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AN ALGEBRAIC PROBLEM ARISING IN BIOMATHEMATICS

Sisilia Sylviani*, Ema Carnia, A. K. Supriatna

Department of Mathematics, Universitas Padjadjaran, Indonesia

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*Corresponding author
sisilia.sylviani@unpad.ac.id

Graphical abstract

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_n \end{pmatrix}.$$

Abstract

This paper discusses a matrix model that describes the dynamics of a population with *m* live stages and lives in *n* patch seen from algebra viewpoint. The matrix *D* describes population growth in a patch or location. The matrix *D* is defined as a matrix obtained from matrix multiplication of a permutation matrix with a block diagonal matrix that its diagonal blocks is matrices with non-negative entries and transpose of a permutation matrix [4]. It will be shown that the permutation matrix contained in *D* has a special form.

Keywords: Block diagonal matrix, permutation matrix, population growth

1.0 INTRODUCTION

One thing that is studied in biomathematics is population model, including a structured population model. One way to build a population model is to use the matrix model. The matrix model can be used to describe the dynamic of a population.

There are several models of matrix used in describing the dynamics of a population e.g. Leslie matrix model and Lefkovicth matrix model. In this paper we will discuss the matrix that describes the dynamics of a population with *m* live stages and in *n* patch from algebra point of view. Focus of this paper is a discussion of some of the properties associated with the matrix model. Some definitions and theorems from algebraic view of point are also discussed here.

First of all, will be discussed first some definitions and theorems that will be used in this paper.

Definition 1[5] (Block diagonal matrices). Let A_j be square matrix with nonnegative entries. The matrix $A_1 \oplus \cdots \oplus A_n$ defined as a block diagonal matrix with diagonal blocks $A_1, A_2, \cdots, A_n$ or can be written as follows

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_n \end{pmatrix}.$$

In order to determine the rate of growth of a population that described in a matrix model, we used spectral radius of the matrix.

Definition 2[5] (The Spectral Radius). Let A be a $n \times n$ matrix. The spectral radius of A is the non-negative number $\rho(A)$ which is defined as follow

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i|.$$

As mentioned briefly above the matrix model discussed in this paper involves a permutation matrix. Definition of the permutation matrix is described below.

Definition 3[5] (Permutation Matrices). If σ is a permutation of n letters, the permutation matrix $P(\sigma)$ is defined to be matrix with entries p_{ij} where

$$p_{ij} = \begin{cases} 1, & if \ \sigma(j) = 1 \\ 0, & otherwise \end{cases}$$

In describing the dynamics of a population that suffered displacement from one location to another we will use sub stochastic column matrix.

Definition 4[5] (Column sub stochastic Matrices). An $n \times n$ matrix S with non-negative entries is column sub stochastic if all its columns sum are less than or equal to +1, or can be written as

$$S = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix},$$

where

$$\sum_{i=1}^{n} |s_{ij}| \le 1, \quad \forall j = 1, \dots, n .$$

Other definition that used in this paper is described as follows.

Definition 5[5] (Kronecker Product). Let $A = (a_{ij})$ is a $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then Kronecker Product $A \otimes B$ is a $mp \times nq$ where

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{11}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

Definition 6 [4]. Matrix E_{mn} is defined as square matrix that its entries are zero except in *m*-th row and *n*-th column, or it can be written as $E_{mn} = (e_{ij})$ where

$$e_{ij} = \begin{cases} 1, & if \ i = m, j = n \\ 0, & otherwise. \end{cases}$$

2.0 POPULATION GROWTH MATRIX MODEL

There are some types of matrix model that can be used to describe the dynamic of a population. Matrix model that can be used in describing the dynamics of stage-structured population with *m* life stages and live in *n* patches is the ***SD*** matrix multiplication model. The matrix ***S*** is a block diagonal matrix with $n \times n$ column substochastic matrices $S_1, S_2, \dots, S_n$ and D illustrate the growth and reproduction within the patches [4].

Let us assume that the population be first going through a growth phase in which individuals survive, reproduce and changing from a life stages to other in one location. For each patch *j*, suppose A_j is a $m \times m$ non-negative matrix of representing the dynamics of growth on the location of *j*. Then vector abundance of that population is describe as follow

$$x(t+1) = Dx(t)$$

where the dynamics on the population in a patch is

$$D = P^t (A_1 \oplus A_2 \oplus \cdots \oplus A_n) P,$$

For a permutation matrix P [4].

3.0 METHODS

The method that we use in this paper is literature study. First we study the definition that associated with population growth matrix. Since the focus of this paper is the *D* matrix that contain in the SD matrix multiplication model, hence we study the SD matrix multiplication model first, and then the *D* matrix model. The *D* matrix model contains permutation matrix *P*. The last stage is finding the suitable form of the permutation matrix such that it can facilitate in finding the *D* matrix model.

4.0 RESULTS AND DISCUSSION

The previous section explained that the matrix *D*, which represents the growth and development of the population in a patch, is defined as follows

$$D = P^t (A_1 \oplus A_2 \oplus \cdots \oplus A_n) P,$$

where *P* is a permutation matrix. In order to obtain a matrix *D*, the shape of the permutation matrix needs to be known. This section will explain how the shape of the permutation matrix so that we can obtain a model that describes the growth and development of the existing population in a patch.

Before moving further, some related properties of the permutation matrix will be explained first.

Lemma 7[5]. Let $A=(a_{ij})$ is a $n\times n$ matrix and $P(\sigma)=(p_{ij})$ is a $n\times n$ permutation matrix. Then

(a) Multiplication of a matrix A at the left side with a permutation matrix $P(\sigma)$ will result in the row exchange in A.

(b) Multiplication of a matrix A at the right side with a permutation matrix $P(\sigma)$ will result in the column exchange in A .

Proof:

(a) Let $A=(a_{ij})$ be any $n\times n$ matrix and $P(\sigma)=(p_{ij})$ is arbitrary $n\times n$ permutation matrix. Using the definition of matrix multiplication on both matrix by positioning matrix A on the left side, then we obtained obtain

$$(P(\sigma)A)_{ik}=\sum_{j=1}^{n}p_{ij}a_{jk}$$

Remember that we have a condition that says $p_{ij}=1$ when $i=\sigma(j)$ and $p_{ij}=0$ otherwise. It means that there is only one term in the sum above is nonzero, that is

$$(P(\sigma)A)_{ik}=p_{ij}a_{jk} \text{ when } i=\sigma(j)$$

In other words, the (j,k) entry of A moves to the $(\sigma(j),k)$ position. Since this happens for every k , then the entire *j*-th row of A moves to the $\sigma(j)$-th row.

(b) Proof in this section can be done in a way that is similar to part (a). Let $A=(a_{ij})$ be arbitrary $n\times n$ matrix and $P(\sigma)=(p_{ij})$ is any $n\times n$ matrix. Using the definition of matrix multiplication on both matrix by positioning matrix A on the right side, then we obtained obtain

$$(AP(\sigma))_{ik}=\sum_{i=1}^{n}a_{ki}p_{ij}$$

Remember that we have a condition that says $p_{ij}=1$ when $i=\sigma(j)$ or $j=\sigma^{-1}(i)$ and $p_{ij}=0$ otherwise. It means that there is only one term in the sum above is nonzero, that is

$$(AP(\sigma))_{kj}=a_{ki}p_{ij} \text{ when } i=\sigma(j)$$

In other words, the (k,i) entry of A moves to the $(k,\sigma^{-1}(i))$ position. Since this happens for every k , then the entire *i*-th column of A moves to the $\sigma^{-1}(i)$-th row.

Broadly speaking, multiplying a matrix with a permutation matrix will result in the exchange of rows or columns. The Lemma above holds quite important role in obtaining the permutation matrix forms that exist in the matrix model *D*.

Chi-Kwong Li, *et al*. on his paper entitled "On dispersal and population growth for multistate matrix models," suggests another form of *D*, which results in the following lemma.

Lemma 8[4]. Suppose that the matrix A_j where $j=1,\dots,n$ is a $m\times m$ non-negative matrix. *D* is a matrix of the form $D=\sum_{j=1}^{n}A_j\otimes E_{jj}$, where E_{jj} is a matrix that described in Definition 6.

Furthermore, Let A_1 and A_2 be nonnegative matrix representing the growth dynamics in location 1 and 2. Consider

$$A_1=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \text{and } A_2=\begin{pmatrix}e & f\\ g & h\end{pmatrix}.$$

Then based on lemma 8 we obtain:

$$\begin{aligned}D&=A_1\otimes E_{11}+A_2\otimes E_{22}\\ &=\begin{pmatrix}aE_{11} & bE_{11}\\ cE_{11} & dE_{11}\end{pmatrix}+\begin{pmatrix}eE_{22} & fE_{22}\\ gE_{22} & hE_{22}\end{pmatrix}\\ &=\begin{pmatrix}a&0&b&0\\0&0&0&0\\c&0&d&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&e&0&f\\0&0&0&0\\0&g&0&h\end{pmatrix}\\ &=\begin{pmatrix}a&0&b&0\\0&e&0&f\\c&0&d&0\\0&g&0&h\end{pmatrix}\\ &=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a&0&b&0\\c&0&d&0\\0&e&0&f\\0&g&0&h\end{pmatrix}\end{aligned}$$

$$=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a&b&0&0\\c&d&0&0\\0&0&e&f\\0&0&g&h\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}A_1&0\\0&A_2\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

From the result above, we obtain that for $j=1,2$ the permutation matrix for $D=P^t\left(A_1\oplus A_2\right)P$ is

$$P=\begin{pmatrix}E_{11}&E_{21}\\E_{12}&E_{22}\end{pmatrix}.$$

In a similar way, we can also obtain the form of a matrix of permutations for $j=1,\dots,n$, which is

$$P=\begin{pmatrix}E_{11}&E_{21}&\cdots&E_{n1}\\E_{12}&E_{22}&\cdots&E_{n2}\\\vdots&\vdots&\ddots&\vdots\\E_{1n}&E_{2n}&\cdots&E_{nn}\end{pmatrix}.$$

Hence, by obtaining the permutation matrix form it will facilitate in obtaining the matrix model that describes the population dynamics on a patch.

Chi-Kwong Li and S.J Schreiber in [4] give a good example of a population that can be described by the model discussed in the paper. Consider a population of juveniles and reproductively mature adults salmon living in two spatial locations (a juveniles salmon develop in fresh water and an adults salmon become reproductively mature in the ocean).

Let us assume that in location 1 (in this case a freshwater river), all adults produce two juveniles before dying but juveniles can't become reproductively mature adults. In location 2 (in this case the ocean), all juveniles become reproductively mature adults but progeny produced by the adults in location 2 can't survive (since salmon can't develop in salt water).

We obtain

$$A_1=\begin{pmatrix}0&2\\0&0\end{pmatrix}$$

and

$$A_2=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Then we can easily obtain the matrix model D where and

$$P=\begin{pmatrix}E_{11}&E_{21}\\E_{12}&E_{22}\end{pmatrix}$$

Hence

$$D=P^t\left(A_1\oplus A_2\right)P$$

$$=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0&2&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0&0\\2&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}.$$

Then we have the matrix model for the case, that is

$$D=\begin{pmatrix}0&0&0&0\\2&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}.$$

From the example above we can see that using the permutation matrix formula we obtain, the matrix model ***D*** *can be* easily obtain too.

5.0 CONCLUSION

We have discussed a matrix model that describes a population growth in a patch, denoted by *D*. The Matrix *D* is defined as follows

$$D=P^t\left(A_1\oplus A_2\oplus\cdots\oplus A_n\right)P,$$

for a permutation matrix P and $A_1,A_2,\dots,A_n$ is an $m\times m$ non-negative matrix. The matrix A_j where $j=1,\dots,n$ represents population growth dynamics in patch j. It has been shown that the permutation matrix P is a block matrix in which its entries are

$$P=\begin{pmatrix}E_{11}&E_{21}&\cdots&E_{n1}\\E_{12}&E_{22}&\cdots&E_{n2}\\\vdots&\vdots&\ddots&\vdots\\E_{1n}&E_{2n}&\cdots&E_{nn}\end{pmatrix},$$

with matrix $E_{mn}=\left(e_{ij}\right)$ where

$$e_{ij} = \begin{cases} 1, & if \ \ i = m, j = n \\ 0, & otherwise. \end{cases}$$

By obtaining the permutation matrix form it will facilitate in obtaining the matrix model that describes the population dynamics on a patch.

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