Jurnal Teknologi

Full Paper

A NEW CLASS OF BINARY APPROXIMATING SUBDIVISION SCHEMES

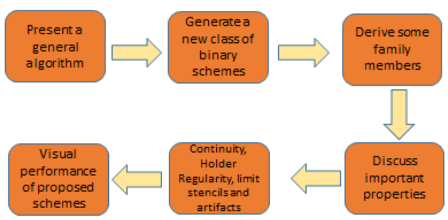
Ghulam Mustafa*, Pakeeza Ashraf, Noreen Saba

Department of Mathematics, The Islamia University of Bahawalpur, Pakistan

Article history
Received
25 October 2015
Received in revised form
14 December 2015
Accepted
9 February 2016

*Corresponding author
ghulam.mustafa@iub.edu.pk

Graphical abstract



Abstract

In this article, we present a general algorithm to generate a new class of binary approximating subdivision schemes and give derivation of some family members. We discuss important properties of derived schemes such as: convergence, continuity, Hölder regularity, degree of polynomial generation and reproduction, support, limit stencils and artifacts. Furthermore, visual performance of proposed schemes has also been presented.

Keywords: Approximating scheme; continuity; polynomial reproduction; artifacts

1.0 INTRODUCTION

Subdivision scheme is a technique in the field of Computer Aided Geometric Design (CAGD) to create smooth curves and surfaces. In the process of subdivision, we take the control polygon and apply the subdivision schemes in which series of successive iterations are performed in order to find the points on curve. It has found many applications in CAGD because of its efficiency, simplicity and flexibility of algorithms. Lane-Riesenfeld [10] and Hormann and Sabin [9] presented subdivision schemes based on B-spline. Cashman et al. [2] presented generalization of Lane Riesenfeld scheme to generate a family of schemes. Ashraf et al. [1] introduced variation on Lane-Riesenfeld method to generate schemes. Dubuc [8] generalized the schemes of de Rham [12] and Chaikin [3]. Conti and Romani [5] used de Rham transform to introduce a class of dual *m*-ary schemes. Mustafa et al. [11] introduced a class of dual and primal schemes. In our framework, we develop a well-designed algorithm that generates a class of binary approximating schemes. The proposed class of schemes is categorized by a parameter. Greater values of parameter give schemes with wider mask and support. Degree of polynomial generation of proposed schemes goes up as value of parameter is increased while proposed schemes have linear polynomial reproduction for each value of parameter. We find out that continuity and Hölder regularity of proposed schemes increase gradually as we increase value of parameter. Moreover we also determine that artifact magnitude decreases as we increase value of parameter. In Section 2, we present an algorithm to design a class of subdivision schemes which depends on a parameter. In Section 3, degree of polynomial generation and reproduction of proposed schemes are analyzed. In Section 4, continuity and Hölder regularity of some of proposed schemes are discussed. In Section 5, artifact analysis and limit stencil analysis are carried out. Applications and summary are included in last section.

2.0 GENERATION OF SUBDIVISION SCHEMES

In this section we present the algorithm for the generation of binary approximating subdivision schemes. Now consider two subdivision schemes, 3-point binary approximating subdivision scheme [16] is given by

$$f_{2i}^{k+1}=\frac{9}{32}f_{i-1}^{k}+\frac{22}{32}f_{i}^{k}+\frac{1}{32}f_{i+1}^{k},$$
$$f_{2i+1}^{k+1}=\frac{1}{32}f_{i-1}^{k}+\frac{22}{32}f_{i}^{k}+\frac{9}{32}f_{i+1}^{k}, \qquad (1)$$

and 4-point binary interpolating subdivision scheme [6] is given by

$$f_{2i}^{k+1}=f_{i}^{k},$$
$$f_{2i+1}^{k+1}=-\frac{1}{16}f_{i-1}^{k}+\frac{9}{16}f_{i}^{k}+\frac{9}{16}f_{i+1}^{k}-\frac{9}{16}f_{i+2}^{k}, \qquad (2)$$

Laurent polynomial of subdivision scheme (1) is

$$\beta(z)=\frac{1}{32}(1+9z+22z^2+22z^3+9z^4+z^5), \qquad (3)$$

and Laurent polynomial of subdivision scheme (2) is

$$\alpha(z)=\frac{1}{32}(-1+9z^2+16z^3+9z^4-z^6). \qquad (4)$$

General Laurent polynomial can be written as

$$\alpha(z)=\alpha_{even}(z^2)+z\alpha_{odd}(z^2).$$

From (4), we have

$$\alpha_{even}(z)=\left(\frac{1+z}{2}\right)\left(\frac{-1+10z-z^2}{8}\right). \qquad (5)$$

Also (3) can be factorized as

$$\beta(z)=\left(\frac{1+z}{2}\right)^3\left(\frac{1+6z+z^2}{4}\right). \qquad (6)$$

Now we introduced the family of schemes named: $H=(H_l:l=0,1,2...)$,where general member H_l has the Laurent polynomial of the form

$$P_l(z)=\left(\alpha_{even}(z)\right)^l\beta(z). \qquad (7)$$

By substituting (5) and (6) in (7), we get

$$P_l(z)=\left(\frac{1+z}{2}\right)^{l+3}\left(\frac{-1+10z-z^2}{8}\right)^l\left(\frac{1+6z+z^2}{4}\right). \qquad (8)$$

We can easily derive the subdivision schemes H_l and their masks by substituting $l=0,1,2....,$ in (8).

2.1 Derivation Of Subdivision Schemes

Here, we derive 3-point, 5-point, 6-point and 8-point binary approximating subdivision schemes by substituting $l=0,1,2,3$ in (8) respectively.

2.1.1 3-Point Binary Approximating Subdivision Scheme H₀

By substituting $l=0$ in (2.8), we get the Laurent polynomial of scheme H_0 as follows

$$P_0(z)=\frac{1}{32}(1+9z+22z^2+22z^3+9z^4+z^5), \qquad (9)$$

whose mask is given by

$$\alpha_0=\frac{1}{32}\{1,9,22,22,9,1\},$$

and we obtain the scheme H_0 as

$$f_{2i}^{k+1}=\frac{9}{32}f_{i-1}^{k}+\frac{22}{32}f_{i}^{k}+\frac{1}{32}f_{i+1}^{k},$$
$$f_{2i+1}^{k+1}=\frac{1}{32}f_{i-1}^{k}+\frac{22}{32}f_{i}^{k}+\frac{9}{32}f_{i+1}^{k}. \qquad (10)$$

2.1.2 5-Point Binary Approximating Subdivision Scheme H₁

By substituting $l=1$ in (2.8), we get the Laurent polynomial of scheme H_1 as follows

$$P_1(z)=\frac{1}{512}(-1+68z^2+256z^3+378z^4+256z^5+68z^6-z^8), \qquad (11)$$

whose mask is given by

$$\alpha_1=\frac{1}{32}\{-1,0,68,256,378,256,68,0,-1\},$$

and we obtain the following scheme H_1

$$f_{2i}^{k+1}=\frac{1}{512}(0f_{i-2}^{k}+256f_{i-1}^{k}+256f_{i}^{k}+0f_{i+1}^{k}+0f_{i+2}^{k}),$$
$$f_{2i+1}^{k+1}=\frac{1}{512}(-1f_{i-2}^{k}+68f_{i-1}^{k}+378f_{i}^{k}+68f_{i+1}^{k}-1f_{i+2}^{k}). \qquad (12)$$

2.1.3 6-Point Binary Approximating Subdivision Scheme H2

By substituting $l=2$ in (2.8), we get the Laurent polynomial of scheme H2 as follows

$$P_2(z)=\frac{1}{8192}(1-9z-77z^2+357z^3+2538z^4 +5382z^5+5382z^6+2538z^7 +357z^8-77z^9-9z^{10}+z^{11}), \quad (13)$$

whose mask is given by

$$\alpha_2=\frac{1}{8192}\{1,-9,-77,357,2538,5382, 5382,2538,357,-77,-9,1\},$$

and we obtain the following scheme H2

$$f_{2i}^{k+1}=\frac{1}{8192}(-9f_{i-2}^k+357f_{i-1}^k+5382f_i^k +2538f_{i+1}^k-77f_{i+2}^k+f_{i+3}^k), \quad (14)$$

$$f_{2i+1}^{k+1}=\frac{1}{8192}(f_{i-2}^k-77f_{i-1}^k+2538f_i^k +5382f_{i+1}^k+357f_{i+2}^k-9f_{i+3}^k). \quad 4)$$

2.1.4 8-Point Binary Approximating Subdivision Scheme H3

By substituting $l=3$ in (2.8), we get the Laurent polynomial of scheme H2 as follows

$$P_3(z)=\frac{1}{131072}(-1+18z+5z^2-1132z^3-9z^4 +20750z^5+65541z^6+91800z^7 +65541z^8+20750z^9-9z^{10} -1132z^{11}+5z^{12}+18z^{13}-z^{14}), \quad (15)$$

whose mask is given by

$$\alpha_3=\frac{1}{131072}\{-1,18,5,-1132,-9,20750, 65541,91800,65541,20750, -9,-1132,5,18,-1\},$$

and we obtain the following scheme H3

$$f_{2i}^{k+1}=\frac{1}{131072}(18f_{i-4}^k-1132f_{i-3}^k+20750f_{i-2}^k +91800f_{i-1}^k+20750f_i^k-1132f_{i+1}^k +18f_{i+2}^k+0f_{i+3}^k), \quad (2.16) \quad (16)$$

$$f_{2i+1}^{k+1}=\frac{1}{131072}(-1f_{i-4}^k+5f_{i-3}^k-9f_{i-2}^k+65541f_{i-1}^k +65541f_i^k-9f_{i+1}^k+5f_{i+2}^k-f_{i+3}^k).$$

Remark 2.1. Support of basic limit function:

If δ be the initial data such that $\delta_0=1$ for $i=0$ and $\delta_i=1$ for $i\neq 0$ so by applying the convergent subdivision scheme H_l on this data, we get basic limit function $\phi_l=H_l^\infty\delta$ of H_l scheme.

Since the number of non-zero coefficients in the Laurent polynomials of $\beta(z)$ and $\alpha_{even}(z)$ are 6 and 4 respectively then the support of basic limit functions of the schemes corresponding to the polynomial $\beta(z)$ and $\alpha_{even}(z)$ are 5 and 3 respectively. As we know that the Laurent polynomial of the scheme can be obtain by applying $\alpha_{even}(z)$, l times on $\beta(z)$ therefore the support of basic limit function of the scheme with Laurent polynomial $P_l(z)$ *is* $5l+3$.

3.0 POLYNOMIAL GENERATION ANDREPRODUCTION OFSCHEMES

Here we discuss degree of polynomial generation and reproduction of H_l schemes.

3.1 Polynomial Generation Of H_l Schemes

Polynomial generation of degree *n* is the ability of subdivision scheme to generate the full space of polynomials of up to *n*.

Theorem 3.1. Degree of polynomial generation of H_l schemes is l + 2.

Proof. Since Laurent polynomial of general member H_l is given by

$$P_l(z)=\left(\frac{1+z}{2}\right)^{l+3}\left(\frac{-1+10z-z^2}{8}\right)^l\left(\frac{1+6z+z^2}{4}\right)$$

Since number of common factors is $l+3$, so by [4], degree of polynomial generation of H_l schemes is $l+2$.

3.2 Polynomial Reproduction Of H_l Schemes

Here we use the algebraic condition (14) and Lemma 4.2 of [4] on the symbol of H-schemes to find the degree of polynomial generation and reproduction.

Theorem 3.2. The binary scheme H_l reproduces linear polynomial if

$$P_l^k(1) = 2\prod_{j=0}^{k-1}(\tau_l - j) \ \ and \ \ P_l^k(-1) = 0, \ \ k = 0,1,$$ where

$$\tau_l = \frac{P_l^1(1)}{2}$$ is parametric shift.

Proof. By differentiating (8), we have

$$P_l^1(1) = \frac{(1+z)^{l+2}\left(-(3l+5)z^4 + 24z^3 + (114l+246)z^2 + (72l+64)z + 9l - 9\right)}{2^{4l+5}\left(-1+10z-z^2\right)^{1-l}}.$$

It is easy to see that $P_l^k(-1) = 0, \ k = 0,1,$. Now from (8), we get $P_l^0(1) = P_l(1) = 2$, also $2\prod_{j=0}^{-1}(\tau_l - j) = 2$ so this implies $P_l^0(1) = 2\prod_{j=0}^{-1}(\tau_l - j)$.

Similarly for $k = 1$, we have $P_l^1(1) = 2\prod_{j=0}^{0}(\tau_l - j)$, which completes the proof.

4.0 CONTINUITY AND HÖLDER REGULARITY ANALYSIS OF SUBDIVISION SCHEMES

In this section, we present the continuity and Hölder regularity analysis of subdivision schemes H_l.

4.1 Continuity Analysis Of Subdivision Schemes

We present the continuity analysis of subdivision schemes H_l by using method of [7].

Theorem 4.1. The 3-point binary subdivision scheme H0 is C2 continuous.

Proof. Since Laurent polynomial (9) of the scheme H0 is given by

$$P_0(z) = \left(\frac{1+z}{2}\right)^3 b(z),$$

where $b(z) = \frac{1}{4}(1 + z + z^2)$.

Let S_b be the scheme corresponding to the symbol $b(z)$. Since

$$\left\| \frac{1}{2}S_b \right\|_\infty = \frac{1}{2}\max\left\{ \sum_{j\in\mathbb{Z}} |b_{2j}|, \sum_{j\in\mathbb{Z}} |b_{2j+1}| \right\},$$

then, we have

$$\left\| \frac{1}{2}S_b \right\|_\infty = \frac{1}{2}\max\left\{ \frac{2}{4}, \frac{6}{4} \right\} = \frac{3}{4} < 1.$$

Therefore by ([7], Corollary 4.11), the scheme H_0 is C^2.

Table 1 presents continuity of the scheme H_0 and some other members of the family

4.2 Hölder Regularityanalysis Of Subdivision Schemes

Hölder regularity is an extension of convergence and continuity. Hölder regularity analysis is done by using Rioul's [13] method.

Theorem 4.2. The lower bound and the upper bound on the Hölder regularity of the scheme H_0 is 2.4150.

Proof. The Laurent polynomial (9) of the scheme H_0 can be written as

$$P_0(z) = \left(\frac{1+z}{2}\right)^3 b(z),$$

where $b(z) = \frac{1}{4}(1 + z + z^2)$. (4.1)

From (4.1) $d_0 = \frac{1}{4}, d_1 = \frac{6}{4}, d_2 = \frac{1}{4}$, (i.e. non-zero coefficients of z in $b(z)$], $m = 3$ (i.e. number of factors in $P_0(z)$), $q = 2$ (i.e. number of non-zero coefficients of z in $b(z)$, start counting from 0). The matrices D_0 and D_1 can be computed by using the relations

$$(D_0)_{ij} = d_{q+i-2j}, \quad and$$

$$(D_1)_{ij} = d_{q+i-2j+1}, \quad for \ \ i,j = 1,\ldots,q$$

Thus D_0 and D_1 are given by

$$D_0 = \frac{1}{4}\begin{pmatrix} 6 & 0 \\ 1 & 1 \end{pmatrix} \quad and \quad D_1 = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix}.$$

As

$$max\{\rho(D_0), \rho(D_1)\} \le \mu \le max\{\|D_0\|_\infty, \|D_1\|_\infty\},$$

which implies

$$max(1.5, 1.5) \le \mu \le max(1.5, 1.5).$$

So lower bound on the Hölder regularity is given by $3 - log_2(1.5) = 2.147$ and also upper bound on Hölder regularity is given by $3 - log_2(1.5) = 2.147$.

Table 1 shows the continuity of other members of H_l schemes. From this table we conclude that as we increase parameter *l*, level of continuity and Hölder continuity of H_l schemes go up steadily.

Table 1 *Comparison of continuity analysis of the H_l schemes.*

l	Continuity	Hölder continuity	
		Lower bound	Upper bound
0	2	2.4150	2.4150
1	3	3.0458	3.1457
2	3	3.7711	3.8381
3	4	4.4483	4.5026

5.0 LIMIT STENCIL AND ARTIFACT ANALYSIS OF SUBDIVISION SCHEMES

In this section, we present limit stencil and artifact analysis of some of the proposed schemes.

5.1 Limit Stencils Of Subdivision Schemes

A stencil which gives a point on the limit curve in the form of the original control points is called limit stencil. The limit stencil evaluate points on the limit curve itself with a relatively small number of calculations. We obtain limit stencil by using

$$p^{\infty} = B\left(\lim_{j\to\infty} D^j\right)B^{-1}p^0,$$

Where

$$\lim_{j\to\infty} D^j = D^{\infty}, \text{ so } p^{\infty} = BD^{\infty}B^{-1}p^0, \qquad (17)$$

Where *B* is the matrix of eigenvectors corresponding to eigenvalues and *D* is diagonal matrix of eigenvalues of subdivision matrix of the scheme.

Theorem 5.1. Limit stencil of 3-point binary approximating subdivision scheme H_0 is L_0 = (0.0209,0.4791,0.4791,0.0209).

Proof. By the Laurent polynomial (9), the subdivision matrix of scheme H_0 is given by

$$A_0 = \begin{pmatrix} 0.2813 & 0.6875 & 0.0313 & 0 & 0 \\ 0.0313 & 0.6875 & 0.2813 & 0 & 0 \\ 0 & 0.2813 & 0.6875 & 0.0313 & 0 \\ 0 & 0.0313 & 0.6875 & 0.2813 & 0 \\ 1 & 0 & 0.2813 & 0.6875 & 0.0313 \end{pmatrix}.$$

Eigenvalues of A_0 are

$$\lambda_0 = 1,\ \lambda_1 = 0.5001,\ \lambda_2 = 0.2500,$$
$$\lambda_3 = 0.1874,\ \lambda_4 = 0.0313.$$

The matrix of eigenvectors corresponding to the above eigenvalues is

$$B_0 = \begin{pmatrix} 0 & 0.4472 & 0.4472 & -0.2280 & 0.2944 \\ 0 & 0.4472 & 0.1491 & 0.0326 & -0.0128 \\ 0 & 0.4472 & -0.1491 & -0.0326 & -0.0128 \\ 0 & 0.4472 & -0.4472 & 0.2280 & 0.2944 \\ 1 & 0.4472 & -0.7453 & 0.9454 & 0.9090 \end{pmatrix}.$$

We can define the diagonal matrix D_0 as

$$D_0 = \begin{pmatrix} 0.0313 & 0 & 0 & 0 & 0 \\ 0 & 1.000 & 0 & 0 & 0 \\ 0 & 0 & 0.5001 & 0 & 0 \\ 0 & 0 & 0 & 0.1874 & 0 \\ 0 & 0 & 0 & 0 & 0.2500 \end{pmatrix}.$$

By diagonalization of matrix A_0, we get $A_0 = B_0 D_0 B_0^{-1}$ where

$$B_0^{-1} = \begin{pmatrix} 0.2005 & -1.6014 & 3.6019 & -3.2010 & 1 \\ 0.0467 & 1.0714 & 1.0714 & 0.0467 & 0 \\ 0.3357 & 2.3465 & -2.3465 & -0.3357 & 0 \\ -1.5342 & 4.6017 & -4.6017 & 1.5342 & 0 \\ 1.6274 & -1.6274 & -1.6274 & 1.6274 & 0 \end{pmatrix}.$$

By substituting values in (5.1), we have

$$\begin{pmatrix} p_{-2}^{\infty} \\ p_{-1}^{\infty} \\ p_0^{\infty} \\ p_1^{\infty} \\ p_2^{\infty} \end{pmatrix} = \begin{pmatrix} 0.0209 & 0.4791 & 0.4791 & 0.0209 & 0 \\ 0.0209 & 0.4791 & 0.4791 & 0.0209 & 0 \\ 0.0209 & 0.4791 & 0.4791 & 0.0209 & 0 \\ 0.0209 & 0.4791 & 0.4791 & 0.0209 & 0 \\ 0.0209 & 0.4791 & 0.4791 & 0.0209 & 0 \end{pmatrix} \begin{pmatrix} p_{-2}^{0} \\ p_{-1}^{0} \\ p_0^{0} \\ p_1^{0} \\ p_2^{0} \end{pmatrix}.$$

Thus the limit stencil of 3-point binary approximating subdivision scheme H_0 is given by

$$L_0 = (0.0209, 0.4791, 0.4791, 0.0209). \qquad (5.2)$$ (18)

Similarly, we can find out limit stencil of other H_l-schemes for $l = 1, 2..$. In Table 2, limit stencils of some

Similarly, we can find out limit stencil of other H_l-schemes for $l = 1, 2..$. In Table 2, limit stencils of some of the proposed schemes are presented.

Table 2 Limit stencils of H_l schemes for $l = 0, 1, 2$ *and* 3

L	Limit stencils
0	L_0 = (0.0209,0.4791,0.4791,0.0209)
1	L_1 = (−0.0015,0.1726,0.6578,0.1726,−0.0015)
2	L_2 = (−0.0045,0.1426,0.7237,0.1426,−0.0045)
3	L_3 = (0.0001,−0.0064,0.1909,0.6306,0.1909,−0.0064,0.0001)

5.2 Artifact Analysis Of Subdivision Schemes

In this section, we discuss the unwanted features presented in the limit curve that cannot be removed by the movement of initial control points. These features are called artifact.

Theorem 5.2. The amount of artifact presented in the limit curve generated by the scheme denoted by H_0 is

$$M_0(\upsilon) = 0.0832\zeta^8 + 0.5\zeta^6 + 0.4168\zeta^4,$$

where $\zeta = \sin\left(\frac{\pi\upsilon}{2}\right)$, $\upsilon = \frac{1}{n}$ and n represents the initial number of control points of the polygon.

Proof. The Laurent polynomial of limit stencil L_0 can be written as

$$L_0(z) = 0.0208 + 0.4792z + 0.4792z^2 + 0.0208z^3. \qquad (19)$$

After multiplying Laurent polynomial $P_0(z)$ of scheme H_0 and (19), we get

$$P_0(z)L_0(z) = \frac{1}{2^5}(0.0208 + 0.6664z + 5.2496z^2 + 15.3336z^3 + 21.4592z^4 + 15.3336z^5 + 5.2496z^6 + 0.6664z^7 + 0.0208).$$

Ths implies

$$P_0(z)L_0(z) = \frac{1}{2^5}(0.0208(1+z)^8 + 0.5(1+z)^6 z + 1.6672\,(1+z)^4 z^2). \qquad (20)$$

For symmetrized version of (20), we multiply (20) by z^{-4} and get

$$\hat{P}(z) = \frac{1}{2^5}(0.0208\frac{(1+z)^8}{z^4} + 0.5\frac{(1+z)^6}{z^4}z + 1.6672\,\frac{(1+z)^4}{z^4}z^2).$$

The above expression can be written as

$$\hat{P}(z) = \frac{1}{2^5}\left\{ 0.0208\left(\frac{1+z}{z^{\frac{1}{2}}}\right)^8 + 0.5\left(\frac{1+z}{z^{\frac{1}{2}}}\right)^6 + 1.6672\left(\frac{1+z}{z^{\frac{1}{2}}}\right)^4 \right\},$$

which implies that

$$\hat{P}(z) = (2)^3(0.0208)\left(\frac{1+z}{2z^{\frac{1}{2}}}\right)^8 + (2)0.5\left(\frac{1+z}{2z^{\frac{1}{2}}}\right)^6 + (2)^{-1}(1.6672)\left(\frac{1+z}{2z^{\frac{1}{2}}}\right)^4,$$

By writing above expression as polynomial in $\gamma = \frac{1+z}{2z^{1/2}}$,

$$G_0(\gamma) = 0.1664\gamma^8 + \gamma^6 + 0.8336\gamma^4. \qquad (21)$$

Thus magnitude of artifact in the limit curve of scheme H_0 is given by

$$M_0(\upsilon) = 0.0832\zeta^8 + 0.5\zeta^6 + 0.4168\zeta^4,$$

where $\zeta = \sin\left(\frac{\pi\upsilon}{2}\right)$, $\upsilon = \frac{1}{n}$.

In the same way we can prove the following theorems.

Theorem 5.3. The amount of artifact presented in the limit curve generated by the scheme denoted by H_1 is

$$M_1(\upsilon) = 0.0060\zeta^{12} - 0.1960\zeta^{10} + 0.2619\zeta^8 + 0.6903\zeta^6 - 0.2318\zeta^4$$

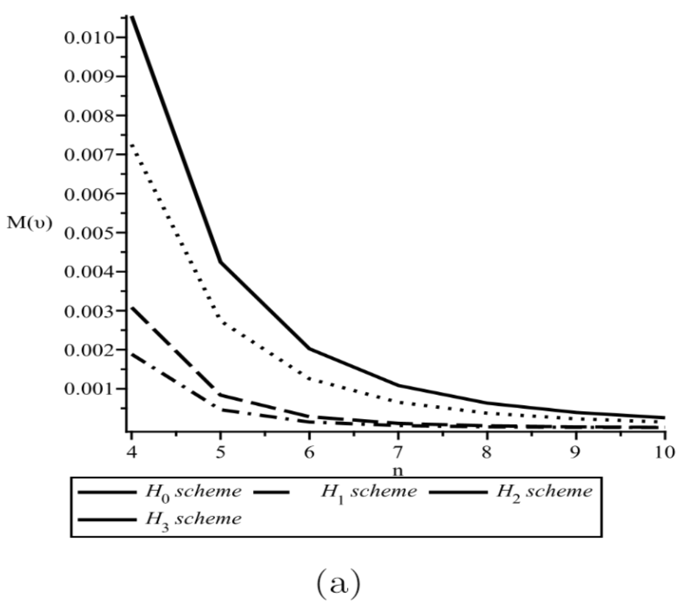
Theorem 5.4. The amount of artifact presented in the limit curve generated by the scheme denoted by H_2 is

$$M_2(\upsilon) = -0.0028\zeta^{16} + 0.0404\zeta^{14} - 0.3090\zeta^{12} - 0.4530\zeta^{10} + 0.5418\zeta^8 + 0.9126\zeta^6.$$

Theorem 5.5. The amount of artifact presented in the limit curve generated by the scheme denoted by H_3 is

$$M_3(\upsilon) = -0.0004\zeta^{20} + 0.0102\zeta^{18} - 0.1176\zeta^{16} + 0.4560\zeta^{14} - 0.8496\zeta^{12} - 0.4543\zeta^{10} + 1.4674\zeta^8 + 0.3979\zeta^6 - 0.0000019226\zeta^4 + 0.00000016475\zeta^2 - 0.0000000022885.$$

Similarly, we can compute artifact presented in the other members of family of schemes. In Figure 1(a), the magnitudes of artifact against the number of control points n in the limit curve of the schemes H_0, H_1, H_2 and H_3 are plotted. The graph of figure shows that the magnitude of artifact decreases by increasing the number of initial control points while it decreases for increasing l with fixed number of initial control points.



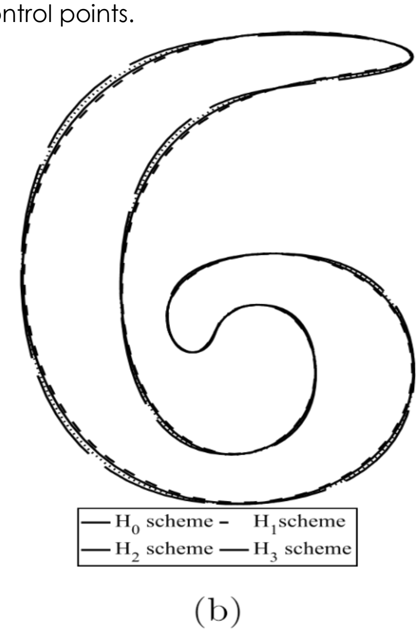
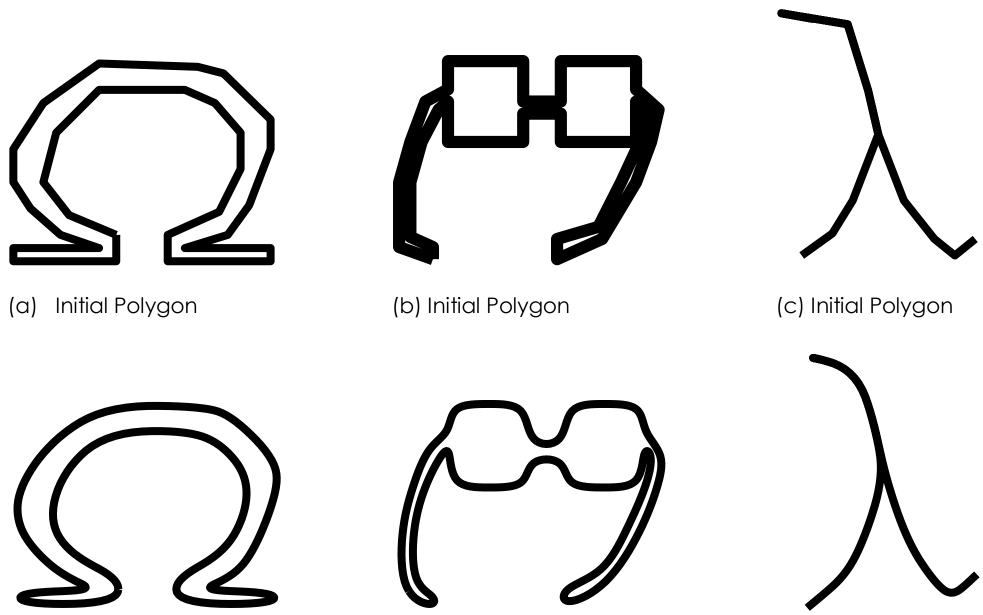


Figure 1 (a) presents magnitudes of artifact in the limit curves of the schemes H_0, H_1, H_2 and H_3 and (b) presents limit curves generated by the schemes H_0, H_1, H_2 and H_3.

6.0 APPLICATION AND SUMMARY

In this section, we present brief summary of work done so far. Comparison between limit curves produced by the schemes H_l for $l = 0, 1, 2$ and 3 are shown in Figure 1 (b). Figure 2 (a-c) show the initial control polygon and Figure 2 (d-f) are the limit curves obtained by H_0 and H_1 schemes at third subdivision level.

In this paper a class of binary subdivision schemes is offered with the help of two binary schemes. A parameter "l" is used to classify members of the proposed family. It is proved that each member of the proposed family has linear polynomial reproduction. It is also shown that continuity and Hölder regularity of proposed schemes increase gradually as we increase parameter l while magnitude of artifacts presented in the limit curve decreases. Furthermore, limit stencil analysis is done. Applications of proposed schemes are shown through several example curves.



(a) Initial Polygon (b) Initial Polygon (c) Initial Polygon

(d) H_0 scheme (e) H_1 scheme (f) H_0 scheme

Figure 2 (a-c) present initial control polygon and (d-f) are the limit curves obtained by H_0 and H_1 schemes at third subdivision level.

Acknowledgement

This work is supported by NRPU (P. No. 3183) Pakistan.

References

[1] Ashraf, P., Mustafa, G., & Deng, J. 2014. A Six-Point Variant On The Lane-Riesenfeld Algorithm. *Journal of Applied Mathematics. 1-7*

[2] Cashman, T. J., Hormann, K., & Reif, U. 2013. Generalized Lane–Riesenfeld Algorithms. *Computer Aided Geometric Design.* 30(4): 398-409.

[3] Chaikin, G. M. 1974. An Algorithm For High-Speed Curve Generation. *Computer Graphics And Image Processing*. 3(4): 346-349.

[4] Conti, C., & Hormann, K. 2011. Polynomial Reproduction For Univariate Subdivision Schemes Of Any Arity. *Journal of Approximation Theory*. 163(4): 413-437.

[5] Conti, C., & Romani, L. 2013. Dual Univariate M-Ary Subdivision Schemes Of De Rham-Type. *Journal of Mathematical Analysis and Applications.* 407(2): 443-456.

[6] Poret, S., Jony, R. D. and Gregory, S. 2009. Image Processing for Color Blindness Correction. *IEEE Toronto International Conference*. 1–6.

[7] Deslauriers, G., & Dubuc, S. 1989. Symmetric Iterative Interpolation Processes. *Constructive Approximation*. 49-68.

[8] Dyn, N., & Levin, D. 2002. Subdivision Schemes In Geometric Modelling. *Acta Numerica.* 11: 73-144.

[9] Dubuc, S. 2011. De Rham Transforms For Subdivision Schemes. *Journal of Approximation Theory*. 163(8): 966-987.

[10] Hormann, K., & Sabin, M. A. 2008. A Family Of Subdivision Schemes With Cubic Precision. *Computer Aided Geometric Design*. 25(1): 41-52.

[11] Lane, J. M., & Riesenfeld, R. F. 1980. A Theoretical Development For The Computer Generation And Display Of Piecewise Polynomial Surfaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 2(1): 35-46.

[12] Mustafa, G., Ashraf, P., & Aslam, M. 2014. Binary Univariate Dual And Primal Subdivision Schemes. *SeMA Journal*. 65(1): 23-35.

[13] De Rham, G. 1947. Un Peude Mathematiques A Proposed Une Courbe Plane. Revwe De Mathematiques Elementry II, Oevred Completes. 2: 678-689.

[14] Rioul, O. 1992. Simple Regularity Criteria For Subdivision Schemes. *SIAM Journal on Mathematical Analysis*. 23(6): 1544-1576.

[15] Shannon, C. and weaver, w. 1949. The Mathematical Theory Of Communication, University Of Illinois Press.

[16] Sabin, M. A., Augsdörfer, U. H., & Dodgson, N. A. 2005. Artifacts In Box-Spline Surfaces. In Mathematics Of Surfaces XI. 350-363. *Springer Berlin Heidelberg*.

[17] Siddiqi, S. S., & Ahmad, N. 2007. A New Three-Point Approximating C2 Subdivision Scheme. *Applied Mathematics Letters.* 20(6): 707-711.