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AN EFFECTIVE NUMERICAL METHOD FOR SOLVING THE NONLINEAR SINGULAR LANE-EMDEN TYPE EQUATIONS OF VARIOUS ORDERS

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Abstract

The Lane-Emden type equations are employed in the modeling of several phenomena in the areas of mathematical physics and astrophysics. These equations are categorized as non-linear singular ordinary differential equations on the semi-infinite domain. In this paper, the generalized fractional order of the Chebyshev orthogonal functions (GFCFs) of the first kind have been introduced as a new basis for Spectral methods, and also presented an effective numerical method based on the GFCFs and the collocation method for solving the nonlinear singular Lane-Emden type equations of various orders. Obtained results have compared with other results to verify the accuracy and efficiency of the presented method.

Keywords: Fractional order of the Chebyshev functions, Lane-Emden type equations, Isothermal gas sphere equation, Collocation method, Nonlinear ODE

1.0 INTRODUCTION

In this section, some basic definitions and theorems which are useful for our method have been introduced [1].

Definition 1. For any real function $f(t)$, $t > 0$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, is said to be in space C_μ, $\mu \in \Re$, and it is in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$.

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense by the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as [2, 3, 54]:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D^m f(s) ds, \quad \alpha > 0,$$

for $m - 1 < \alpha \le m, m \in N, t > 0$ and $f \in C_{-1}^m$.

Some properties of the operator D^α are as follows. For $f \in C_\mu,\ \mu \ge -1,\ \alpha, \beta \ge 0,\ \gamma \ge -1,\ N_0 = \{0,1,2,\dots\}$ and constant C:

$$(i) D^\alpha C = 0,$$
$$(ii) D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t),$$

$$(iii)\ D^\alpha t^\gamma = \begin{cases} 0 & \gamma \in N_0 \ and\ \gamma < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & otherwise. \end{cases}$$

$$(iv)\ D^\alpha \left(\sum_{i=1}^n c_i f_i(t) \right) = \sum_{i=1}^n c_i D^\alpha f_i(t), \qquad where\ c_i \in R.$$

Definition 3. Suppose that $f(t) \in C(0, \eta]$ and $w(t)$ is a weight function, then we define:

$$\| f(t) \|_w^2 = \int_0^\eta f^2(t) w(t) dt.$$

Theorem 1. (*Generalized Taylor's formula*) Suppose that $f(t) \in C[0, \eta]$ and $D^{k\alpha} f(t) \in C[0, \eta]$, where $k = 0,1,\dots,m$, $0 < \alpha \le 1$ and $\eta > 0$. Then we have

$$f(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} f(\xi), \qquad (1)$$

with $0 < \xi \le t, \forall t \in [0, \eta]$. And thus

$$|f(t) - \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+)| \le M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}, \quad (2)$$

where $M_\alpha \ge |D^{m\alpha} f(\xi)|$.

Proof: See Ref. [4].

In case of $\alpha = 1$, the generalized Taylor's formula in the Eq. (1) reduces to the classical Taylor's formula.

The organization of the paper is expressed as follows: in section 2, the methodology used is expressed. In section 3, the proposed method is applied to some types of Lane-Emden equations and then the obtained results are considered. Finally, a conclusion is provided.

2.0 METHODOLOGY

In this section, the mathematical Preliminaries for our method have been considered.

2.1 Mathematical Preliminaries on Lane-Emden Type Equations

In this section, the mathematical Preliminaries on Lane-Emden type equations of various orders have been expressed.

2.1.1 The Lane-Emden Type Equations of Second Order

The study of singular boundary value problems modeled by second-order nonlinear ordinary differential equations (ODEs) have attracted many mathematicians and physicists. One of the important equations in this category is the following Lane-Emden type equation:

$$y''(t) + \frac{k}{t} y'(t) + f(t, y(t)) = h(t), \qquad k, t > 0, \quad (3)$$

with the boundary conditions:

$$y(0) = y_0, \quad y'(0) = y_1, \quad (4)$$

where k, y_0 and y_1 are real constants, $f(t,y)$ and $h(t)$ are some given continuous real-valued functions. For special forms of $f(t,y)$, the well-known Lane-Emden equations occur in several models of non-Newtonian fluid mechanics, mathematical physics, astrophysics, etc. For example, when $f(t,y) = q(y)$, the Lane-Emden equations occur in modeling several phenomena in mathematical physics and astrophysics, such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [5, 6].

The Eq. (3) can be written as follows [7]:

$$t^{-2} \frac{d}{dt}\left(t^2 \frac{dy}{dt}\right) + f(t, y(t)) = h(t), \qquad k, t > 0. \quad (5)$$

2.1.2 The Lane-Emden Type Equations of Third and Fourth Orders

According to Eq. (5), in general, we can achieve: $t^{-k} \frac{d}{dt}\left(t^k \frac{d}{dt}\right) y + f(t, y(t)) = h(t)$,

$$y(0) = y_0, y'(0) = y_1, \quad (6)$$

where k is called the shape factor.

To consider the Lane-Emden type equations of higher orders, the Eq. (6) is used as follows [8, 9]:

$$t^{-k} \frac{d^m}{dt^m}\left(t^k \frac{d^n}{d^n t}\right) y + f(t, y(t)) = h(t). \quad (7)$$

1. To determine third-order equations, it is obvious that:

$$m + n = 3, \qquad m, n \ge 1,$$

namely $\{m = 2, n = 1\}$, or $\{m = 1, n = 2\}$. Therefore

(a) For m=2, n=1:

$$y''' + \frac{2k}{t} y'' + \frac{k(k-1)}{t^2} y' + f(t, y(t)) = h(t), \quad (8)$$
$$y(0) = y_0, y'(0) = y''(0) = 0.$$

(b) For m=1, n=2:

$$y''' + \frac{k}{t} y'' + f(t, y(t)) = h(t), \quad (9)$$
$$y(0) = y_0, y'(0) = y''(0) = 0.$$

2. To determine fourth-order equations, it is obvious that:

$$m + n = 4, \qquad m, n \ge 1,$$

namely $\{m = 3, n = 1\}$, $\{m = 2, n = 2\}$ or $\{m = 1, n = 3\}$. Therefore

(a) For m=3, n=1:

$$y^{(4)} + \frac{3k}{t} y''' + \frac{3k(k-1)}{t^2} y'' + \frac{k(k-1)(k-2)}{t^3} y' + f(t, y(t)) = h(t), \quad (10)$$
$$y(0) = y_0, y'(0) = y''(0) = y'''(0) = 0,$$

(b) For m=2, n=2:

$$y^{(4)} + \frac{2k}{t} y''' + \frac{k(k-1)}{t^2} y'' + f(t, y(t)) = h(t), \quad (11)$$
$$y(0) = y_0, y'(0) = y_1, y''(0) = y'''(0) = 0,$$

(c) For m=1, n=3:

$$y^{(4)} + \frac{k}{t} y''' + f(t, y(t)) = h(t), \quad (12)$$
$$y(0) = y_0, y'(0) = y_1, y''(0) = y_2, y'''(0) = 0,$$

Recently, some researchers obtained approximations for Lane-Emden equations, for example, Wazwaz [8, 9, 10] by using ADM, Chowdhury and Hashim [11], Bataineh *et al*. [12], Singh *et al*. [13], Van Gorder [14] by using HPM, Yildirim and Ozis [15] and Dehghan and Shakeri [16] by using VIM, Boubaker and Van Gorder [17] by using Boubaker polynomials expansion scheme, Marzban *et al*. [18] by using hybrid functions, Parand *et al*. in [19] by using a Hermite functions collocation method, in [20] by using Bessel orthogonal functions collocation method, in [21] by using Rational Chebyshev functions of the second kind collocation method, Hosseini and Abbasbandy [55] by using combination of the Spectral Method and Adomian Decomposition Method, and Azarnavid *et al*. [56] by using Picard-Reproducing Kernel Hilbert Space Method, and other methods [22, 23, 24, 25, 26, 27, 28, 29, 30].

2.2 Generalized Fractional order of the Chebyshev Functions (GFCFs)

In this section, first, the GFCFs have been introduced, and then some properties and convergence of them for our method have been expressed.

2.2.1 The Chebyshev Functions

The Chebyshev polynomials have many properties, for example orthogonal, recursive, simple real roots, complete in the space of polynomials. For these reasons, many researchers have employed these polynomials in their research [31, 32, 33, 34, 35, 36].

The number of researchers using some transformations extended Chebyshev polynomials to various domains, for example by using $x=\frac{t-L}{t+L},\ L>0$ the rational Chebyshev functions on semi-infinite domain [37, 38, 39, 40, 41, 42, 43, 44, 45], by using $x=\frac{t}{\sqrt{t^2+L}}, L>0$ the rational Chebyshev functions on infinite domain [46], and by using $x=1-2(\frac{t}{\eta})^{\alpha},\ \eta,\alpha>0$ the generalized fractional order of the Chebyshev functions (GFCF) of the first kind on the domain $[0,\eta]$ [47] are introduced.

2.2.2 The GFCFs Definition

Using transformation $x=1-2(\frac{t}{\eta})^{\alpha}$, $\alpha,\eta>0$ on classical Chebyshev polynomials of the first kind, the GFCFs are defined in the interval $[0,\eta]$, and are denoted by ${}_{\eta}FT_n^{\alpha}(t)=T_n(1-2(\frac{t}{\eta})^{\alpha})$ [47].

The analytical form of ${}_{\eta}FT_n^{\alpha}(t)$ of degree $n\alpha$ given by

$$\begin{aligned}{}_{\eta}FT_n^{\alpha}(t)&=\sum_{k=0}^{n}(-1)^k\frac{n2^{2k}(n+k-1)!}{(n-k)!\,(2k)!}(\frac{t}{\eta})^{\alpha k}\\&=\sum_{k=0}^{n}\beta_{n,k,\eta,\alpha}\,t^{\alpha k},\quad t\in[0,\eta],\end{aligned}\tag{13}$$

where

$$\beta_{n,k,\eta,\alpha}=(-1)^k\frac{n2^{2k}(n+k-1)!}{(n-k)!(2k)!\eta^{\alpha k}}\quad and\quad \beta_{0,k,\eta,\alpha}=1.\tag{14}$$

The GFCFs are orthogonal with respect to the weight function $w(t)=\frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^{\alpha}-t^{\alpha}}}$ in the interval $[0,\eta]$ as follow:

$$\int_0^{\eta}{}_{\eta}FT_n^{\alpha}(t)\ {}_{\eta}FT_m^{\alpha}(t)w(t)dt=\frac{\pi}{2\alpha}c_n\delta_{mn}.\tag{15}$$

where δ_{mn} is the Kronecker delta, $c_0=2$, and $c_n=1$ for $n\geq 1$.

3.3 Approximation of Functions

Any function $y(t)$, $t\in[0,\eta]$, can be expanded as follows:

$$y(t)=\sum_{n=0}^{\infty}a_n\ {}_{\eta}FT_n^{\alpha}(t),$$

where using the property of orthogonality in the GFCFs:

$$a_n=\frac{2\alpha}{\pi c_n}\int_0^{\eta}{}_{\eta}FT_n^{\alpha}(t)y(t)w(t)dt,\qquad n=0,1,2,\cdots.$$

In practice, we have to use first m-terms GFCFs and approximate $y(t)$:

$$y(t)\simeq y_m(t)=\sum_{n=0}^{m-1}a_n\ {}_{\eta}FT_n^{\alpha}(t)=A^T\Phi(t),\tag{16}$$

with

$$A=[a_0,a_1,\cdots,a_{m-1}]^T,\tag{17}$$

$$\Phi(t)=[{}_{\eta}FT_0^{\alpha}(t),\ {}_{\eta}FT_1^{\alpha}(t),\cdots,\ {}_{\eta}FT_{m-1}^{\alpha}(t)]^T.\tag{18}$$

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 2. Suppose that $D^{k\alpha}f(t)\in C[0,\eta]$ for $k=0,1,\ldots,m$, and ${}_{\eta}F_m^{\alpha}$ is the subspace generated by $\{{}_{\eta}FT_0^{\alpha}(t),{}_{\eta}FT_1^{\alpha}(t),\ldots,{}_{\eta}FT_{m-1}^{\alpha}(t)\}$. If $f_m=A^T\Phi$ (in Eq. (16)) is the best approximation to $f(t)$ from ${}_{\eta}F_m^{\alpha}$, then the error bound is presented as follows

$$\| f(t)-f_m(t)\|_w\leq\frac{\eta^{m\alpha}M_{\alpha}}{2^m\Gamma(m\alpha+1)}\sqrt{\frac{\pi}{\alpha.m!}},$$

where $M_{\alpha}\geq|D^{m\alpha}f(t)|, t\in[0,\eta]$.

Proof. See Ref. [47].

Theorem 3. The generalized fractional order of the Chebyshev function ${}_{\eta}FT_n^{\alpha}(t)$, has precisely n real zeros on interval $(0,\eta)$ in the form

$$t_k=\eta\left(\frac{1-\cos(\frac{(2k-1)\pi}{2n})}{2}\right)^{\frac{1}{\alpha}},\qquad k=1,2,\ldots,n.$$

Moreover, $\frac{d}{dt}\ {}_{\eta}FT_n^{\alpha}(t)$ has precisely $n-1$ real zeros on interval $(0,\eta)$ in the following points:

$$t'_k=\eta\left(\frac{1-\cos(\frac{k\pi}{n})}{2}\right)^{\frac{1}{\alpha}},k=1,2,\ldots,n-1.$$

Proof. See Ref. [47].

2.3 Application of the Method

In this section, the GFCFs collocation method to solve some well-known Lane-Emden type equations of various orders for different values of $f(t,y)$, y_0, y_1, y_2 and k is applied.

1. The second-order Lane-Emden type equations

For satisfying the boundary conditions, we satisfy the conditions Eq. (4) by multiplying the operator Eq. (16) by t^2 and adding it to y_0 and $y_1 t$ as follows:

$$\widehat{y_m}(t) = y_0 + y_1 t + t^2 y_m(t). \tag{19}$$

Now, $\widehat{y_m}(t) = y_0$ and $\frac{d}{dt}\widehat{y_m}(t) = y_1$, when t tends to zero, so the conditions in the Eq. (4) are satisfied.

To apply the collocation method, we construct the residual function by substituting $\widehat{y_m}(t)$ in the Eq. (19) for $y(t)$ in Lane-Emden type Eq. (3):

$$Res(t) = \frac{d^2}{dt^2}\widehat{y_m}(t) + \frac{k}{t}\frac{d}{dt}\widehat{y_m}(t) + f(t, \widehat{y_m}(t)) - h(t). \tag{20}$$

2. The third-order Lane-Emden type equations

For satisfying the boundary conditions, we satisfy the conditions in the Eqs. (8) and (9) as follows:

$$\widehat{y_m}(t) = y_0 + t^3 y_m(t).$$

Now, $\widehat{y_m}(t) = y_0$ and $\frac{d}{dt}\widehat{y_m}(t) = \frac{d^2}{dt^2}\widehat{y_m}(t) = 0$, when t tends to zero.

We construct the residual functions:

(a) For m=2, n=1:

$$Res(t) = \widehat{y_m}''' + \frac{2k}{t}\widehat{y_m}'' + \frac{k(k-1)}{t^2}\widehat{y_m}' + f(t, \widehat{y_m}(t)) - h(t). \tag{21}$$

(b) For m=1, n=2:

$$Res(t) = \widehat{y_m}''' + \frac{k}{t}\widehat{y_m}'' + f(t, \widehat{y_m}(t)) - h(t). \tag{22}$$

3. The fourth-order Lane-Emden type equations

For satisfying the boundary conditions, we satisfy the conditions in the Eqs. (10), (11) and (12) as follows:

(a) For m=3, n=1: $\widehat{y_m}(t) = y_0 + t^4 y_m(t)$.

(b) For m=2, n=2: $\widehat{y_m}(t) = y_0 + y_1 t + t^4 y_m(t)$.

(c) For m=1, n=3: $\widehat{y_m}(t) = y_0 + y_1 t + \frac{y_2}{2}t^2 + t^4 y_m(t)$.

We construct the residual functions:

(a) For m=3, n=1:

$$Res(t) = \widehat{y_m}^{(4)} + \frac{3k}{t}\widehat{y_m}''' + \frac{3k(k-1)}{t^2}\widehat{y_m}'' + \frac{k(k-1)(k-2)}{t^3}\widehat{y_m}' + f(t, \widehat{y_m}(t)) - h(t). \tag{23}$$

(b) For m=2, n=2:

$$Res(t) = \widehat{y_m}^{(4)} + \frac{2k}{t}\widehat{y_m}''' + \frac{k(k-1)}{t^2}\widehat{y_m}'' + f(t, \widehat{y_m}(t)) - h(t). \tag{24}$$

(c) For m=1, n=3:

$$Res(t) = \widehat{y_m}^{(4)} + \frac{k}{t}\widehat{y_m}''' + f(t, \widehat{y_m}(t)) - h(t). \tag{25}$$

The equations to obtain the coefficient $\{a_i\}_{i=0}^{m-1}$ arise from equalizing $Res(t)$ to zero on m collocation points:

$$Res(t_i) = 0, \qquad i = 0,1,\dots,m-1. \tag{26}$$

In this study, the roots of the GFCFs in the interval $[0,\eta]$ (Theorem 3) are used as collocation points. By solving the obtained set of equations by a suitable method (e.g. Newton's method), we have the approximating function $\widehat{y_m}(t)$.

It is worthwhile to note that it is common to solve a system of nonlinear equations, is applying the Newton's method. The main difficulty with such a system is how we can choose an initial approximation to handle Newton's method. We have had reason to believe that the best way to discover the proper initial approximation (or initial approximations) is to solve the system analytically for the very small m (by means of symbolic software programs, such as Mathematica or Maple) and, then, we can find proper initial approximations, and particularly the multiplicity of solutions of such system. This action has been done by starting from proper initial approximations with the maximum number of ten iterations. In the present method, due to be added the fractional power, the order of complexity increases, but in many differential equations, accuracy of computations increases with m less.

And also consider that all of the computations have been done by Maple 2015.

3.0 RESULTS AND DISCUSSION

In this section, using the present methods, some nonlinear singular Lane-Emden type equations of various orders are solved and then the obtained results are considered.

3.1 Examples for the Second-order Lane-Emden Type Equations

Example 1 (The standard Lane-Emden equation): For $f(t,y) = y^M, k = 2, y_0 = 1$ and $y_1 = 0$, the Eq. (3) is the standard Lane-Emden equation, which was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamic [7, 19, 20, 51]:

$$y''(t) + \frac{2}{t}y'(t) + y^M(t) = 0, \qquad t > 0, \tag{27}$$
$$y(0) = 1, y'(0) = 0,$$

where $M \geq 0$ is a constant. For $= 0,\ 1$, and 5, the Eq. (27) has the exact solutions, respectively:

$$y(t) = 1 - \frac{1}{3!}t^2, \quad y(t) = \frac{sin(t)}{t}, \quad y(t) = \left(1 + \frac{t^2}{3}\right)^{-\frac{1}{2}}. \tag{28}$$

In other cases, there is not any exact analytical solution. Therefore, we apply the GFCF collocation method to solve the standard Lane-Emden Eq. (27), for $M = -0.5, 0.5, 1.5, 2, 2.5, 3,$ and 4.

We construct the residual function as follows:

$$Res(t) = \frac{d^2}{dt^2}\widehat{y_m}(t) + \frac{2}{t}\frac{d}{dt}\widehat{y_m}(t) + (\widehat{y_m}(t))^M. \quad (29)$$

Therefore, to obtain the coefficient $\{a_i\}_{i=0}^{m-1}$, $Res(t)$ is equalized to zero at m collocation point. By solving this set of nonlinear algebraic equations, we can find the approximating function $\widehat{y_m}(t)$.

Tables 1 - 8 show comparing the obtained solutions $y(t)$ by the present method and some well-known methods in other papers, for the standard Lane-Emden equations with $M = -0.5,\ 0.5,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5$ and 4 respectively. These tables also show the residual function $Res(t)$ in some points. Table 9 shows comparing the obtained zeros of the standard Lane-Emden equations by the present method and the values given by Horedt [7], Parand *et al.* [19] and Parand *et al.* [20] for $M = -0.5,\ 0.0,\ 0.5,\ 1.0,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5$, and 4. It is seen that using low number of points, the obtained results are very good compared to other methods, and for various values of M, the accurate results are calculated. Figure 1 shows the graphs of the standard Lane-Emden type equations for $M = -0.5$, $0.0,\ 0.5,\ 1.0,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5$ and 4.

Table 1 Obtained values of $y(t)$ for standard Lane-Emden $M = -0.5$ by the present method for example 1 ($m = 10, \alpha = 0.75$)

t	Horedt [7]	Present	Error	Res(t)
0.100	0.9983329	0.9983353	0.0000024	1.4360e-4
0.500	0.9580681	0.9579185	0.0001495	8.9803e-4
1.000	0.8288357	0.8288176	0.0000180	1.2076e-3
2.000	0.2320758	0.2318148	0.0002609	8.2943e-3
2.208	8.8001e-4	8.3136e-4	0.0000486	1.8312e-3

Table 2 Obtained values of $y(t)$ for standard Lane-Emden $M = 0.5$ by the present method for example 1 ($m = 15, \alpha = 0.75$)

t	Horedt [7]	Present	Error	Res(t)
0.10	0.9983338	0.9983338	7.0717e-9	7.4826e-6
0.50	0.9585943	0.9585942	2.4441e-8	3.2593e-5
1.00	0.8375345	0.8375352	7.4162e-7	1.2657e-4
2.00	0.4025795	0.4025796	1.4812e-7	1.4373e-4
2.70	2.6741e-2	2.6738e-2	2.7973e-6	9.3612e-3
2.75	1.3502e-3	1.3504e-3	1.4657e-7	6.5629e-3

Table 3 Obtained values of $y(t)$ for standard Lane-Emden $M = 1.5$ by the present method for example 1 ($m = 15, \alpha = 0.75$)

t	Horedt [7]	Present	Error	Res(t)
0.10	0.9983346	0.9983345	1.7679e-8	1.0538e-9
0.50	0.9591039	0.9591038	5.0043e-8	3.4754e-7
1.00	0.8451698	0.8451697	4.3996e-8	7.6167e-7
3.00	0.1588576	0.1588575	6.0018e-9	7.0922e-6
3.60	1.1090e-2	1.1091e-2	5.0493e-8	1.2344e-5
3.65	7.6392e-4	7.6393e-4	7.3142e-9	2.4625e-5

Table 4 Obtained values of $y(t)$ for standard Lane-Emden $M = 2$ by the present method for example 1 ($m = 15, \alpha = 0.75$)

t	Horedt [7]	Parand [20]	Present	Res(t)
0.10	0.9983350	0.99833499854	0.99833499986	1.0838e-7
0.50	0.9593527	0.95935271580	0.95935271585	3.1003e-7
1.00	0.8486541	0.84865411140	0.84865409603	6.6652e-7
3.00	0.2418241	0.24182408305	0.24182406641	1.7998e-6
4.00	4.88401e-2	4.88401499e-2	4.884014079e-2	2.6970e-6
4.30	6.81094e-3	6.81094327e-3	6.810947394e-3	2.7737e-6
4.35	3.66030e-4	3.66030179e-4	3.660339568e-4	2.3105e-6

Table 5 Obtained values of $y(t)$ for standard Lane-Emden $M = 2.5$ by the present method for example 1 ($m = 10, \alpha = 0.75$)

t	Horedt [7]	Parand [20]	Present	Res(t)
0.10	0.9983354	0.99833541418	0.99833503458	4.8147e-6
0.50	0.9595978	0.95959775446	0.95960162974	3.9807e-5
1.00	0.8519442	0.85194419912	0.85194342182	2.5267e-6
4.00	0.1376807	0.13768073303	0.13766004942	1.2495e-4
5.00	2.90191e-2	2.90191866e-2	2.902408137e-2	3.0697e-4
5.30	4.25954e-3	4.25954353e-3	4.258764232e-3	2.7243e-4
5.355	2.10089e-5	2.10089382e-5	2.100708657e-5	1.6482e-4

Table 6 Obtained values of $y(t)$ for standard Lane-Emden $M = 3$ by the present method for example 1 ($m = 10, \alpha = 0.75$)

t	Horedt [7]	Parand [20]	Present	Res(t)
0.10	0.9983358	0.99833582956	0.99833883172	1.372e-4
0.50	0.9598391	0.95983906994	0.95979086302	3.922e-4
1.00	0.8550576	0.85505756858	0.85506225670	2.129e-3
5.00	0.1108198	0.11081983504	0.11074136512	2.184e-3
6.00	4.37379e-2	4.37379838e-2	4.373437433e-2	1.415e-3
6.80	4.16778e-3	4.25954876e-3	4.171491113e-3	1.374e-3
6.896	3.60111e-5	3.60111453e-5	3.602801805e-5	2.177e-3

Table 7 Obtained values of $y(t)$ for standard Lane-Emden $M = 3.5$ by the present method for example 1 ($m = 8, \alpha = 0.90$)

t	Horedt [7]	Present	Error	Res(t)
0.10	0.9983362	0.9983391	0.0000029	6.7541e-4
0.50	0.9600768	0.9597179	0.0003588	1.5668e-3
1.00	0.8580096	0.8575136	0.0004959	1.8181e-5
5.00	0.1786843	0.1834231	0.0047388	2.7972e-3
9.00	1.18031e-2	1.20298e-2	0.0002267	3.9381e-3
9.50	7.47234e-4	7.30544e-4	0.0000166	9.2213e-4
9.53	1.20772e-4	1.18150e-4	0.0000026	1.1934e-3

Table 8 Obtained values of $y(t)$ for standard Lane-Emden $M = 4$ by the present method for example 1 ($m = 15, \alpha = 0.75$)

t	Horedt [7]	Parand [20]	Present	Res(t)
0.10	0.9983367	0.9985876	0.9983371	2.347e-5
0.50	0.9603109	0.9605160	0.9602977	7.612e-4
1.00	0.8608138	0.8610072	0.8608802	4.359e-4
5.00	0.2359227	0.2358368	0.2357450	9.300e-4
10.0	5.96727e-2	0.0596105	5.983709e-2	7.000e-4
14.0	8.33052e-3	0.0083058	8.360725e-3	3.053e-4
14.9	5.76418e-4	0.0005759	5.765300e-4	1.721e-4

Table 9 Obtained zeros of standard Lane-Emden equations, by the present method for several M for example 1

M	m	Horedt [7]	Parand [19]	Parand [20]	Present
-0.5	10	2.20858842	-	-	2.208588452
0.0	20	2.44948974	-	-	2.449489742
0.5	15	2.75269805	-	-	2.752698013
1.0	40	3.14159265	-	-	3.141592653
1.5	15	3.65375374	3.65375374	3.653753749	3.653753762
2.0	15	4.35287460	4.35287460	4.352874595	4.352874625
2.5	10	5.35527546	5.35527546	5.355275468	5.355275468
3.0	10	6.89684862	6.89684862	6.896848619	6.896848534
3.5	08	9.53580534	-	-	9.535805274
4.0	15	14.9715463	4.9715463	14.97154637	14.97154183

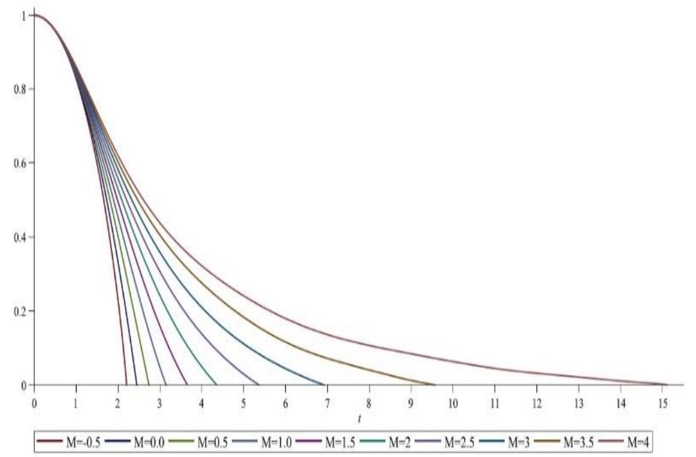


Figure 1 Obtained graphs of solutions of Lane-Emden standard equation, by the present method for several M for example 1

Example 2 (The isothermal gas spheres equation): For $f(t,y) = e^{y}, y_0 = 0$, and $y_1 = 0$, the Eq. (3) is the isothermal gas sphere equation [19]:

$$y''(t) + \frac{2}{t}y'(t) + e^{y(t)} = 0, \qquad t > 0, \qquad (30)$$
$$y(0) = 0, y'(0) = 0,$$

This model can be used to treat the isothermal gas sphere. For a thorough discussion of Eq. (30), see Davis [6], Van Gorder [14]. This equation has been solved by some researchers, for example Wazwaz [10] and Chowdhury and Hashim [11] by using ADM and HPM, respectively, Parand *et al*. [19] by using the Hermite collocation method, and Parand *et al*. [20] by using Bessel orthogonal functions collocation method. We construct the residual function as follows:

$$Res(t) = \frac{d^2}{dt^2}\widehat{y_m}(t) + \frac{2}{t}\frac{d}{dt}\widehat{y_m}(t) + e^{\widehat{y_m}(t)}.$$

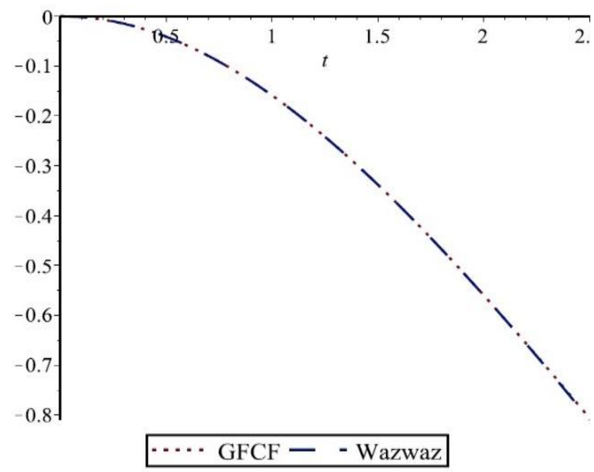
A series solution obtained by Wazwaz [10], Liao [52], Singh *et al*. [13] and Ramos [53] by using ADM, ADM, MHAM and series expansion respectively:

$$y(t) \simeq -\frac{1}{6}t^2 + \frac{1}{5.4!}t^4 - \frac{8}{21.6!}t^6 + \frac{122}{81.8!}t^8 - \frac{61.67}{495.10!}t^{10} + \cdots.$$

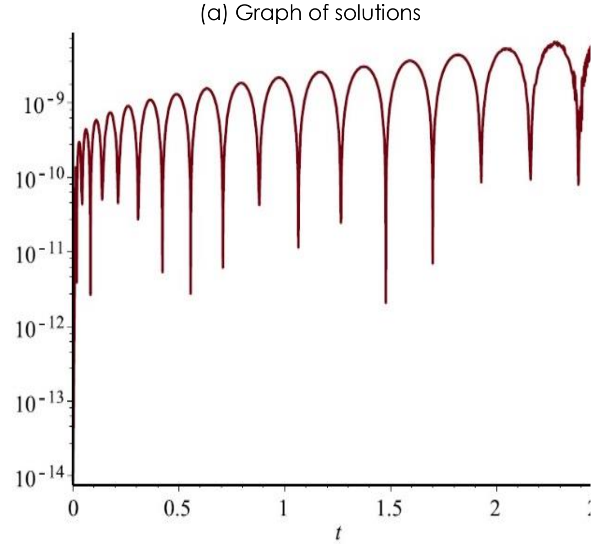
Tables 10 shows the comparison of $y(t)$ obtained by the present method and those obtained by Wazwaz [10] and Parand *et al.* [19] and [20]. The resulting graph of the isothermal gas spheres equation in comparison to the present method and those obtained by Wazwaz [10] and the Log graph of the residual error of approximate solution of the isothermal gas spheres equation are shown in Figure 2.

Table 10 Obtained values of $y(t)$ for the isothermal gas spheres equation for example 2 ($m = 30, \alpha = 0.75$)

t	Parand [19]	Parand [20]	Wazwaz [10]	Present	Res(t)
0.1	-0.0016664188	-0.0016658338	-0.0016658339	-0.0016658338	4.9e-10
0.2	-0.0066539713	-0.0066533671	-0.0066533671	-0.0066533671	4.1e-10
0.5	-0.0411545150	-0.0411539572	-0.0411539568	-0.0411539572	1.96e-9
1.0	-0.1588281737	-0.1588276775	-0.1588273537	-0.1588276775	1.96e-9
1.5	-0.3380198308	-0.3380194247	-0.3380131103	-0.3380194247	1.08e-9
2.0	-0.5598233120	-0.5598230043	-0.5599626601	-0.5598230043	4.25e-9
2.5	-0.8063410846	-0.8063408705	-0.8100196713	-0.8063408705	7.45e-9



(a) Graph of solutions



(b) Graph of the residual error

Figure 2 Graphs of solutions and the residual error of the isothermal gas spheres equation for example 2

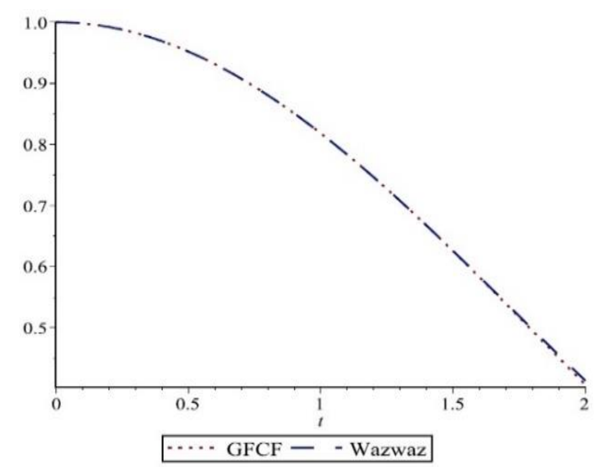
Example 3: For $f(t,y) = sinh(y), y_0 = 1$ and $y_1 = 0$, Eq. (3) will be one of the Lane-Emden type equations [19, 20]:

$$y''(t) + \frac{2}{t}y'(t) + sinh(y(t)) = 0, \qquad t > 0, \qquad (31)$$
$$y(0) = 1, y'(0) = 0,$$

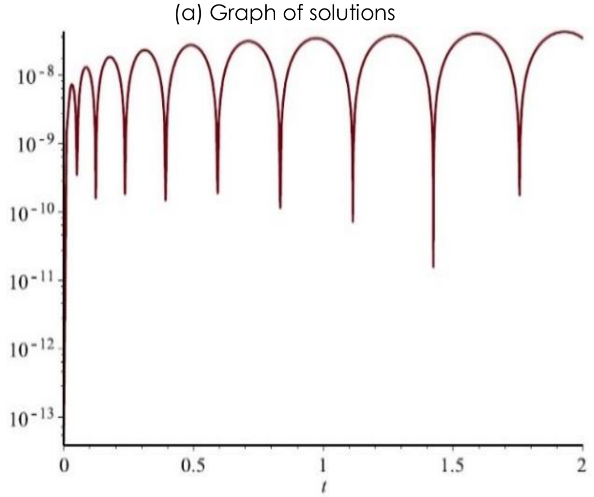
A series solution obtained by Wazwaz [10] by using Adomian Decomposition Method (ADM) is:

$$y(t) \simeq 1 - \frac{e^2-1}{12e}t^2 + \frac{1}{480}\frac{e^4-1}{e^2}t^4 - \frac{1}{30240}\frac{2e^6-3e^4+3e^2-2}{e^3}t^6 + \frac{1}{26127360}\frac{61e^8-104e^6+104e^2-61}{e^4}t^8 + \cdots.$$

Table 11 shows the comparison of $y(t)$ obtained by the present method and those obtained by Wazwaz [10] and Parand *et al*. [19]. The resulting graph of the Eq. (31) in comparison to the present method and those obtained by Wazwaz [10] and the Log graph of the residual error of approximate solution are shown in Figure 3. This graph shows that the present method has an appropriate convergence rate.



(a) Graph of solutions



(b) Graph of the residual error

Figure 3 Graphs of solutions and the residual error for example 3

Table 11 Obtained values of $y(t)$ for Lane-Emden equation by the present method for example 3 ($m = 20, \alpha = 0.75$)

t	Parand [19]	Wazwaz [10]	Present	Res(t)
0.1	0.9981138095	0.9980428414	0.9980428414	1.07e-8
0.2	0.9922758837	0.9921894348	0.9921894347	1.51e-8
0.5	0.9520376245	0.9519611019	0.9519610925	2.71e-8
1.0	0.8183047481	0.8182516669	0.8182429282	3.30e-8
1.5	0.6254886192	0.6258916077	0.6254387632	2.68e-8
2.0	0.4066479695	0.4136691039	0.4066228877	3.43e-8

Example 4: For $f(t,y) = sin(y), y_0 = 1$ and $y_1 = 0$, the Eq. (3) will be one of the Lane-Emden type equations that we want to solve [19, 20]:

$$y''(t) + \frac{2}{t}y'(t) + sin(y(t)) = 0, \qquad t > 0, \qquad (32)$$
$$y(0) = 1, y'(0) = 0,$$

A series solution obtained by Wazwaz [10] by using ADM is:

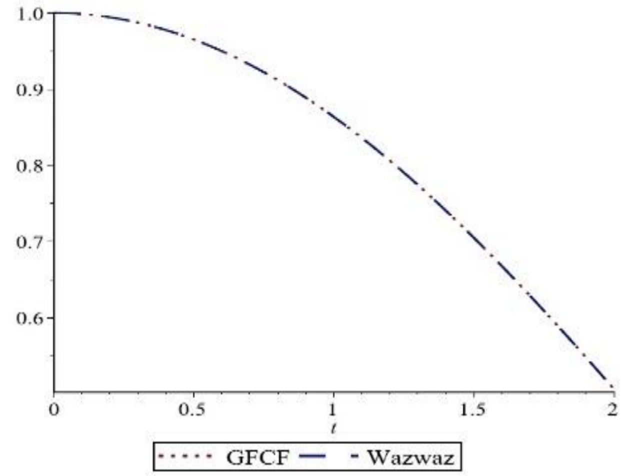
$$y(t) \simeq 1 - \frac{1}{6}k_1t^2 + \frac{1}{120}k_1k_2t^4 + k_1(\frac{1}{3024}k_1^2 - \frac{1}{5040}k_2^2)t^6 + k_1k_2(-\frac{113}{3265920}k_1^2 + \frac{1}{362880}k_2^2)t^8 + k_1(\frac{1781}{898128000}k_1^2k_2^2 - \frac{1}{399168000}k_2^4 - \frac{19}{23950080}k_1^4)t^{10} + \cdots,$$

where $k_1 = sin(1)$ and $k_2 = cos(1)$.

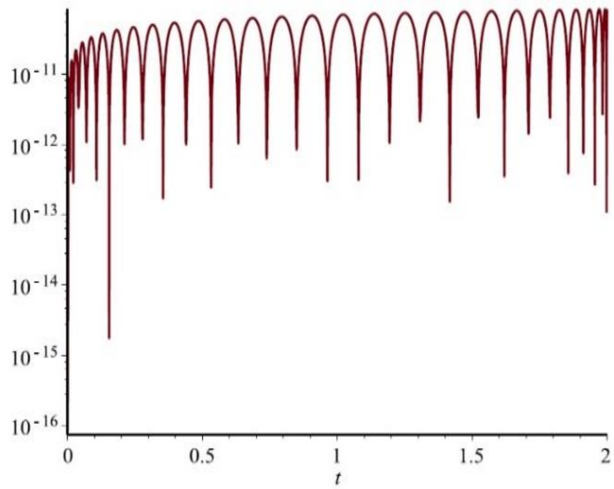
Table 12 shows the comparison of $y(t)$ obtained by the present method and those obtained by Wazwaz [10]. In order to compare the present method with those obtained by Wazwaz [10] and Parand *et al.* [19]. The resulting graph of the Eq. (32) in comparison to the present method and those obtained by Wazwaz [10] and the Log graph of the residual error of approximate solution are shown in Figure 4. This graph shows that the present method has an appropriate convergence rate.

Table 12 Obtained values of $y(t)$ for Lane-Emden equation by the present method for example 4 ($m = 30, \alpha = 0.75$)

t	Parand [19]	Wazwaz [10]	Present	Res(t)
0.1	0.9986051425	0.9985979358	0.9985979273	1.81e-11
0.2	0.9944062706	0.9943962733	0.9943962648	2.40e-11
0.5	0.9651881683	0.9651777886	0.9651777801	5.82e-11
1.0	0.8636881301	0.8636811027	0.8636811255	5.82e-11
1.5	0.7050524103	0.7050419247	0.7050452334	4.78e-11
2.0	0.5064687568	0.5063720330	0.5064636272	8.34e-11



(a) Graph of solutions



(b) Graph of the residual error

Figure 4 Graphs of solutions and the residual error for example 4

3.2 Examples for the Third-order Lane-Emden Type Equations

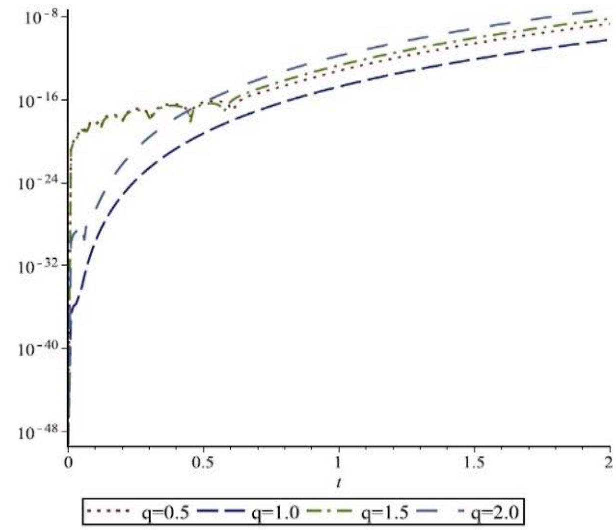
Example 5: For $f(t,y) = y^q, q \in R$ and $k = 4, y_0 = 1$, the Eq. (8) will be one of the third-order Lane-Emden type equations [8]:

$$y''' + \frac{8}{t}y'' + \frac{12}{t^2}y' + y^q = 0, \quad (33)$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

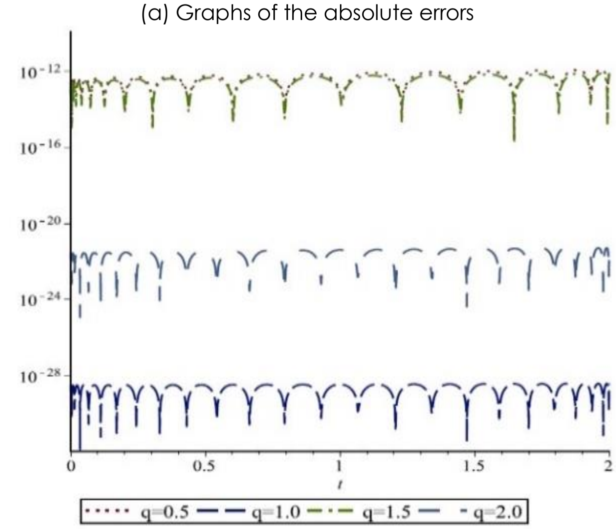
A series solution obtained by Wazwaz [8] by using ADM is:

$$y(t) \simeq 1 - \frac{1}{90}t^3 + \frac{q}{38880}t^6 - \frac{q(17q-12)}{230947200}t^9 + \frac{q(679q^2 - 1182q + 528)}{2909934720000}t^{12} + \ldots$$

The resulting graph of the Eq. (33) in comparison to the present method and those obtained by Wazwaz [8] and the Log graph of the residual error of approximate solution with $m = 25, \alpha = 0.75$ and various values of q are shown in Figure 5.



(a) Graphs of the absolute errors



(b) Graph of the residual error

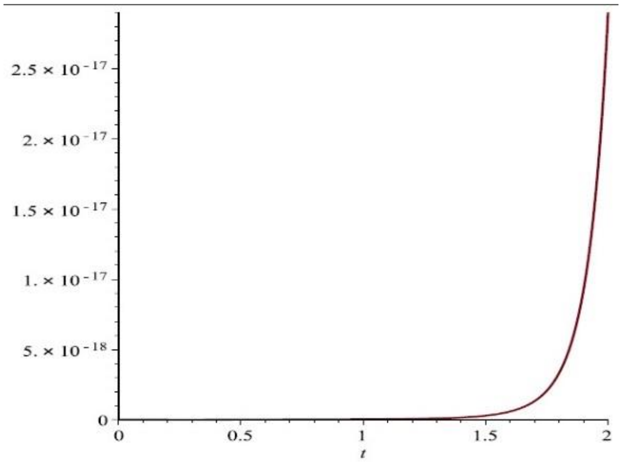
Figure 5 Graphs of the absolute errors and the residual errors for example 5

Example 6: For $f(t,y) = -9(4 + 10t^3 + 3t^6)y, k = 2$, and $y_0 = 1$, the Eq. (8) will be one of the third-order Lane-Emden type equations [8]:

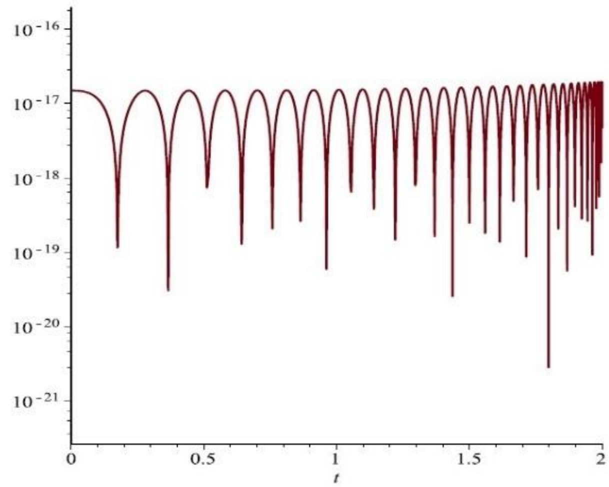
$$y''' + \frac{4}{t}y'' + \frac{2}{t^2}y' - 9(4 + 10t^3 + 3t^6)y = 0, \quad (34)$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

An analytical solution obtained by Wazwaz [8] by using ADM is $y(t) = e^{t^3}$.

The resulting graph of the Eq. (34) in comparison to the present method and analytical solution, and the Log graph of the residual error of approximate solution with $m = 30$ and $\alpha = 3$ are shown in Figure 6.



(a) Graphs of the absolute errors



(b) Graph of the residual error

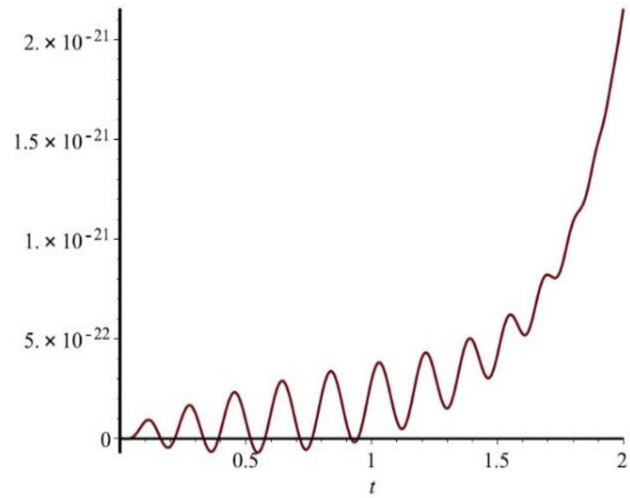
Figure 6 Graphs of the absolute errors and the residual errors for example 6

Example 7: For $f(t,y) = -(10+10t^3+t^6)y, k=4,$ and $y_0 = 1$, the Eq. (9) will be one of the third-order Lane-Emden type equations [8]:

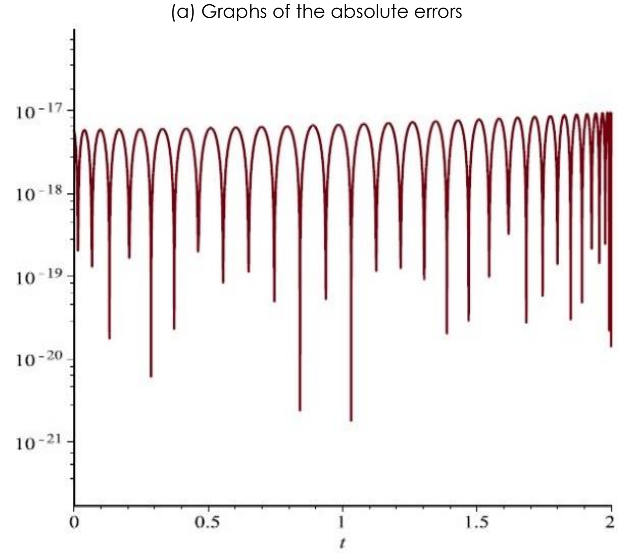
$$y''' + \frac{4}{t}y'' - (10+10t^3+t^6)y = 0, \qquad (35)$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

An analytical solution obtained by Wazwaz [8] by using ADM is $y(t) = e^{\frac{t^3}{3}}$.

The resulting graph of Eq. (35) in comparison to the present method and analytical solution, and the Log graph of the residual error of approximate solution with $m = 30$ and $\alpha = 1.5$ are shown in Figure 7.



(a) Graphs of the absolute errors



(b) Graph of the residual error

Figure 7 Graphs of the absolute errors and the residual errors for example 7

3.3 Examples for the Fourth-order Lane-Emden Type Equations

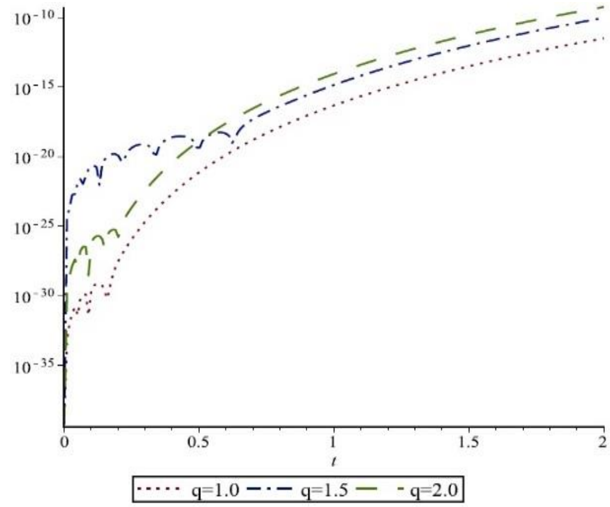
Example 8: For $f(t,y) = y^q, q \in R, k = 4$, and $y_0 = 1$, the Eq. (10) will be one of the fourth-order Lane-Emden type equations [9]:

$$y^{(4)} + \frac{12}{t}y''' + \frac{36}{t^2}y'' + \frac{24}{t^3}y' + y^q = 0, \qquad (36)$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

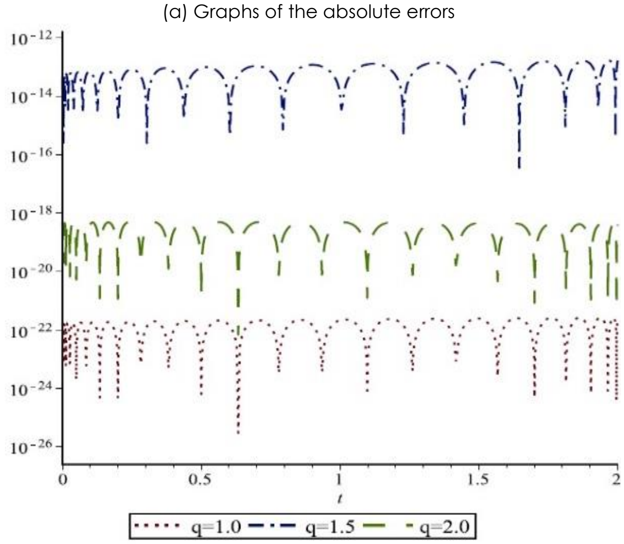
A series solution obtained by Wazwaz [9] by using ADM is:

$$y(t) \simeq 1 - \frac{1}{840}t^4 + \frac{q}{6652800}t^8 - \frac{q(40q-33)}{1525620096000}t^{12} + \ldots$$

The resulting graph of Eq. (36) in comparison to the present method and those obtained by Wazwaz [9] and the Log graph of the residual error of approximate solution with $m = 25, \alpha = 0.50$ and various values of q are shown in Figure 8.



(a) Graphs of the absolute errors



(b) Graph of the residual error

Figure 8 Graphs of the absolute errors and the residual errors for example 8

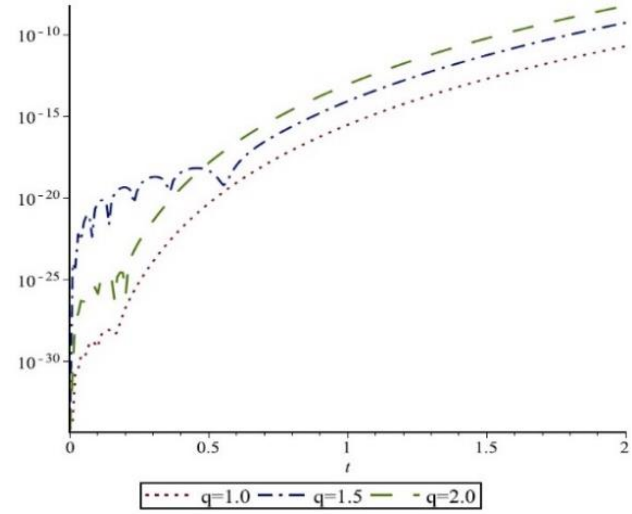
Example 9: For $f(t, y) = y^q$, $q \in R$, $k = 4$, $y_0 = 1$, and $y_1 = 0$, the Eq. (11) will be one of the fourth-order Lane-Emden type equations [9]:

$$y^{(4)} + \frac{8}{t} y''' + \frac{12}{t^2} y'' + y^q = 0, \qquad (37)$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

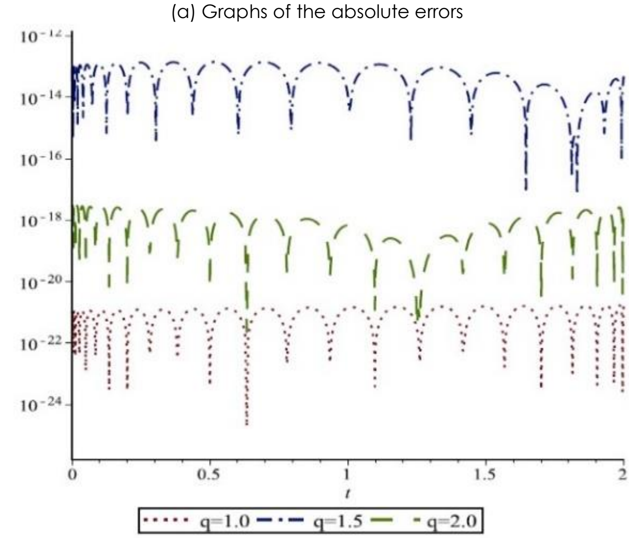
A series solution obtained by Wazwaz [9] by using ADM is:

$$y(t) \simeq 1 - \frac{1}{360} t^4 + \frac{q}{1814400} t^8 - \frac{q(8q-7)}{43589145600} t^{12} + \ldots$$

The resulting graph of Eq. (37) in comparison to the present method and those obtained by Wazwaz [9] and the Log graph of the residual error of approximate solution with $m = 25, \alpha = 0.50$ and various values of q are shown in Figure 9.



(a) Graphs of the absolute errors



(b) Graph of the residual error

Figure 9 Graphs of the absolute errors and the residual errors for example 9

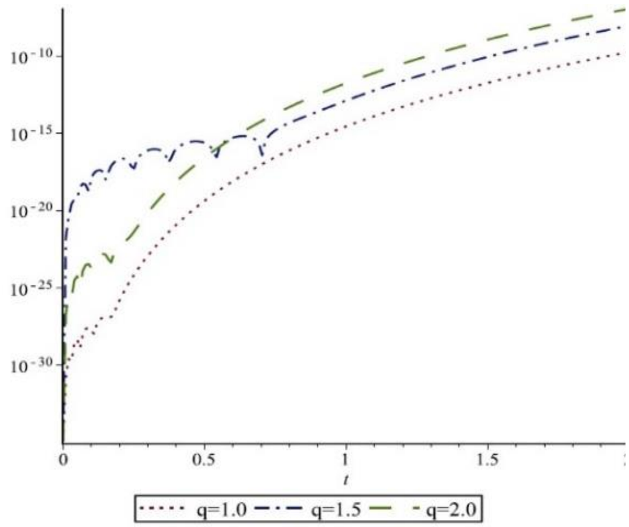
Example 10: For $f(t, y) = y^q, q \in R, k = 4, y_0 = 1, y_1 = 0$, and $y_2 = 0$, the Eq. (12) will be one of the fourth-order Lane-Emden type equations [9]:

$$y^{(4)} + \frac{4}{t} y''' + y^q = 0, \qquad (38)$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

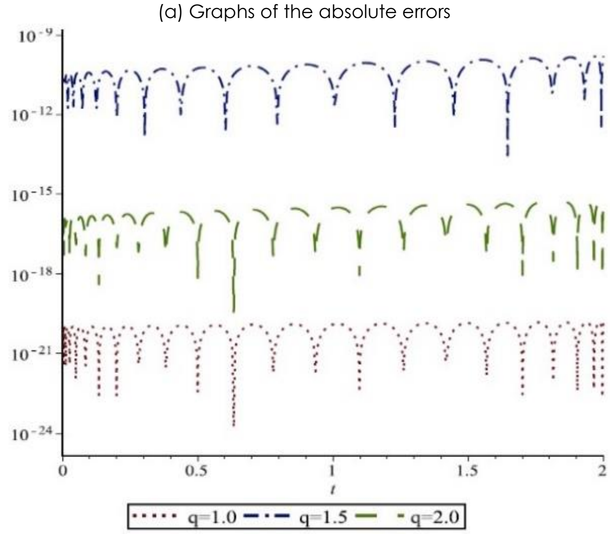
A series solution obtained by Wazwaz [9] by using ADM is:

$$y(t) \simeq 1 - \frac{1}{120} t^4 + \frac{q}{362880} t^8 - \frac{q(68q-63)}{31135104000} t^{12} + \ldots$$

The resulting graph of Eq. (38) in comparison to the present method and those obtained by Wazwaz [9] and the Log graph of the residual error of approximate solution with $m = 25, \alpha = 0.50$ and various values of q are shown in Figure 10.



(a) Graphs of the absolute errors



(b) Graph of the residual error

Figure 10 Graphs of the absolute errors and the residual errors for example 10

4.0 CONCLUSION

The main goal of the paper was to introduce a new orthogonal basis, namely the generalized fractional order Chebyshev orthogonal functions (GFCFs) to construct an approximation to the solution of non-linear Lane-Emden type equations of various orders. The presented results show that the introduced basis for the collocation spectral method is efficient and applicable. Our results have better accuracy with lesser m as compared to other results, and in most cases, the present method has the absolute and the residual errors are better. Comparison was made of the exact solution, the numerical solutions of Parand *et al.* [19, 20], the analytical solution of Wazwaz [9, 10], the numerical solution of Horedt [7] and the present method. It has been shown that the present method has provided an acceptable approach to solve non-linear Lane-Emden type equations of various orders.

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