A DESIGN PROGRAM FOR THE LINEAR EQUALIZER THAT MINIMIZES THE MEAN SQUARE ERROR

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Abstract

A design program with detailed illustrative example, for the linear equalizer that minimizes the mean square error due to intersymbol interference in its output signal, is presented. The results are evaluated for many types of distortion channels which have been selected from a wide range of common signal distortions. This includes the various combinations of amplitude and phase distortions.

A synchroneous serial digital baseband signal is assumed throughout. The digital signal is transmitted over the linear time invariant baseband channel whose impulse response is known. The practical implementation of the filters and the techniques on the automatic or adaptive adjustment of the equalizer taps are not considered. The aim of the paper is to show the basic principles of the linear equalizer that minimizes the mean square error, with the aid of the design program and the example.

The design of the linear equalizer is based on a statistical criterior in time domain and the study is confined to simple transversal equalizers whose tap gains do not vary except in response to a change in a channel.

1. Introduction

The need for higher speed data transmission system to furnish the computer communications is rising at a very rapid rate and the requirement is met primarily by the utilization of the telephone channels developed for voice communication. The digital signals are sent over the telephone lines, which normally have a bandwidth of 300 to 3000Hz by translating the binary data to audio frequency signals and back, with the help of a modem (a modulator — demodulator system). A modem transmitter collects a number of bits of data at a time and encode them to symbols for transmission at a signaling rate. The practical telephone channels usually reproduce an output signal, which is a corrupted and transformed (distorted) version of the input signal. The corruption of the waveform may be additive or multiplicative, etc, because of the unavoidable background thermal noise, impulse noise and fades. The transformation of the waveforms by the channel is mainly due to the frequency translation, nonlinear or harmonic distortion and time dispersion.

In digital communication systems, the effect of each symbol (or pulse) over a time dispersion channel extends beyond the time interval allocated for that symbol. When a series of symbols are sent over such a channel, the distortion causes the received signals at the receiver to overlap each other, resulting in intersymbol interference. The distortion, therefore, limits the transmission rate and is one of the serious obstacles to reliable high speed data transmission over the low background noise channels of limited bandwidth. Evidently, in order to have a high speed data transmission system, a rather accurate compensation of the distortion, known as equalization, is required to reduce the intersymbol interference introduced by the channel. The received signal which has been subjected to linear distortion is sampled and fed through the linear equalizer that corrects the distortion and hence eliminates the intersymbol interference. The purpose of the linear equalizer, placed in the path of the received signal, is to reduce the intersymbol interference component to a minimum, so that the probability of correct detection of the signal is maximized. Intersymbol interference is encountered in all pulse modulation systems, including frequency shift keying (FSK), phase shift keying (PSK) and quadrature amplitude modulation (QAM), etc. However, its effect can be described most easily by a baseband digital binary signals.

Linear equalization of distorted digital signals for the baseband channels have been studied very extensively.[1-12] An equalizer for the digital signal is basically a digital filter that corrects the distortion introduced by the channel and restores the received signal into a replica of the transmitted signals. Usually a linear equalizer is a feedforward transversal filter in which the output signal is obtained by summing a series of delayed versions of the input signal, weighted by a set of weights called the tap gains.

The linear equalizers may be designed to minimize the peak distortion,[7,8] the mean square distortion[7,8] or the mean square error.[1,7,8] A program and the design technique presented here is based on the equalizer that minimizes the mean square difference between the actual and the ideal sample values at the output, for a given number of taps, when a continuous stream of data elements is being received in the presence of noise. The advantage in tolerance to additive white Gaussian noise over the arrangement with no equalizer, is calculated for many types of distortion channels. The different values of the sampled impulse response of the channel have been selected to give a fairly wide range of signal distortions, which includes various combinations of amplitude and phase distortions.

The study and the design program is based on the simple transversal equalizers, whose tap gains do not vary with time. The practical implementation of the equalizers is not considered. It is assumed that the channel is either time invariant or else varies only slowly with time and that the equalizer is held correctly to the channel. The practical channels that can be equalized by the design techniques presented here includes the telephone lines, high frequency radio links, 600 ohms pair and coaxial cable links and the multimode fiber optic communication links. With a little modification, the technique may also be used in the digital transmission of signals for seismic applications[2,3] and the sidelobe reduction of pulses for radar applications.[4].

2. Model of Data Transmission System

The model of data transmission system is shown in Fig. 1. A synchroneous serial binary signal is introduced at the input to the transmitter filter and the binary pulses are regularly spaced at intervals of T seconds. The impulses at time $t = iT$ have the value $s_i = \pm 1$

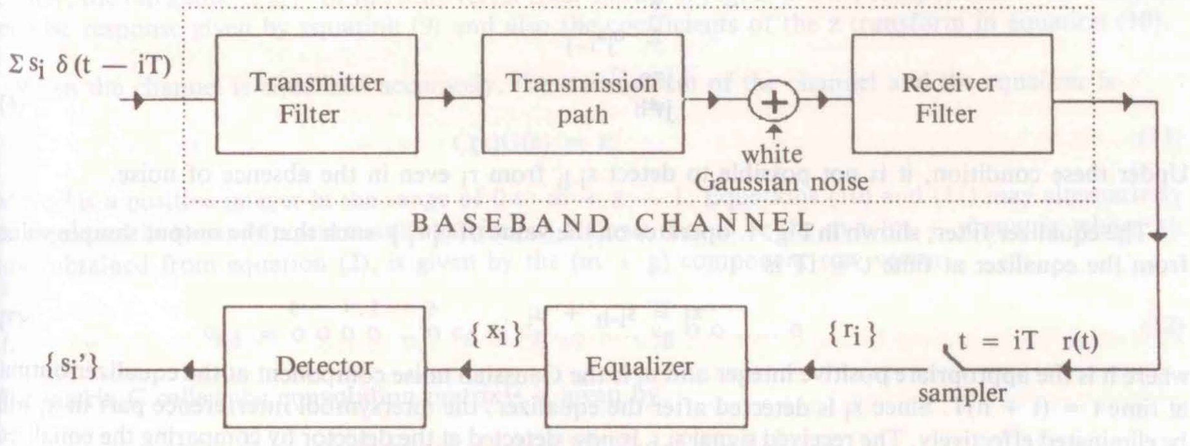

Fig. 1. Model of data transmission system.

Each impulse $s_i\,\delta(t - iT)$ is a binary polar signal element, where $\delta(t)$ is a unit positive impulse at time $t = 0$. The $\{ s_i \}$ values are statistically independent and equally likely to have either binary values. The transmitter filter, the transmission path and the receiver filter, together form a linear baseband channel whose impulse response c(t) is assumed to be known at the receiver. It is assumed that c(t) has a finite duration of less than $(k + 1)T$ seconds, where k is a positive integer. Therefore, the first transmitted signal element $s_i\,\delta(t - iT)$, at the input to the baseband channel at the time $t = iT$, gives the waveform $s_i c(t - iT)$ at the output of the channel.

A stationary white Gaussian noise with zero mean and flat power spectral density σ^2 (frequency independent) is added to the output of the transmission path to give the Gaussian noise[11] waveform w(t) at the output of the receiver filter. The waveform at the output of the receiver filter is

$$r(t) = \sum_{i}^{l} s_i c(t - iT) + w(t) \tag{1}$$

where l is a large positive integer. The received waveform r(t) is sampled once per data symbol at the time instants $\{iT\}$ to give the received samples $\{ r_i \}$.

If a unit impulse is fed to the baseband channel in the absence of noise, the resultant waveform at the input to the samples is c(t). The corresponding sample values at the sampler output from the sampled impulse response of the channel are given by (g + 1) component row vector

$$C = c_o c_1 c_2 c_3 \ldots \ldots \ldots c_g \quad (2)$$

where c_i is the sample value at time t = iT. Since there is no delay in transmission, the z transform of the impulse response of the channel is

$$C(z) = c_o + c_1 z^{-1} + c_2 z^{-2} + \ldots \ldots + c_g z^{-g} \quad (3)$$

where z^{-i} represents a delay of iT seconds.

The sampled impulse response of the channel is assumed to be known to the receiver and the receiver make use of this knowledge in the process of equalization. The sample values r_i for the ith signal element in absence of noise at time t = iT are

$$c_o s_i,\ c_1 s_i,\ c_2 s_i,\ \ldots\ c_g s_i,\ 0,\ 0,0,\ \ldots\ldots \quad (4)$$

Thus, the z transform of the ith received signal element, after sampling is $s_i z^{-i} C(z)$. When there is no attenuation or delay, $c_o = 1$ and $c_j = 0$ for j = 0, 1, 2, ... g. When a continuous stream of signal elements are received in presence of noise, then

$$r_i = \sum_{j=o}^{g} c_j s_{i-j} + w_i \quad (5)$$

It is evident from equation (5) that, in addition to the noise component, the wanted signal $s_{i-h} c_h$ (h is a positive integer between 0 and g) also contains an intersymbol interference component

$$\sum_{\substack{j=o \\ j \neq h}}^{g} c_j s_{i-j} \quad (6)$$

Under these condition, it is not possible to detect s_{i-h} from r_i even in the absence of noise.

The equalizer filter, shown in Fig. 1. operates on the value of $\{ r_i \}$ such that the output sample value from the equalizer at time t = iT is

$$x_i = s_{i-h} + u_i \quad (7)$$

where h is the appropriate positive integer and u_i is the Gaussian noise component at the equalizer output at time t = (i + h)T. Since x_i is detected after the equalizer, the intersymbol interference part in x_i will be eliminated effectively. The received signal s_{i-h} is now detected at the detector by comparing the equalized signal x_i with the decision threshold of zero. That is, if $x_i > o$ then s_{i-h} is detected as 1 and if $x_i < o$ then s_{i-h} is detected as -1.

A high signal to noise ratio is assumed throughout and the impulse response of the channel is such that it can be equalized accurately by the linear filter. This requires that the zeros (roots) of the z transform of the sampled impulse response of the channel do not lie on the unit circle in the z plane.

3. Equalizer for Minimum Mean Square Error.

Feedforward transversal filters[4-12] are very widely used as the linear equalizer and such a filter with m taps is shown in Fig. 2. The equalizer operates on sample values and the signals (sample values) shown in Fig. 2. are those present at the time instant t = iT. Each store, represented by the symbol T, introduces a delay equal to the sampling interval of T seconds. A multiplier g_j is connected to each store so that the output signal r_{i-j} from the store will be multiplied by g_j to give $r_{i-j} g_j$. The output signals from the amplifiers are added together to give the final output at time t = iT, of

$$x_i = \sum_{j=o}^{m-1} r_{i-j} g_j \quad (8)$$

Let the sampled impulse response of the transversal filter given by the m component row vector.

$$G = g_0\, g_1\, g_2 \ldots\ldots\ldots g_{m-1} \tag{9}$$

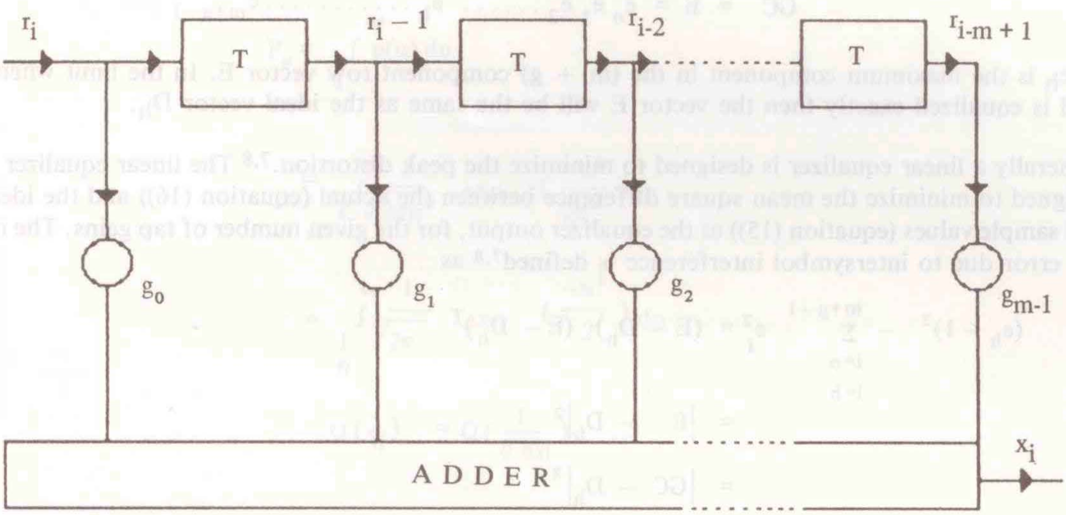

Fig. 2. Linear feedforward m taps transversal equalizer.

In terms of z transform it may be represented as

$$G(z) = g_o + g_1 z^{-1} + g_2 z^{-2} + \ldots\ldots\ldots + g_{m-1} z^{-(m-1)} \tag{10}$$

Clearly, the tap gains $\{g_i\}$ of the transversal filter shown in Fig. 2. are the components of the sampled impulse response given by equation (9) and also the coefficients of the z transform in equation (10).

When the channel is equalized accurately, the z transform of the channel and the equalizer is

$$C(z)G(z) = z^{-l} \tag{11}$$

where l is a positive integer in the range of 0 to m + g — 1. Equations (10) and (11) may alternatively be expressed in terms of vectors and matrices as follows. Let C be the m x (m + g) matrix whose ith row, obtained from equation (2), is given by the (m + g) component row vector

$$c_{i-1} = \overbrace{0\;0\;0\;0 \ldots 0}^{\longleftarrow\ i-1\ \longrightarrow}\; c_0\; c_1\; c_2 \ldots\ldots\ldots c_g\; 0\; 0 \ldots\ldots 0 \tag{12}$$

The matrix C called the convolution matrix[12] is given by

$$C = \begin{bmatrix} c_o & c_1 & c_2 & \ldots & c_g & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & c_o & c_1 & c_2 & \ldots & c_g & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & c_o & c_1 & \ldots & & c_g & 0 & \ldots & 0 & 0 & 0 \\ \ldots & & & & & & & & & & \\ \ldots & & & & & & & & & & \\ 0 & 0 & 0 & \ldots & & & & c_{g-2} & c_{g-1} & c_g & 0 \\ 0 & 0 & 0 & \ldots & & & & & c_{g-2} & c_{g-1} & c_g \end{bmatrix} \tag{13}$$

The z transform representation of equation (11) may now be expressed alternatively in terms of the m component row vector G and the m x (m + g) matric C as

$$\sum_{i=o}^{m-1} g_i c_i = GC = D_h \tag{14}$$

where D_h is the (m + g) component row vector given by

$$D_h = \overbrace{0\;0\;0 \ldots\ldots 0}^{\longleftarrow\ h\ \longrightarrow}\; 1\; 0\; 0 \ldots\ldots\ldots 0\; 0 \tag{15}$$

In equation (14) D_h is the desired or ideal sampled impulse response of the equalized channel and with the finite feedforward transversal filter the equation will not, in general be satisfied exactly. Instead it may be given by the actual values as

$$GC = E = e_o \, e_1 \, e_2 \cdots\cdots e_h \cdots\cdots\cdots\cdots e_{m+g-1} \tag{16}$$

where e_h is the maximum component in the (m + g) component row vector E. In the limit when the channel is equalized exactly then the vector E will be the same as the ideal vector D_h.

Generally a linear equalizer is designed to minimize the peak distortion.[7,8] The linear equalizer may be designed to minimize the mean square difference between the actual (equation (16)) and the ideal or desired sample values (equation (15)) at the equalizer output, for the given number of tap gains. The mean square error due to intersymbol interference is defined[7,8] as

$$(e_h - 1)^2 - \sum_{\substack{i=o \\ i=h}}^{m+g-1} e_i^2 = (E - D_h)\;(E - D_h)^T$$

$$= |E - D_h|^2$$

$$= |GC - D_h|^2$$

since E = GC from equation (16). (17)

In equation (17), $|GC - D_h|$ is the length of the error vector $GC - D_h$ and is therefore the distance between the vectors $GC - D_h$ in the (m + g) dimensional vector spaces containing these vectors. It can be seen from equation (12) that the m component row vectors C_i are linearly independent so that m x (m + g) matrix of C has rank m. From equation (14), the vector GC is a point in the m dimensional subspace spanned by m C_i. Therefore $GC - D_h$ is minimum when GC a point in the m dimensional subspace is at the minimum distance from D_h. By the projection theorem GC is the orthogonal projection of D_h on to the m dimensional subspace. Thus, the m vectors given by the rows of C are orthogonal to the error vector $(GC - D_h)$ so that

$$(GC - D_h)\;(C^T) = 0$$

$$GCC^T = D_h C^T$$

$$G = D_h C^T (CC^T)^{-1} \tag{18}$$

In equation (18) G is given by the (h + 1) th row of the (m + g) x m matrix $C^T(CC^T)^{-1}$. To obtain the optimum equalizer, the vector G must be determined for each value of h in the range 0 to m + g − 1 and the vector G for the required equalizer selected as that which gives the minimum value of $|GC - D_h|$.

Under these conditions, the output sample value from the linear equalizer at time t = (i + l)T, is

$$x_{i+l} = s_i + u_{i+l} \tag{19}$$

where $u_{i+l} = \sum_{k=o}^{m-1} g_i w_{i+l-k}$ and l is a positive integer as given in equation (11).

Since the sample values $\{w_i\}$ are statistically independent Gaussian random variables with zero mean and variance σ^2, u_{i+1} is a sample value of a Gaussian random variable with zero mean and variance

$$\eta^2 = \sigma^2 \sum_{k=o}^{m-1} g_k^{\,2} = \sigma^2 \, |G|^2 \tag{20}$$

where $|G|$ is the length of the vector G. Hence the probability density function of u_{i+1} is

$$p(u) = \frac{1}{\sqrt{2\pi\eta^2}} \exp\left(\frac{-u^2}{2\eta^2}\right) \tag{21}$$

At the detector, the equalized signal s_i is detected as 1 when x_{i+1} is positive and s_i is taken as — 1 when x_{i+1} is negative. An error occurs when u_{i+l} has the magnitude greater than 1 and opposite sign to s_i. The probability of error in the detection of s_i is given by

$$P_e = \int_1^{\infty} p(u)\,du$$

$$= \int_1^{\infty} \frac{1}{\sqrt{2\pi\eta^2}} \exp\left(\frac{-u^2}{2\eta^2}\right) du$$

$$= \int_{\frac{1}{\eta}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du \tag{22}$$

$$= Q\left(\frac{1}{\eta}\right) = Q\left(\frac{1}{\sigma\,|G|}\right) \tag{23}$$

$$\text{where } Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du \tag{24}$$

When the channel is accurately equalized, $|G|$ has a fixed value, dependent only on the channel response so that the tolerance to the noise is determined by the channel.

4. The Structure of The Program

In this section the main features of the design algorithm, based on the previous theoretical results are described. The main constituent of the program is the design by using matrix transformations on the sample values involved. Because the m component vectors and the matrices are basically an array of one dimensional or two dimensional values, the processing of the impulse repsonse of the distortion channel by the linear equalizer filter is readily programable on the computer. The manipulations on the vectors and matrices are achieved by using the basic matrix programs as subroutines and these may be called many times during the execution.

In the main program, the sample values of the five component row vector, giving the sampled impulse-response of the channel are initially entered in the form of input data. The input five component row vector is normalized to have a unit length. This restricts the transmitted signal energy to unity and the noise performances of the equalizers may readily be compared for different distortion channels. This also allows the channel to introduce distortion but no attenuation or gain. The five component row vector is extended to n components by adding the sample values of zeros for the rest of the (n — 5) terms. The (m x (m + g)) convolution matrix C, as described in equation (13) is formed as the two dimensional matrix Y of IV rows and N columns. The operations for transpose, multiplication and inverse are called from their respective subroutines provided and the vector G for the tap gains of the equalizer is evaluated by using equation (18).

The length of the vector G as required in equation (20) gives the factor for the noise variance for the Gaussian random variable with zero mean. The output of the linear equalizer in tandem with the distortion channel is determined and the vector G scaled by a factor so that the maximum component of the equalized signal will be unity. For given values of probability of error P_e (equation (23)), the value of the noise variance is compared to the case where there is no equalization and the tolerance to additive white Gaussian noise of the linear equalizer relative to no distortion is computed.

The loop in the main program is then repeated for N times by using different positions of h component for the desired signal vector D_h. The vector G that gives the minimum value of $|GC - D_h|$ is selected as the optimum equalizer. The complete flow diagram of the main program is shown in Fig. 3.

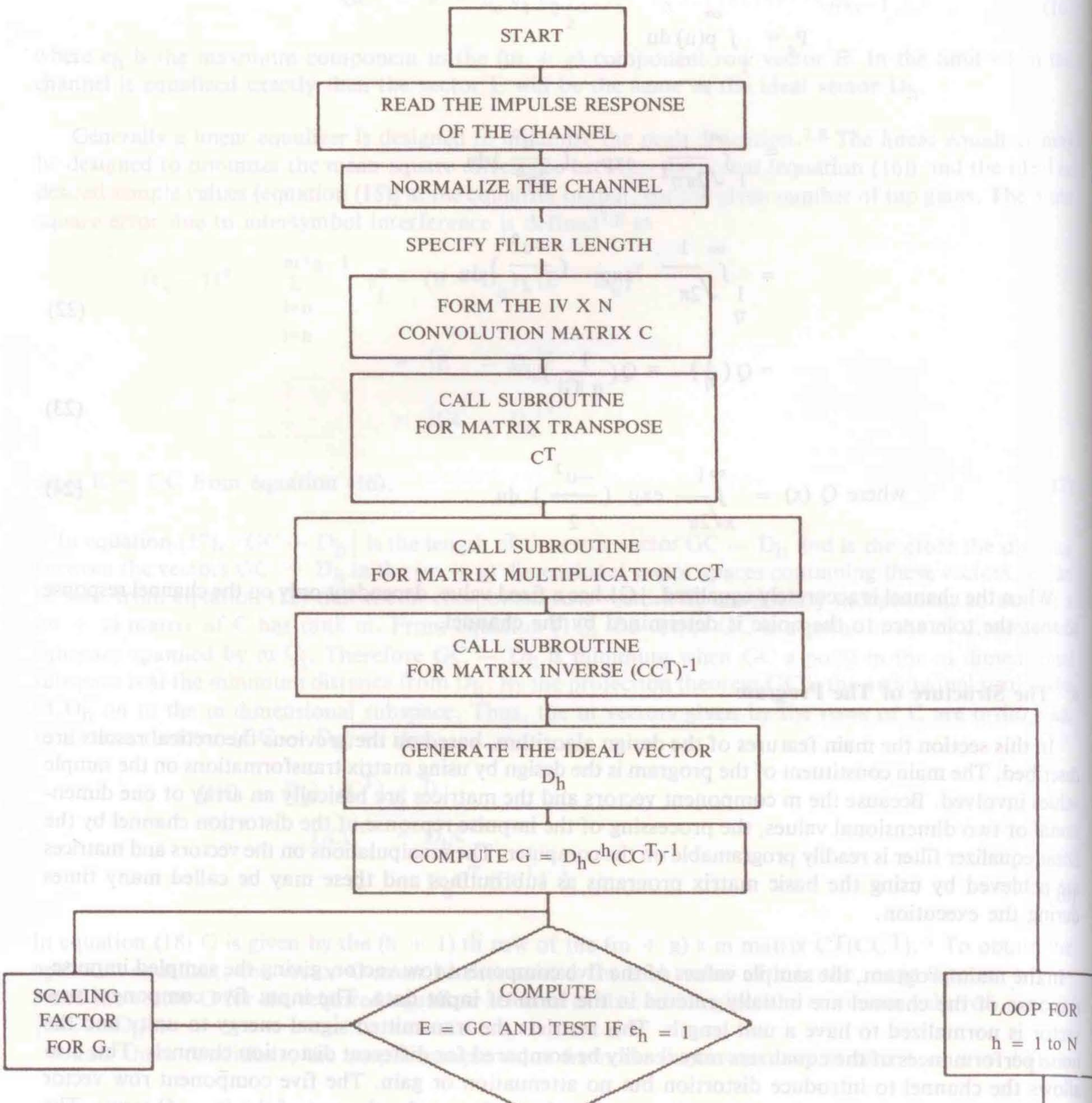

Fig. 3. Flow diagram of the main program.

5. A Design Example

A design example is now presented for the sampled impulse response of the baseband channel which may be represented by a five component row vector as

$$C = 0.2 \quad 0.4 \quad 0.8 \quad 0.4 \quad 0.2 \tag{25}$$

The vector C is normalized to have a length of unity. The distortion channel is equalized by a 17-tap linear feedforward transversal filter, which is designed to minimize the mean square error due to the intersymbol inteference at its output. The convolution matrix C of the order of 17 X 21 is given by

MATRIX OF ORDER 17 BY 21

0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0	0.0
0:0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.4	0.8	0.4	0.2

(26)

The matrix C^T (the transpose matrix of C) has the order of 21 x 17 so that the product matrix CC^T and the inverse $(CC^T)^{-1}$ will be a square matrix of order 17 x 17. The matrix $(CC^T)^{-1}$ is given by

MATRIX OF ORDER 17 BY 17

3.1	−3.2	0.3	1.6	−1.2	0.0	0.6	−0.3	−0.1	0.2	−0.1	0.0	0.0	0.0	0.0	0.0	0.0
−3.2	6.4	−3.5	−1.4	2.9	−1.2	−0.6	0.9	−0.3	−0.2	0.2	0.0	−0.1	0.1	0.0	0.0	0.0
0.3	−3.5	6.4	−3.4	−1.5	2.9	−1.1	−0.7	0.9	−0.2	−0.3	0.2	0.0	−0.1	0.1	0.0	0.0
1.6	−1.4	−3.4	7.3	−4.0	−1.5	3.2	−1.3	−0.7	1.0	−0.3	−0.3	0.3	0.0	−0.1	0.1	0.0
−1.2	2.9	−1.5	−4.0	7.8	−4.0	−1.7	3.3	−1.3	−0.8	1.0	−0.3	−0.3	0.3	0.0	−0.1	0.0
0.0	−1.2	2.9	−1.5	−4.0	7.8	−4.0	−1.7	3.3	−1.3	−0.8	1.0	−0.3	−0.3	0.2	0.0	0.0
0.6	−0.6	−1.1	3.2	−1.7	−4.0	7.9	−4.0	−1.7	3.3	−1.3	−0.8	1.0	−0.3	−0.3	0.2	−0.1
−0.3	0.9	−0.7	−1.3	3.3	−1.7	−4.0	7.9	−4.0	−1.8	3.3	−1.3	−0.8	1.0	−0.2	−0.2	0.2
−0.1	−0.3	0.9	−0.7	−1.3	3.3	−1.7	−4.0	7.9	−4.0	−1.7	3.3	−1.3	−0.7	0.9	−0.3	−0.1
0.2	−0.2	−0.2	1.0	−0.8	−1.3	3.3	−1.8	−4.0	7.9	−4.0	−1.7	3.3	−1.3	−0.7	0.9	−0.3
−0.1	0.2	−0.3	−0.3	1.0	−0.8	−1.3	3.3	−1.7	−4.0	7.9	−4.0	−1.7	3.2	−1.1	−0.6	0.6
0.0	0.0	0.2	−0.3	−0.3	1.0	−0.8	−1.3	3.3	−1.7	−4.0	7.8	−4.0	−1.5	2.9	−1.2	0.0
0.0	−0.1	0.0	0.3	−0.3	−0.3	1.0	−0.8	−1.3	3.3	−1.7	−4.0	7.8	−4.0	−1.5	2.9	−1.2
0.0	0.1	−0.1	0.0	0.3	−0.3	−0.3	1.0	−0.7	−1.3	3.2	−1.5	−4.0	7.3	−3.4	−1.4	1.6
0.0	0.0	0.1	−0.1	0.0	0.2	−0.3	−0.2	0.9	−0.7	−1.1	2.9	−1.5	−3.4	6.4	−3.5	0.3
0.0	0.0	0.0	0.1	−0.1	0.0	0.2	−0.2	−0.3	0.9	−0.6	−1.2	2.9	−1.4	−3.5	6.4	−3.2
0.0	0.0	0.0	0.0	0.0	0.0	−0.1	0.2	−0.1	−0.3	0.6	0.0	−1.2	1.6	0.3	−3.2	3.1

(27)

The 17 component row vector whose component values are the tap gains of the linear transversal filter, are computed for different values of h as described in the flow diagram of the program. The 21 sets of row vectors, each having 17-components as computed according to equation (18), are collectively expressed in matrix form as follows:

ROW VECTORS FOR G TO SEARCH FOR OPTIMUM

MATRIX OF ORDER 21 BY 17

0.6	−0.6	0.1	0.3	−0.2	0.0	0.1	−0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.6	0.0	−0.6	0.4	0.1	−0.2	0.1	0.1	−0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.8	−0.7	0.1	0.1	−0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
−0.8	2.1	−0.8	−0.3	0.4	−0.1	−0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	−0.9	2.1	−0.8	−0.3	0.4	−0.1	−0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.3	−0.3	−0.8	2.3	−0.9	−0.3	0.5	−0.1	−0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
−0.2	0.4	−0.3	−0.9	2.3	−0.9	−0.4	0.5	−0.1	−0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0
0.0	−0.1	0.4	−0.3	−0.9	2.3	−0.9	−0.3	0.5	−0.1	−0.1	0.1	0.0	0.0	0.0	0.0	0.0
0.1	−0.1	−0.1	0.5	−0.3	−0.9	2.3	−0.9	−0.3	0.5	−0.1	−0.1	0.1	0.0	0.0	0.0	0.0
0.0	0.1	−0.1	−0.1	0.5	−0.3	−0.9	2.3	−0.9	−0.4	0.5	−0.1	−0.1	0.1	0.0	0.0	0.0
0.0	0.0	0.1	−0.1	−0.1	0.5	−0.3	−0.9	2.3	−0.9	−0.3	0.5	−0.1	−0.1	0.1	0.0	0.0
0.0	0.0	0.0	0.1	−0.1	−0.1	0.5	−0.4	−0.9	2.3	−0.9	−0.3	0.5	−0.1	−0.1	0.1	0.0
0.0	0.0	0.0	0.0	0.1	−0.1	−0.1	0.5	−0.3	−0.9	2.3	−0.9	−0.3	0.5	−0.1	−0.1	0.1
0.0	0.0	0.0	0.0	0.0	0.1	−0.1	−0.1	0.5	−0.3	−0.9	2.3	−0.9	−0.3	0.4	−0.1	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.1	−0.1	−0.1	0.5	−0.4	−0.9	2.3	−0.9	−0.3	0.4	−0.2
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	−0.1	−0.1	0.5	−0.3	−0.9	2.3	−0.8	−0.3	0.3
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	−0.1	−0.1	0.4	−0.3	−0.8	2.1	−0.9	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	−0.1	−0.1	0.4	−0.3	−0.8	2.1	−0.8
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	−0.1	0.1	0.1	−0.7	1.2
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	−0.1	0.1	0.1	−0.2	0.1	0.4	−0.6	0.0	0.6
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	−0.1	0.1	0.0	−0.2	0.3	0.1	−0.6	0.6

(28)

In the above matrix (28), the hth row G_h represents a row vector for the values of the tap gains of the transversal filter equalizer that will equalize the channel to give the actual values, as described by equation (16). The actual values E_h for the sampled impulse response of the channel and the equalizer in tandem, is now computed for each row G_h of the matrix G in equation (28). The 21 different row vectors of the actual values E_h are presented for comparison for all the different possible positions of h. This is given in matrix form by equation (29). Each row vector from equation (29) may be compared with the respective desired values D_h and the minimum value of $|GC - D_h|$ or $|E - D_h|$ is calcuated for each value of h. From the results of $|E - D_h|$ as given in equation (30), the minimum value for the length of the vector is given by E_{11}, the 11th row of the matrix (29), so that the corresponding 11th row G_{11} of the matrix (28) is selected as the component values of the tap gains of the transversal equalizer filter shown in Fig. 2.

MATRIX OF ORDER 21 BY 21

0.1	0.1	0.2	−0.2	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.2	0.4	0.1	−0.2	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.4	0.7	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
−0.2	0.1	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	−0.2	0.1	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.1	0.1
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.1	−0.2	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.1	−0.2
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.7	0.4	0.2
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	−0.2	0.1	0.4	0.2	0.1
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	−0.2	0.2	0.1	0.1

(29)

COMPARISON OF | GC – D | VALUES

7.3523	3.2959	0.3639	0.0377	0.0092	0.0018	0.0016	0.0003	0.0001
0.0001	0.0001	0.0003	0.0016	0.0018	0.0092	0.0377	0.3639	3.2959
7.3524								

(30)

Thus, $$GC = z^{-11} \tag{31}$$

so that s_i is detected from the sign of the value

$$x_{i+11} = s_i + u_{i+11} \tag{32}$$

at the output of the filter at time $t = (i + 11)T$.

The 17-component row vector G_{11}, selected from the matrix (28) may be written accurately as,

$$\begin{aligned} G_{11} = & -.009 \quad -.015 \quad .089 \quad -.096 \quad -.130 \quad .487 \quad -.349 \\ & -.896 \quad 2.345 \quad -.896 \quad -.349 \quad .487 \quad -.130 \quad -.096 \\ & .089 \quad -.015 \quad -.009 \end{aligned} \tag{33}$$

In equation (32) u_{i+11} is a Gaussian random variable with zero mean and variance,

$$\eta^2 = \sigma^2 \left|G_{11}\right| \tag{34}$$

$$= 7.854\,\sigma^2$$

so that $$\eta = 2.802\,\sigma \tag{35}$$

From (23) the probability of error in the detection of s_i from x_{i+11} is approximately

$$P_e = Q\left(\frac{k}{\eta}\right) = Q\left(\frac{k}{2.8\sigma}\right) \tag{36}$$

If there is no equalizer filter to correct the distortion channel, the ith received signal element is

$$s_i z^{-i} C(z) = s_i\,(.2z^{-i} + .4z^{-i-1} + .8z^{-i-2} + .4z^{-i-3} + .2z^{-i-4}) + u_i \tag{37}$$

In equation (37), it can be seen that the distortion channel represented by the five component row vector C has a length of unity. Thus, the channel introduces distortion, but no attenuation or gain. If the distortion channel is now replaced by one which introduces no distortion or attenuation, then the transmitted signal level will be unchanged so that in presence of noise, the received signal for the ith signal element will be

$$r_i = s_i + u_i \tag{38}$$

Thus, s_i will be detected as 1 when $r_i > 0$ and $s_i = -1$ when $r_i < 0$. An error in the detection of s_i occurs when u_i has magnitude greater than ½ and has a opposite sign to s_i. At high signal to noise ratio, the average probability of error in the detection of s_i from r_i is approximately given equation (23) as

$$P_e = Q\left(\frac{1}{\sigma}\right) \tag{39}$$

By comparing equations (36) and (39) it can be seen that for a given probability of error, P_e, the value of σ in the case of the system with linear equalizer is $\frac{1}{2.802}$ times that of the system with no equalization. This means the linear equalizer decreases the tolerance in additive white Gaussian noise by,

$$\begin{aligned} 20\log_{10}\eta &= 20\log_{10}\frac{(2.802)\sigma}{\sigma} \text{ db.} \\ &= 8.951 \text{ db.} \end{aligned} \tag{40}$$

From the study of the row vectors E in matrix (29), for the actual values of the equalized signal, and the lengths of the error vectors given in equation (30), the equalization of the channel by the 11th row vector G_{11} is fairly accurate. Furthermore, the mean square difference between the actual and the desired signals at the output of the equalizer has a minimum value of 0.0001.

6. Assessment of The Results

By making use of the theoretical analysis and the design program presented, the tolerance to additive white Gaussian noise of the linear equalizer is evaluated for different types of distortion channels. The program is tested for 23 types of distortion channels and the results are given in Table 1. The channels are equalized accurately, as described in the design example and a high signal to noise ratio is assumed. The different sample values for the impulse response of the distortion channels, listed in column A of Table 1, have been selected to include pure amplitude distortion, pure phase distortion and various combinations of amplitude and phase distortions. The 5-component row vectors for the sampled impulse response of the channels are chosen to have unit lengths, so that the channels are allowed to have distortion, but no gain or attenuation. The roots (zeros) of the z transform of the impulse response of the channel may have real or complex values and may lie inside or outside the unit circle in the z plane, but not on it.

	A						B		C		D
CHANNEL DISTORTION							NOISE VARIANCE η^2		η		20LOG η
1	0.196	0.392	0.785	0.392	0.196	* * * *	7.854	* * * *	2.802	* * * *	8.951
2	0.167	0.471	0.707	0.471	0.167	* * * *	103.224	* * * *	10.160	* * * *	20.138
3	0.137	0.457	0.684	0.479	0.274	* * * *	34.103	* * * *	5.840	* * * *	15,328
4	0.203	0.339	0.749	0.406	0.343	* * * *	20.244	* * * *	4.499	* * * *	13.063
5	0.265	−0.486	0.728	−0.368	0.169	* * * *	22.622	* * * *	4.756	* * * *	13.545
6	0.182	−0.571	0.642	−0.428	0.214	* * * *	44.546	* * * *	6.674	* * * *	16.488
7	0.274	0.479	0684	0.457	0.137	* * * *	27.577	* * * *	5.251	* * * *	14.405
8	0.273	0.447	0.682	−0.455	−0.161	* * * *	1.200	* * * *	1.096	* * * *	0.793
9	0.152	0.597	0.643	−0.429	−0.152	* * * *	1.608	* * * *	1.268	* * * *	2.064
10	0.307	0.510	0.702	−0.314	−0.234	* * * *	1.623	* * * *	1.274	* * * *	2.104
11	0.250	0.500	1.000	0.500	0.250	* * * *	7.903	* * * *	2.811	* * * *	8.978
12	0.214	−0.428	0.642	−0.571	0.182	* * * *	72.193	* * * *	8.497	* * * *	18.585
13	0.161	0.974	0.161	0.000	0.000	* * * *	1.255	* * * *	1.120	* * * *	0.985
14	0.262	0.929	0.262	0.000	0.000	* * * *	2.059	* * * *	1.435	* * * *	3.136
15	0.348	0.870	0.348	0.000	0.000	* * * *	6.109	* * * *	2.472	* * * *	7.859
16	0.378	0.845	0.378	0.000	0.000	* * * *	15.507	* * * *	3.938	* * * *	11.905
17	0.219	0.750	0.625	0.000	0.000	* * * *	11.897	* * * *	3.449	* * * *	10.754
18	0.625	0.750	0.219	0.000	0.000	* * * *	11.897	* * * *	3.449	* * * *	10.754
19	0.575	0.776	0.259	0.000	0.000	* * * *	17.684	* * * *	4.205	* * * *	12.476
20	0.262	0.928	−0.262	0.000	0.000	* * * *	1.010	* * * *	1.005	* * * *	0.041
21	0.348	0.870	−0.348	0.000	0.000	* * * *	1.031	* * * *	1.015	* * * *	0.132
22	0.575	0.776	−0.259	0.000	0.000	* * * *	1.409	* * * *	1.187	* * * *	1.490
23	0.625	0.750	−0.219	0.000	0.000	* * * *	1.772	* * * *	1.331	* * * *	2.484

Table. 1. No. of db reduction in tolerance to additive white Gaussian noise by the linear equalizer compared to the case with no equalization.

The results listed in column D of table 1. gives the number of decibels increase in noise level, which is necessary to maintain a given low probability of error when the channel with the corresponding sampled impulse response is replaced by one introducing no distortion or attenuation and the transmitted signal level is held constant. Depending on the distortion characteristics, it is found that the design technique can increase the noise level ranging from few decibels to over 20 decibels. However, a fairly accurate equalization is achieved by the equalizer, resulting in a complete reduction in intersymbol interference. At high signal to noise ratio, the intersymbol interference is more dominant compared to noise, so that it is more important to reduce the intersymbol interference than the noise.

7. Conclusion

A design technique with the flow diagram of the program has been presented for the linear equalizer that minimizes the mean square error. The design idea is based on moving a set of equalizer taps closer to a unique optimum set corresponding to the mean square error criterior. The test runs of the program for different values of the sampled impulse response of the channel shows that for those with roots (zeros) far away from the unit circle in the z plane, the channel can be equalized accurately and the search for minimum is convergent. However for the roots very near to the unit circle, the results lead to a relatively large error vector and greater potential for unstability. With the accurate equalization of the channel, the tap gains, and hence the tolerance to additive white Gaussian noise of the transversal equalizer filter, are effectively fixed by the channel and the reduction in tolerance to additive white Gaussian noise is often associated with the linear equalizer compared to non equalization arrangement.

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