order to obtain a reliable and applicable solution to the configuration of problem. **Figure 2** shows the sketch of c_o - curves at 5th order approximation.

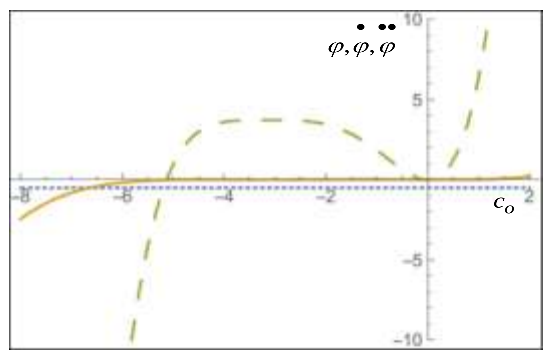


Figure 2 The c_o - curves according to the 5th order approximation. Dashed Point: $\varphi(0.01,0.01)$, Solid Line: $\dot{\varphi}(0.01,0.01)$ and Dashed Line: $\ddot{\varphi}(0.01,0.01)$

Liao [15], pointed out that the valid region of c_o lies on the horizontal line segment (refer Figure 2).Based on Figure 2, the admissible convergence interval of HAM solution is $-4\le c_o\le -2$. After a detailed computation using MATHEMATICA Version 10, c_o is determined at -3. Each points of c_o in the interval of admissible convergence will be applied in MATHEMATICA to identify whether the chosen c_o could describe the flow of water wave over an obstacle or not. The final chosen c_o is assumed to describe the water wave phenomenon. HAM solution when $c_o=-3$, is found at 5th order approximation is shown in equation (3.2).

$$\varphi(x,t)=\varphi_0(x,t)+\varphi_1(x,t)+\ldots\ldots\ldots+\varphi_5(x,t) \quad (3.1)$$

$$\varphi(x,t) = -\frac{2e^{x}}{\left(1+e^{x}\right)^{2}} - \frac{1}{2\left(1+e^{x}\right)^{5}} 3e^{-\frac{x^{2}}{4}} t\left(\frac{2}{3}e^{x+\frac{x^{2}}{4}} - \frac{58}{3}e^{2x+\frac{x^{2}}{4}} + \frac{58}{3}e^{3x+\frac{x^{2}}{4}} - \frac{2}{3}e^{4x+\frac{x^{2}}{4}} - \frac{x}{20} - \frac{e^{x}x}{4} - \frac{1}{2}e^{2x}x - \frac{1}{2}e^{3x}x - \frac{1}{4}e^{4x}x - \frac{1}{20}e^{5x}x\right) + \frac{1}{32\left(1+e^{x}\right)^{8}} 3e^{-\frac{x^{2}}{4}} t\left(\frac{4}{5} + \frac{32e^{x}}{5} + \frac{112e^{2x}}{5} + \frac{224e^{3x}}{5} + 56e^{4x} + \frac{224e^{5x}}{5} + \frac{112e^{6x}}{5} + \frac{32e^{7x}}{5} + \frac{4e^{8x}}{5} + \ldots \right. \quad (3.2)$$

The following Figure 3 and 4 shows the 2D and 3D plot obtained through HAM solution using equation (3.2) with an auxiliary parameter $c_o = -3$, respectively.

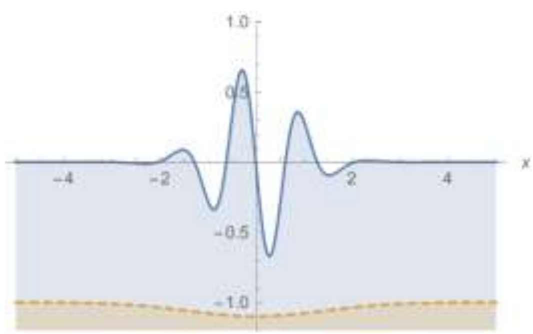


Figure 3 2D Plot of HAM solution for equation (2.1) and $n = 2$ at t = 4. Bottom dotted line: Hole, $b(x)$ and Upper line: water elevation $\varphi(x,4)$

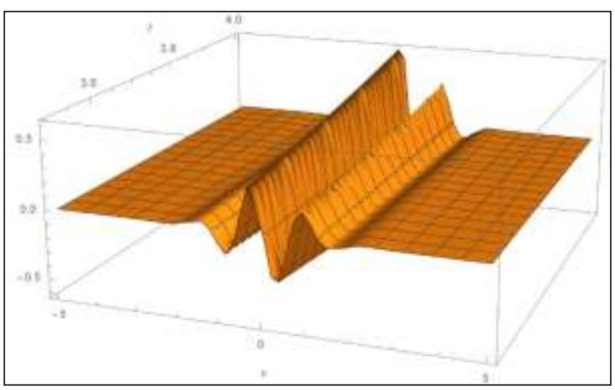


Figure 4 3D Plot of HAM solution for equation (2.1) and $n = 2$ for $3.5 \le t \le 4$.

Figure 3 shows the flow of waves over a hole at t=4 and geometry of sea bed shown in the bottom line. The bottom topography in Figure 1(a), clearly shows that the sea bed topography has a shaped of inverse bell. The deepest part of the hole is centric at x=0 which is 0.1. From the left side (x<0), it is found that the depth of hole increasing in a slow manner and once it reach centric points (x=0), the depth of hole decreasing again over x>0. Figure 3 shows that fKdV with a hole sea bed geometry and rises of multi soliton solitary waves. It is observed that three solitary waves exhibits over the centric hole. The peak each of solitary waves exists at the x-coordinate of x=-1.5, x=-0.25 and x=0.9. Particularly the middle wave reached the highest amplitude in the region of deepest hole. Figure 4 shows flow of waves over non-flatten topography for $3.5 \le t \le 4$. The flow seems very stable and it reaches maximum elevation at the center of hole. Wu [22] described the features of negative forcing by suggesting that negative forcing results in two or more harmonics and local wave excited to a relatively quite large amplitude within the negative region then settles to a smaller height. Wu's description of negative forcing perfectly agreed with HAM results in Figure 3 and 4 as three solutions solitary waves rises while the middle waves reach maximum height at the bottom of hole and finally the waves settle at the edge of the hole geometry. The solution of fKdV equation incorporated with the hole topography found to be interesting as it exhibits multi solitary wave over the hole and the maximum elevation of waves occurs at the deepest hole of the seabed.

3.2 Case 2 : *n* =8

HAM solution of fKdV equation (2.1) is obtained at 4th-order approximation for $n = 8$. Figure 5 shows the sketch of c_o - curves at 4th order approximation.

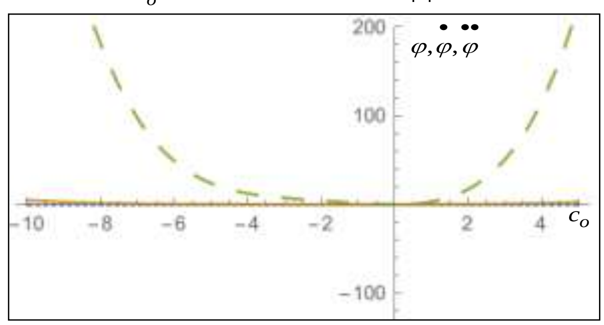


Figure 5 The c_o - curves according to the 5th order approximation. Dashed Point: $\varphi(0.01,0.01)$, Solid Line: $\dot{\varphi}(0.01,0.01)$ and Dashed Line: $\ddot{\varphi}(0.01,0.01)$

Based on Figure 5, the admissible convergence interval of HAM solution is $-3 \le c_o < 0$. Using MATHEMATICA Version 10, c_o is determined at -0.4. Equation (3.4) represents HAM solution at 5th order approximation with $c_o = -0.4$.

$$\varphi(x,t) = \varphi_0(x,t) + \varphi_1(x,t) + \varphi_2(x,t) + \varphi_3(x,t) + \varphi_4(x,t) \quad (3.3)$$

$$\varphi(x,t) = -\frac{2e^{x}}{(1+e^{x})^{2}} - \frac{1}{(1+e^{x})^{5}} 0.8e^{-\frac{x^{8}}{4}} t\left(\frac{1}{6}e^{x+\frac{x^{8}}{4}} - \frac{29}{6}e^{2x+\frac{x^{8}}{4}} + \frac{29}{6}e^{3x+\frac{x^{8}}{4}} - \frac{1}{6}e^{4x+\frac{x^{8}}{4}} - \frac{x^{7}}{20} - \frac{e^{x}x^{7}}{4} - \frac{1}{2}e^{2x}x^{7} - \frac{1}{2}e^{3x}x^{7} - \frac{1}{4}e^{4x}x^{7} - \frac{1}{20}e^{5x}x^{7}\right) + \frac{1}{(1+e^{x})^{8}} + \ldots \quad (3.4)$$

The following Figure 6 and **7** shows the 2D and 3D plots obtained through HAM solution using equation (3.4) with an auxiliary parameter $c_o = -0.4$ respectively.

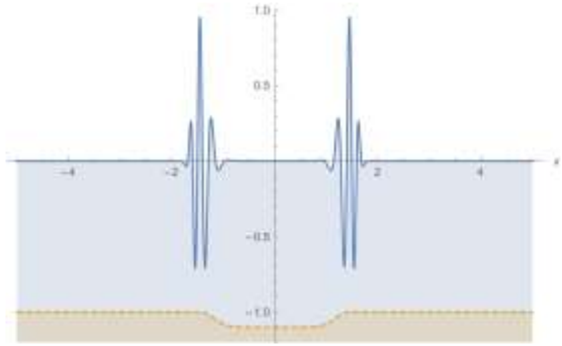


Figure 6 2D Plot of HAM solution for equation (2.1) and $n=8$ at t = 4. Bottom dotted line: Hole, $b(x)$ and Upper line: water elevation $\varphi(x,4)$

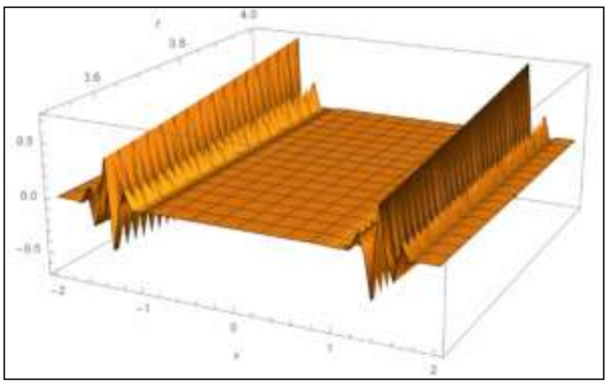


Figure 7 3D Plot of HAM solution for equation (2.1) and $n=8$ for $3.5\le t\le 4$.

Figure 6 shows the flow of waves over a hole at *t*=4. The bottom line in Figure 6 represents sea bed geometry. Bottom topography is clearly shown in Figure 1(b) which the sea floor has deep cut at the edge of hole. Furthermore, the base of hole is flattened and wider with approximately width 2 units. The HAM solution in Figure 6 highlighted that solitary waves rises at the both side edge of hole. At the initial stage from left hand side, there is a sudden rise of a solitary wave when the water nearing edge of hole. Then there is a sudden wake of huge waves continuously with approximately height of 1 unit. Finally, a third solitary wave rise with medium height. It is found that the flows of water across the region $-1\le x\le 1$ does not show any rise of waves. Then again, there is a rise of wave in the medium height in the region of $1\le x\le 2$. Subsequently this situation is followed by a huge wave, and finally with a wake of a small amplitude solitary wave. The existence of three solitary waves that rises in the slopping region of both edges of bottom hole looks symmetric. It can be observed that the waves exhibits particularly at region of $-2\le x\le -1$ are seems reflected at the line of x=0 and producing similar waves at right hand side region of $1\le x\le 2$. Figure 7 shows that solutions created at the edge of seabed hole and the waves are entirely flattened at the base of the hole and over the centric point at x=0. Based on observation of Figures 6 and 7, it can be concluded the wave rises when there is a change in the slope of the bottom topography. This suggests that the waves exhibit in a solitary manner when the sea bottom topography has changed over slope in the sediment. Ee et al. [14] found that trapped waves are only formed over the hole if the hole is very wide and wave-trains observed if there is positive or negative step. The result of HAM produces good agreement with Ee et al. [14] observation as multi solitary waves formed over the step down and step up of the hole and no trapped waves found as the width of the hole is small.

4.0 CONCLUSION

In this work, the free surface critical flow over a hole has been observed using shallow water forced Korteweg-de Vries (fKdV) model. The fKdV equation has a physical relation to sea bed geometry. Two different types of hole are observed using the forcing term of fKdV. The first observed case contains a hole in the shape of inverse bell and the second case; consider a hole which has flattened base with a sharp edge on both sides. The shallow water fKdV equation is solved using HAM and approximated solution obtained shows a good agreement with flow over a hole physically. The results of HAM shows the existence of multi solitary waves when the waves flow over the hole. HAM solution shows that the waves rise in periodical form when there is a change in slope of seabed geometry. It is observed that the wave shows no action on a flattened base of seabed geometry. This emphasis demonstrates that the slope of the bottom topography plays a significant role in exhibiting the water wave profile.

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