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DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING SIXTH-ORDER BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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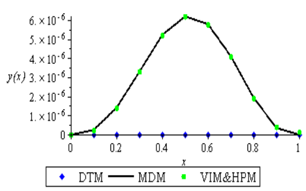
Graphical abstract



Abstract

In this study, sixth-order boundary value problems for linear and nonlinear differential equations have been solved by using Differential Transformation Method (DTM). The numerical solutions are given in several examples. For each example, the solution given by DTM is compared with the exact solution. Absolute relative error (ARE) for each iteration can be computed. Therefore, the maximum absolute relative error (MARE) of the DTM can be obtained. To show that the solution given by the DTM has higher level of accuracy, the absolute relative error of the DTM has been compared with the other methods such as Adomian decomposition method with Green's function, modified decomposition method (MDM), homotopy perturbation method (HPM), Variational Iteration Method (VIM) and Quintic B-Spline Collocation Method. Comparison graphs are given at the end of this paper. The obtained result shows that the proposed method is able to provide better approximation in term of accuracy.

Keywords: Absolute error, differential transformation method, ordinary differential equations, series solution, sixth-order boundary value problems

1.0 INTRODUCTION

We consider the following sixth-order boundary value problems of the form [1]

$$
\begin{gathered}
y^{(6)}(x)+l(x)y(x)=p(x), \quad a<x<b,\\
y(a)=\alpha_0, \qquad y(b)=\alpha_1\\
y'(a)=\gamma_0, \qquad y'(b)=\gamma_1,\\
y''(a)=\delta_0, \qquad y''(b)=\delta_1
\end{gathered}
$$

where functions $f(x)$ and $g(x)$ are continuous on interval $[a,b]$ *while* α_i, γ_i and δ_i, i=0,1 are finite real constants.

In 1986, the idea of differential transformation method (DTM) in solving initial value problems (IVPs) of linear and nonlinear differential equations arisen from an electric circuit analysis was proposed by Zhou [2]. In recent years, DTM has been applied successfully to many problems in linear and nonlinear boundary value problems and initial value problems, for examples, see Yaghoobi and Tarobi [3], Othman and Mahdy [4], Keskin and Oturanc [5] Smarda *et al*. [6] and V.S. Erturk *et al*. [7].

This transformation technique acts as a useful tool to obtain analytical solutions of differential equations which is based on Taylor series expansion. In this method, we constructed the analytical solution in the form of polynomial. The DTM provides higher order of accuracy and approximates to exact solutions which are differentiable. However, the DTM is different from traditional high order Taylor series since it needs more computations for the derivatives functions, Odibat *et* *al*. [8]. DTM can also be applied to high order differential equations therefore, it is an alternative way to get Taylor series solution for the given differential equations for examples, see Hussin *et al*. [9], Hussin and Kilicman [10], Hussin and Kilicman [11], Hussin and Kilicman [12], Husin *et al*. [13,14] and Kilicman and Oltun [15] .

In this paper, we solved sixth-order boundary value problems by DTM and found the maximum error of each method. Then we compared the results of DTM with other methods.

2.0 DIFFERENTIAL TRANSFORMATION METHOD

The following fundamental definitions of differential transformation method are introduced:

Definition 1 The function $y(x)$ for the r-th derivative of one dimensional differential transform is defined as follows:

$$Y(r) = \frac{1}{r!}\left[\frac{d^r y(x)}{dx^r}\right]_{x=x_0}, \quad (1)$$

where $Y(r)$ is the transformed function and $y(x)$ is the original function.

Definition 2 The differential transform of inverse function $Y(r)$ is introduced by:

$$y(x) = \sum_{k=0}^{\infty}(x-x_0)^r Y(r) \quad (2)$$

The substitution of equation (1) into equation (2) yields the following equation:

$$\boldsymbol{y(x) = \sum_{r=0}^{\infty}(x-x_0)^r \frac{1}{r!}\left[\frac{d^r y(x)}{dx^r}\right]_{x=x_0}} \quad (3)$$

where the equation (3) is the Taylor series of $y(x)$ at $x = x_0$. The following basic operations of differential transformation can be deduced from equations (1) and (2) , see [2, 3, 4, 5]:

Theorem 1 Original function $y(x) = q(x) \pm p(x)$, then,
$$Y(r) = Q(r) \pm P(r).$$

Theorem 2 Original function $y(x) = \alpha q(x)$,then,
$$Y(r) = \alpha Q(r).$$

Theorem 3 Original function $y(x) = \frac{dq(x)}{dx}$ then,
$$Y(r) = (r+1)Q(r+1).$$

Theorem 4 Original function $y(x) = \frac{d^2q(x)}{dx^2}$ then,
$$Y(r) = (r+1)(r+2)Q(r+2).$$

Theorem 5 Original function $y(x) = \frac{d^bq(x)}{dx^b}$ then,
$$Y(r) = (r+1)(r+2)\cdots(r+b)Q(r+b).$$

Theorem 6 Original function $y(x) = q(x)p(x)$ then,
$$Y(r) = \sum_{l=0}^{r} P(l)Q(r-l).$$

Theorem 7 Original function $y(x) = x^b$ then
$Y(r) = \delta(r-b)$ where,
$$\delta(r-b) = \begin{cases} 1, & \text{if } r = b \\ 0, & \text{if } r \neq b \end{cases}$$

Theorem 8 Original function $y(x) = \exp(\lambda x)$ then,
$$Y(r) = \frac{\lambda^r}{r!}.$$

Theorem 9 Original function $y(x) = (1+x)^b$ then,
$$Y(r) = \frac{b(b-1)\cdots(b-r+1)}{r!}.$$

Theorem 10 Original function $y(x) = \sin(jx+\alpha)$ then,
$$Y(r) = \frac{j^r}{r!}\sin\left(\frac{\pi r}{2}+\alpha\right).$$

Theorem 11 Original function $y(x) = \cos(jx+\alpha)$ then,
$$Y(r) = \frac{j^r}{r!}\cos\left(\frac{\pi r}{2}+\alpha\right).$$

The following theorem was proved in Hussin *et al*. [9].

Theorem 12 The linear differential equation boundary value problems $y^{(n)}(x) = y(x) + [a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_mx^m]e^x$ for general differential transformation is given by

$$Y(n+r) = \frac{r!}{(n+r)!}\left[Y(r) + \frac{a_0}{r!} + \sum_{i=1}^{m}\left[a_i\left(\sum_{l=0}^{r}\frac{\delta(l-i)}{(r-l)!}\right)\right]\right],$$

for integer $n \geq 1$.

The next two theorems were given in Hussin and Kilicman[10,11] respectively.

Theorem 13 Let $y(x)$ is transformable, then the solution to the high order differential equation $y^{(n)}(x) = e^{-x}y^{\frac{1}{m}}(x)$ is given by

$$Y(r+n) = \begin{cases} \frac{r!}{(r+n)!}\left[\left(\frac{1}{\left(\frac{1}{m}-1\right)^n}\right)\left(\frac{(-1)^r}{r!}\right)r!\,Y(r)\right], & \text{if } r = 0,2,4,\ldots \\ \frac{r!}{(r+n)!}\left[\left(\frac{1}{\left(1-\frac{1}{m}\right)^n}\right)\left(\frac{(-1)^r}{r!}\right)r!\,Y(r)\right], & \text{if } r = 1,3,5,\ldots \end{cases}$$

Theorem 14 The general differential transformation for nonlinear nth-order boundary value problems, $y^{(n)}(x) = e^{-x}y^m(x)$, is given by

$$Y(n+r) = \frac{r!}{(n+r)!}\left[\sum_{r_m=0}^{r}\sum_{r_{m-1}=0}^{r}\dots\sum_{r_1=0}^{r_2}\left(\frac{(-1)^{r_1}}{r_1!}\right)\left(\left[\prod_{i=2}^{m}Y(r_i - r_{i-1})\right]Y(r-r_m)\right)\right]$$

More details on the DTM proofs are available in Odibat *et al.* [8], Hussin *et al.* [9], and Hussin and Kilicman [10, 11].

3.0 NUMERICAL EXAMPLES

By applying the theorems in the paper in Odibat *et al.* [8], Hussin *et al.* [9], and Hussin and Kilicman [10,11], we provided several examples for sixth-order boundary value problems which would be solved by DTM, MDM, HPM, Adomian decomposition method with Green's function, VIM and Quintic B-Spline Collocation Method.

Example 1.Consider solving the following linear sixth-order boundary value problems which was also solved by Wazwaz [16] by using MDM, Noor and Mohyudin [17] by using HPM and Waleed Al Hayani [18] by using Adomian decomposition method with Green's function

$$y^{(6)}(x) = y(x) - 6e^{x} \qquad 0 \le x \le 1, \qquad (4)$$

subject to the boundary conditions

$$y(0) = 1\,, \quad y''(0) = -1, \quad y^{iv}(0) = -3,$$
$$y\,(1) = 0, \quad y''(1) = -2e, \qquad y^{iv}(0) = -4e \qquad (5)$$

The exact solution is $y(x) = (1-x)e^{x}$.

Solution:

By applying Theorem 1 and Theorem 8, equation (4) is transformed into,

$$Y(r+6) = \frac{k!\left[Y(r) - \frac{6}{r!}\right]}{(r+6)!}. \qquad (6)$$

Then by applying the boundary conditions in (5) to equation (1) at point $x = 0$, we obtained the following transformed boundary conditions :

$$Y(0) = 1, Y(1) = a, Y(2) = -\frac{1}{2}, Y(3) = b,$$
$$Y(4) = -\frac{1}{8}, and\ Y(5) = c.$$

By equation (1), $a = \frac{y'(0)}{1!} = Y(1)$, $b = \frac{y'''(0)}{3!} = Y(3)$ and $c = \frac{y^{(5)}(0)}{5!} = Y(5)$. Now we can easily solve for $Y(n)$, $n \ge 6$, by using the transformed equation (6) together with the transformed boundary conditions,. The values of a, b and c can be evaluated at $x = 1$ from the following system of equations:

$$\sum_{r=0}^{12} Y(r) = 0,$$
$$\sum_{r=0}^{12} r(r-1)Y(r) = -2e.$$
$$\sum_{r=0}^{12} r(r-1)(r-2)(r-3)Y(r) = -4e.$$

The solution of this algebraic system gives $a = -6.665 \times 10^{-7}, b = -0.3333323607$ and $c = -0.03333364136$. Finally, the series solution can be formed by applying the inverse transformation in Definition 2 up to $N = 12$, given as follows :

$$y(x) = 1 - 6.665 \times 10^{-7}x - 0.5x^2 - 0.3333323607x^3 - 0.125x^4 - 0.03333364136x^5 - 0.006944444444x^6 - 0.001190476322x^7 - 0.0001736111111x^8 - 0.2204583929 \times 10^{-4}x^9 - 0.248015873 \times 10^{-5}x^{10} - 2.5052200981 \times 10^{-7}x^{11} - 2.296443269 \times 10^{-8}x^{12}.$$

Example 2 Next, we consider in solving the following linear sixth-order BVP of DTM, problems which were also solved by K.N.S.K Viswanadham and Y.S. Raju[19].

$$y^{(6)}(x) \quad = -(24 + 11x + x^3)e^{x}, \qquad 0 \le\ x \le 1 \qquad (7)$$

subjects to the boundary conditions

$$y(0) = 0, y''(0) = 0\,, y^{(4)}(0) = -8,$$
$$y(1) = 0, y'(1) = -e, y''(1) = -4e \qquad (8)$$

The analytical solution is $y(x) = x(1-x)e^{x}$.

Solution:

By using Theorem 12, equation (7) is transformed into

$$Y(r+6) = \frac{r!}{(r+6)!}\Big[-\frac{24}{r!} - 11\left(\sum_{l=0}^{r}\frac{\delta(l-1)}{(r-l)!}\right) - \left(\sum_{l=0}^{r}\frac{\delta(l-3)}{(r-l)!}\right) - \sum_{l=0}^{r} Y(l)\delta(r-l)\Big]. \qquad (9)$$

On applying the boundary conditions in (8) to equation (1) at $x = 0$, then the following transformed boundary conditions can be obtained:

$$Y(0) = 0, Y(1) = 1, Y(2) = 0, Y(3) = a, Y(4) = b$$
$$\text{and } Y(5) = c.$$

We can easily solve for $Y(n), n \ge 6$ by using the transformed equation (9) together with the transformed boundary conditions. The values of a

and b can be evaluated at $x = 1$ from the following system of equations:

$$\sum_{r=0}^{11} Y(r) = 0,$$
$$\sum_{r=0}^{12} rY(r) = -e.$$
$$\sum_{r=0}^{12} r(r-1)Y(r) = -4e.$$

The solution of this system gives $a = -0.49976329$, $b = -0.334029$, $c = 0.12439903$.

Finally, the series solution can be formed by applying the inverse transformation in Definition 2 up to $N = 11$, given as follows:

$$y(x) = x - 0.499763297x^3 - 0.334029005x^4 - 0.124399035x^5 - 0.03333333333x^6 - 0.007142857143x^7 - 0.001140873016x^8 - 0.1653478291 \times 10^{-3}x^9 - 2.314354715 \times 10^{-5}x^{10} - 3.108268093 \times 10^{-6}x^{11}.$$

Example 3 Then we consider in solving the following nonlinear sixth-order BVP of DTM, which was also solved by Wazwaz [16] by using ADM, Noor and Mohyuddin [17] by using HPM and M. A. Noor *et al*. [20] by using VIM

$$y^{(6)}(x) \quad = e^x y^2(x) \quad 0 < x < 1 \tag{10}$$

subjects to the boundary conditions

$$y(0) = 1, y'(0) = -1, y''(0) = 1, y(1) = e^{-1},$$
$$y'(1) = \quad -e^{-1}, y''(1) = -e^{-1} (11)$$

The exact solution is $y(x) = e^x$.

Solution:

By using Theorem 1and Theorem 8, equation (10) is transformed into:

$$Y(r+6) = \frac{r!\left[\sum_{l=0}^{r}\left(\sum_{s=0}^{l}\frac{Y(l-s)Y(r-l)}{s!}\right)\right]}{(r+6)!}. \tag{12}$$

On applying the boundary conditions in equation (11) to equation (1) at $x = 0$, then the following transformed boundary conditions can be

$$Y(0) = 1, Y(1) = -1, Y(2) = \frac{1}{2}, Y(3) = a,$$
$$Y(4) = b, Y(5) = c.$$

By equation (1), $a = \frac{y'''(0)}{3!} = Y(3)$, $b = \frac{y^{(iv)}(0)}{4!} = Y(4)$ and $c = \frac{y^{(v)}(0)}{5!} = Y(5)$. Now we can easily solve for $Y(n)$, $n \geq 6$ by using the transformed equation (12) together with the transformed boundary conditions. The values of a, b and c can be evaluated at $x = 1$ from the following system of equations:

$$\sum_{r=0}^{12} Y(r) = e^{-1},$$
$$\sum_{r=0}^{12} rY(r) = -e^{-1},$$
$$\sum_{r=0}^{12} r(r-1)Y(r) = e^{-1}.$$

Finally, the series solution can be formed by applying the inverse transformation in Definition 2 up to $N = 12$, given as follows:

$$y(x) = 1 - x + 0.5x^2 - 0.1666666719x^3 + .04166667673x^4 - 0.008333340572x^5 + 0.001388888889x^6 - 0.000198412698x^7 + 2.48015873 \times 10^{-5}x^8 - 2.755732096 \times 10^{-6}x^9 + 2.755733254 \times 10^{-7}x^{10} - 2.50521519 \times 10^{-5}x^{11} + 2.08767569 \times 10^{-9}x^{12}.$$

4.0 RESULTS AND DISCUSSIONS

Numerical results and numerical comparisons for Example 1, Example 2 and Example 3 which are presented in Table 1, Table 2 and Table 3 for DTM error at each point respectively. We note that

$$\text{Absolute Relative Error (ARE)} = \frac{|y_{exact\ solution} - y_{DTM}|}{|y_{exact\ solution}|}$$

For Example 1, the maximum absolute relative error (MARE) of DTM, Adomian decomposition method with Green's function, MDM and HPM are in Table 1(B) for $N = 12$. Then for Example 2, the MARE for DTM and b-spline are in Table 2(B) for $N = 12$. Finally for Example 3, the MARE of DTM, MDM, VIM and HPM are in Table 3(B) for $N = 11$. The DTM has the smallest MARE for each example which are 3.06×10^{-7}at point 0.7, 8.946×10^{-6} at point 0.4 and 6.8×10^{-9} at point 1.0 for Example 1, Example 2 and Example 3 respectively. We can see the error DTM was very small compared to the other methods. Besides that the DTM was also efficient since we could solve the problems easily. The error of the DTM was very small for each example rather than other methods.

Figure 1 represents the error comparison for Example 1 while Figure 2 shows the error comparison for Example 2. Lastly, Figure 3 shows the error comparison for Example 3.

Table (1a) Numerical Results for Example 1

x	Exact solution	DTM solution	Error of DTM	ARE DTM
0.0	1	1	0	0
0.1	0.9946537607	0.9946538264	6.57×10^{-8}	$6.605313586 \times 10^{-8}$
0.2	0.9771220810	0.9771222066	1.256×10^{-7}	$1.285407448 \times 10^{-7}$
0.3	0.9449009909	0.9449011653	1.744×10^{-7}	$1.845696022 \times 10^{-7}$
0.4	0.8950946110	0.8950948186	2.076×10^{-7}	$2.319307898 \times 10^{-7}$
0.5	0.8243604141	0.8243606353	2.212×10^{-7}	$2.471247143 \times 10^{-7}$
0.6	0.7288473064	0.7288475202	2.138×10^{-7}	$2.933399056\times 10^{-7}$
0.7	0.6041256276	0.6041258124	1.848×10^{-7}	$3.058966406 \times 10^{-7}$
0.8	0.4451080497	0.4451081859	1.362×10^{-7}	$3.059931181 \times 10^{-7}$
0.9	0.2459602389	0.2459603116	7.27×10^{-7}	$2.955762294 \times 10^{-7}$

Table (1b) MARE in Example 1

DTM	ADM Green's function	MDM and HPM
3.06×10^{-7}	1.73×10^{-5}	1.77×10^{-3}

5.0 CONCLUSION

The findings of this study showed that they ARE of DTM was very small and this indicates that DTM is very accurate in solving boundary value problems and it is very efficient since we can solve boundary value problems easily with a low cost. Besides that, the MARE of DTM was the smallest compared to other methods. The DTM is the most accurate method compared to Adomian decomposition method with Green's function, HPM, VIM, b-spline method and MDM. Therefore, we can conclude that DTM method is very successful and powerful in numerical solutions of sixth order boundary value problems to solve these kinds of problems. The examples were computed by using Maple 13.

Table (2a) Numerical Results for Example 2

x	Exact solution	DTM solution	Error of DTM	ARE DTM
0.1	0.09946538264	0.09946555576	1.7312×10^{-7}	$1.74050504 \times 10^{-6}$
0.2	0.1954244414	0.1954254118	9.704×10^{-7}	$4.96560201 \times 10^{-6}$
0.3	0.2834703496	0.2834725260	2.1764×10^{-6}	$7.67769893 \times 10^{-6}$
0.4	0.3580379276	0.3580411307	3.2031×10^{-6}	$8.94625891 \times 10^{-6}$
0.5	0.4121803178	0.4121838649	3.5471×10^{-6}	8.6056996×10^{-6}
0.6	0.4373085127	0.4373115657	3.0530×10^{-6}	$6.98134134 \times 10^{-6}$
0.7	0.4228880724	0.4228900442	1.9718×10^{-6}	$4.66269949 \times 10^{-6}$
0.8	0.3560865671	0.3560873854	8.183×10^{-7}	2.2980367×10^{-6}
0.9	0.2213643571	0.2213644211	6.40×10^{-8}	$2.891161018 \times 10^{-7}$

Table (2b) MARE in Example 2

DTM	B-SPLINE
8.946×10^{-6}	1.786×10^{-5}

Table (3a) Numerical Results for Example

x	Exact solution	DTM solution	Error of DTM	ARE DTM
0.0	1	1	0	0
0.1	0.9048374181	0.9048374181	0	0
0.2	0.8187307532	0.8187307531	1.0×10^{-10}	$1.221402758 \times 10^{-10}$
0.3	0.7408182206	0.7408182206	0	0
0.4	0.6703200461	0.6703200459	2.0×10^{-10}	$2.983649394 \times 10^{-10}$
0.5	0.6065306598	0.6065306595	3.0×10^{-10}	$4.94616381 \times 10^{-10}$
0.6	0.5488116361	0.5488116357	4.0×10^{-10}	$7.2884752 \times 10^{-10}$
0.7	0.4965853039	0.4965853032	7.0×10^{-10}	$1.409626895 \times 10^{-9}$
0.8	0.4493289639	0.4493289631	8.0×10^{-10}	$1.780432744 \times 10^{-9}$
0.9	0.4065696598	0.4065696583	1.5×10^{-9}	$3.689404666 \times 10^{-9}$
1.0	0.3678794414	0.3678794389	2.5×10^{-9}	$6.795704568 \times 10^{-9}$

Table (3b) MARE in Example 3

DTM	MDM	VIM	HPM
6.8×10^{-9}	1.0532×10^{-5}	1.0534×10^{-5}	1.0534×10^{-5}

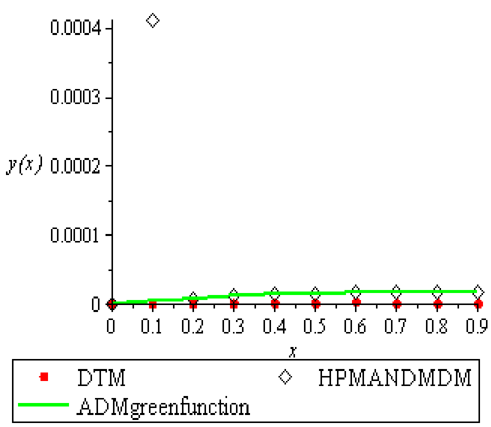


Figure 1 Error comparison for Example 1

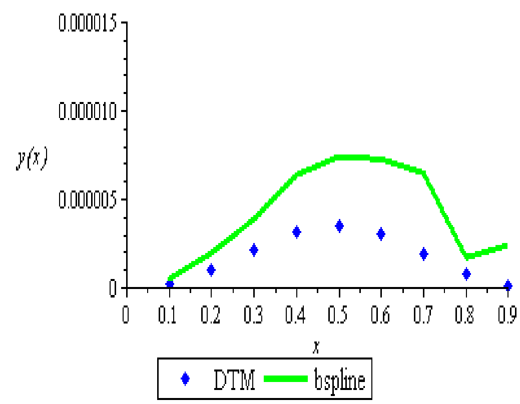


Figure 2 Error comparison for Example 2

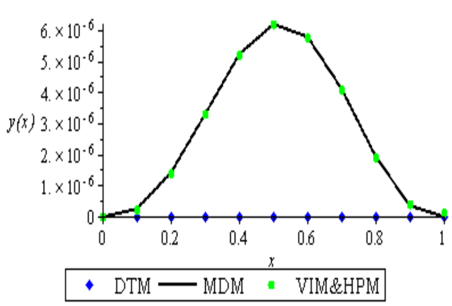


Figure 3 Error comparison for Example 3

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