A THEORETICAL APPROACH IN THE ANALYSIS OF MULTILAYERED CROSS-PLY GRP CYLINDRICAL SHELL

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Synopsis

The constitutive relations for the specially orthotropic lamina under a plane stress state are employed with a linear shell theory, to obtain the governing equations for the multilayered cross-ply cylindrical shell with through thickness symmetry.

The well known Fourier expansion method, used extensively in the analysis of isotropic shell problems, are extended to handle the more complex case of the anisotropic shell subject to a general loading case. Attention is confined to those loads which are symmetric with respect to the generator passing through the lowest point of the shell, that is liquid-filled, self-weight, support loading, etc.

Introduction

The reinforced plastics cylindrical shell is increasingly used in both the process industry and in aircraft and space transportation. The lightness of the product in the latter case is of particular value where up to 50% weight can be saved without loss of strength or material degradation.

In these applications peak stresses (or strains) can occur at a number of locations, for example,

(i) where the shell and its contents are supported, or the shell loaded through a bracket
(ii) where the displacement of the pressuried shell is restrained in anyway
(iii) where there is a discontinuity in the primary loading, such as occurs when a pipeline is partially filled.

The aim of this paper is to provide a theoretical basis for considering these and other similar problems when the shell is manufactured in a multi-layered form with thickness symmetry. The behaviour of a simplified cross-ply lay-up, with each layer specially orthotropic, is examined. In this the principal material axes are aligned in the axial and circumferential coordinates of the cylindrical shell.

Although this work is the first step to solving the more general case of the multilayered anisotropic system it does provide an insight into the behaviour of a multilayered system when subject to various types of loads-examples of which are given in detail[1]. The validity of the analytical approach proposed, has been examined by the author et. al[2] by comparing the results derived for the patch load, with those obtained using a more rigorous approach[3]. For completeness typical values of the results are presented in Figure 2 for a shell of 254 mm radius and 2540 mm length and of radius/thickness = 15. The axial length of the loaded area was varied from approximately 20 mm to 125 mm, with an adjustment in the circumferential length to maintain a constant value of loaded area equal to 3580 mm^2. [The figures were chosen to coinside with those in ref (3)]. It was found that the present method is acceptable for values of moduli ratio that fall within normal shell design.

Theoretical Analysis

Govering differential equations

The constitutive relation for a specially orthotropic lamina under a plane stress state is given by:

$$\begin{bmatrix} \sigma_x \\ \sigma_\phi \\ \tau_{x\phi} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}_k \begin{bmatrix} \Sigma_x \\ \Sigma_\phi \\ \gamma_{x\phi} \end{bmatrix} \quad \ldots (1)$$

where: $Q_{11} = E_{11}/(1-\nu_{12}\nu_{21})$

$Q_{12} = \nu_{12}E_{22}/(1-\nu_{12}\nu_{21}) = \nu_{21}E_{11}/(1-\nu_{12}\nu_{21})$

$Q_{66} = G_{12}$

The strain values Σ_x, Σ_ϕ, $\gamma_{x\phi}$ are defined in terms of u, v and w, for the small strain compatibility relations as gives by Flugge[4]. These equations relate the positive strain and curvature at any distance z from the middle surface as a function of the mid-surface displacements, u, v and w. They are defined as follows:

$$\Sigma_x = \frac{1}{a}\,du - \frac{z}{a^2}\,d^2 w$$

$$\Sigma_\phi = \frac{1}{a}\,\delta v - \frac{z}{a(a+z)}\,\delta^2 w + \frac{1}{(a+z)}\,w$$

$$\gamma_{x\phi} = \frac{1}{(a+z)}\delta u + \frac{(a+z)}{a^2}\,dv - \frac{1}{a}\left(\frac{z}{a} + \frac{z}{a+z}\right) d\delta w \qquad \ldots (2)$$

In these equations the differential operators $\partial/\partial(x/a)$ and ∂/∂_ϕ are represented by d and δ respectively.

The lamina stresses of eqn. (1) can be expressed in terms of u, v and w and their derivatives, using eqn. (2). Assuming the shell to be fabricated as a layered system with through thickness symmetry, it is possible to integrate the stress from eqn. (1) over the thickness to obtain the resultant forces and moments acting on the laminate. These can be written:

$$N_x = \frac{1}{a}(A_{11}\,du + A_{12}\,(\delta v + w)) - \frac{1}{a^2} D_{11} d^2 w$$

$$N_\phi = \frac{1}{a}(A_{12}\,du + A_{22}\,(\delta v + w)) + \frac{1^a}{a^2} D_{22}\,(w + \delta^2 w)$$

$$N_{x\phi} = \frac{1}{a}(A_{66}\,(dv + \delta u)) + \frac{1}{a^2} D_{66}\,(dv - d\delta w)$$

$$N_{\phi x} = \frac{1}{a}(A_{66}\,(dv + \delta u)) + \frac{1}{a^2} D_{66}\,(\delta u + d\delta w)$$

$$M_x = \frac{1}{a^2}(D_{11}\,(d^2 w - du) + D_{12}\,(\delta^2 w - \delta v))$$

$$M_\phi = \frac{1}{a^2}(D_{12}\,d^2 w + D_{22}\,(w + \delta^2 w))$$

$$M_{x\phi} = \frac{1}{a^2}(2\,D_{66}\,(d\delta w - dv))$$

$$M_{\phi x} = \frac{1}{a^2}(D_{66}\,(2\,d\delta w + \delta u - dv)) \qquad \ldots (3)$$

The terms A_{ij} and D_{ij} are the in-plane and bending stiffness for the laminate. They correspond to the summation of the individual layer effects and are defined in the notation.

The stress resultants given in eqn. (3) are shown in Fig. 1.

The equilibrium equations for a thin-walled circular cylindrical vessel can be written as follows:

$$dN_x + \delta N_{\phi x} + p_x a = 0$$

$$a\delta N_\phi + adN_{x\phi} - \delta M_\phi - dM_{x\phi} + p_\phi \cdot a^2 = 0$$

$$\delta M_\phi + d\delta M_{x\phi} + d\delta M_{\phi x} + d^2 M_x + aN_\phi - p_r a^2 = 0$$

$$aN_{x\phi} - aN_{\phi x} + M_{\phi x} = 0 \qquad \ldots (4)$$

The governing differential equations are obtained by substituting the stress resultants from eqn. (3) into the equilibrium eqn. (4) to give the following:

$$(A_{11} d^2 + C1\delta^2)u + C2d\delta v + (A_{12} d + \frac{1}{a^2} D_{66} d\delta^2 - \frac{1}{a^2} D_{11} d^3)w = -p_x a^2$$

$$C2d\delta u + (A_{22}\delta^2 + C3 d^2)v + (A_{12}\delta - C4 d^2 \delta)w = -p_\phi a^2$$

$$(A_{12} d + \frac{1}{a^2} D_{66} d\delta^2 - \frac{1}{a^2} D_{11} d^3) + (A_{22}\delta - C4d^2\delta)v$$

$$+ (C5 + 2\frac{D_{22}}{a^2}\delta^2 + \frac{1}{a^2} D_{22}\delta^4 + \frac{1}{a^2} D_{11} d^4 + C6d^2\delta^2)w = -p_r a^2 \qquad \ldots (5)$$

Writing eqn. (5) in matrix form reveals the symmetry of the relationships :

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = a^2 \begin{bmatrix} -p_x \\ -p_\phi \\ p_r \end{bmatrix}$$

Using gramer's rule, eqn. (5) can be solved to give nine eighth order partial differential equations relating the mid-surface displacements u, v and w to the loading components p_r, p_ϕ and p_x. Those relevant to the radial loading are shown below:

$$(DEXP)\, w = (C7d^2\delta^2 + C8\, d^4 + C9\delta^4)\, p_r .\, a^2$$

$$(DEXP)\, u = (C10d\delta^2 - C11d^3\delta^2 + C12d^3 + C13d^5 - C14d\delta^4)\, p_r a^2$$

$$(DEXP)\, v = -(C15d^2\delta - C16d^4\delta - C17d^2\delta^3 + C18\delta^3)\, p_r a^2$$

where:

$$DEXP1 = C19d^2\delta^2 + C20d^2\delta^4 + C21d^2\delta^6 + C22d^b\delta^2 + C23d^4\delta^4 + C24d^4\delta^2$$

$$+ C25d^2 + C26d^6 + C27d^8 + C28\delta^4 + C29\delta^6 + C30\delta^8 \qquad \ldots (6)$$

The constant C1 etc in eqns. (5) and (6) are functions of extensional and bending stiffness. They are given in detail in Appendix 1.

Fourier expansion solution

A particular solution of eqn. (6) can be obtained by expressing the loading components and the mid-surface displacements in the double Fourier series form show below:

$$p_x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{n,m} \cos n\phi \cos(\lambda\, x/a)$$

$$p_\phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} p_{n,m} \sin n\phi \sin(\lambda\, x/a)$$

$$p_r = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} p_{n,m} \cos n\phi \sin(\lambda\, x/a) \qquad \ldots (7)$$

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} w_{n,m} \cos n\phi \sin(\lambda x/a)$$

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} u_{n,m} \cos n\phi \cos(\lambda x/a)$$

$$v = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} v_{n,m} \sin n\phi \sin(\lambda x/a) \quad \ldots (8)$$

where $\lambda = m\pi a/L$

The choice of the expansions for p_x, p_ϕ and p_r imply that the loading system is symmetric with respect to the generator passing through $\phi = 0$ (see fig. 1). The values for w, u and v eqn. (8) can be used because of the through thickness symmetric nature of the specially orthotropic system. A more general approach for the displacement is presented by the author in ref. (1).

Substituting w, u and v in eqn. (8) and the relevant loading terms, eqn. (7), into eqn. (6), performing the required differential operations and involving orthogonality, a set of algebraic equations are obtained for the coefficients $w_{n,m}$, $v_{n,m}$ and $u_{n,m}$ in terms of the loading $p_{n,m}$.

After deriving these coefficients, eqn. (8) can be used again to obtain expressions for w, u and v, which can be used in eqn. (3) to obtain the stress resultants in series form. These equations are as follows:

$$N_x = \frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} Z4_{n,m} P_{n,m} \cos n\phi \sin(\lambda x/a)$$

$$N_\phi = \frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} Z5_{n,m} P_{n,m} \cos n\phi \sin(\lambda x/a)$$

$$N_{x\phi} = \frac{1}{a} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Z6_{n,m} P_{n,m} \sin n\phi \cos(\lambda x/a)$$

$$N_{\phi x} = \frac{1}{a} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Z7_{n,m} P_{n,m} \sin n\phi \cos(\lambda x/a)$$

$$M_x = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} Z8_{n,m} P_{n,m} \cos n\phi \sin(\lambda x/a)$$

$$M_\phi = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} Z9_{n,m} P_{n,m} \cos n\phi \sin(\lambda x/a)$$

$$M_{x\phi} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Z10_{n,m} P_{n,m} \sin n\phi \cos(\lambda x/a)$$

$$M_{\phi x} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} Z11_{n,m} P_{n,m} \sin n\phi \cos(\lambda x/a) \quad \ldots (9)$$

where:

$$Z4_{n,m} = -A_{11}\lambda Z2_{n,m} + \frac{1}{a^2} D_{11}\lambda^2 Z1_{n,m} + A_{12}(nZ3_{n,m} + Z1_{n,m})$$

$$Z5_{n,m} = -A_{12}\lambda Z2_{n,m} + A_{22}(nZ3_{n,m} + Z1_{n,m}) + \frac{1}{a^2} D_{22} Z1_{n,m}(1-n^2)$$

$$Z6_{n,m} = -A_{66}(\lambda Z3_{n,m} - nZ2_{n,m}) + \frac{\lambda}{a^2} D_{66}(Z3_{n,m} + nZ1_{n,m})$$

$$Z7_{n,m} = -\frac{1}{a^2}(D_{11}\lambda((\lambda Z1_{n,m}) - Z2_{n,m}) + D_{12} n(Z3_{n,m} + nZ1_{n,m}))$$

$$Z8_{n,m} = -\frac{1}{a^2}(D_{11}\lambda^2 Z1_{n,m} - D_{22} Z1_{n,m}(1-n^2))$$

$$Z9_{n,m} = -\frac{1}{a^2}(2D_{66}\lambda(nZ1_{n,m} - Z3_{n,m}))$$

$$Z10_{n,m} = A_{66}(\lambda Z3_{n,m} - nZ2_{n,m}) - \frac{nD_{66}}{a^2}(Z2_{n,m} + \lambda Z1_{n,m})$$

$$Z11_{n,m} = -\frac{1}{a^2} D_{66}(\lambda Z3_{n,m} + n(Z2_{n,m} + 2\lambda Z1_{n,m}))$$

$$Z1_{n,m} = a^2(C7\,n^2\lambda^2 + C8\lambda^4 + C9\,n^4)/DEN_{n,m}$$

$$Z2_{n,m} = a^2(-C10\lambda n^2 - C11\lambda^3 n^2 - C12\lambda^3 + C13\lambda^5 - C14\lambda n^4)/DEN_{n,m}$$

$$Z3_{n,m} = a^2(-C15\lambda^2 n - C16\lambda^4 n - C17\lambda^2 n^2 - C18\,n^3)/DEN_{n,m}$$

$$DEN_{n,m} = C19\lambda^2 n^2 - C20\lambda^2 n^4 + C21\lambda^2 n^6 + C22\lambda^6 n^2 + C23\lambda^4 n^4$$

$$- C24\lambda^4 n^2 + C25\lambda^4 + C26\lambda^8 - C27\lambda^6 + C28\,n^4 - C29\,n^6 + C30\,n^8 \quad \ldots (10)$$

Boundary conditions

The expressions in eqn. (8) imply that certain boundary conditions must exist at the ends of the vessel. Since the origin of the coordinate system is taken at one end of the cylinder (see fig. 1) all the Fourier expansions, or their derivatives containing the term sin $(\lambda x/a)$ vanish at the ends of the cylinder. This implies that:–

(a) The ends cannot deform in the plane of their profile.
(b) No rigid body displacement or rigid body rotation of the ends can occur.
(c) The ends cannot carry applied axial loading.
(d) Generatiors are free to rotate in a plane normal to the profile.

If the vessel ends conform to the above boundary conditions then the Fourier expansion solutions given in eqn. (9), are a complete solution to the problem. No complementary solution need be added to the particular solution since eqn. (9) satisfy both the governing difference eqn. (6) and the boundary conditions of the problem. In practice some deviation from these conditions is likely to occur, for example if the shell is a storage vessel with a flexible end closure, or a pipe with a less than rigid support. In such cases it is still possible to use the results of eqn. (9) with confidence if the local loading applied some distance from the vessel ends.

Fourier series representation of the applied loads

The only unknowns in eqn. (9) are the loading terms $p_{n,m}$. These terms are found by expressing the loading system in double Fourier Series form. This is achieved by multiplying both sides of eqn. (7) by suitable (orthogonal) expressions such as that integration over the surface or the cylinder eliminates all but one of the terms in each Fourier expansion.

To illustrate the method, consider a vessel subject to a radial pressure of p_r over all, or part of its surface. From eqn. (7) this loading is expressed in the form:

$$p_r = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} p_{n,m} \operatorname{Cos} n\phi \operatorname{Sin}(\lambda x/a)$$

$$p_r = \sum_{m=1}^{\infty} (p_{o,m} + \sum_{n=1}^{\infty} p_{n,m} \operatorname{Cos} n\phi) \operatorname{Sin} m\pi x/L$$

Since $\lambda = m\pi a/L$

To illustrate the procedure consider the case n = 0, that is:

$$p_r = \sum_{m=1}^{\infty} p_{o,m} \text{ Sin } m\pi x/L$$

Multiplying both sides of this equation by Sin $(m'\pi x/L)$ dx$d\phi$ and, since the loading is symmetric about the vertical diameter, nitegrating over half of the vessel surface (0 to π) we obtain.

$$\int_o^L \int_o^\pi p_r \text{ Sin } \frac{m'\pi x}{L} \, dx d\phi = \int_o^L \int_o^\pi \sum_{m=1}^{\infty} p_{o,m} \text{ Sin } \frac{m\pi x}{L} \text{ Sin } \frac{m'\pi x}{L} \, dx d\phi$$

Noting that,

$$\int_o^L \text{Sin } m\pi x \text{ Sin } m'\pi x \, dx = 0 \quad \text{when } m \neq m'$$

$$= \tfrac{1}{2} \quad \text{when } m = m'$$

Leads to,

$$p_{o,m} = \frac{2}{L\pi} \int_o^L \int_o^\pi p_r \text{ Sin } \frac{m\pi x}{L} \, dx d\phi \qquad (11a)$$

when,

$$n > 0, \; p_{n,m} = \frac{4}{L\pi} \int_o^L \int_o^\pi p_r \text{ Sin } \frac{m\pi x}{L} \text{ Cos } n\,\phi \, dx d\phi \qquad (11b)$$

Repeating this procedure for the other loading components, p_ϕ and p_x values of $p_{n,m}$ may be obtained for these cases.

A compendium of $p_{n,m}$ solutions is provided by Duthie and Tooth[5], for the full range of load conditions corresponding to the series given in eqn. (7).

Values of stress and strain in the layered system

The strain values are obtained using the compatibility conditions given is eqn. (2), the values for u, v, w from eqn. (8) and $u_{n,m}$, $v_{n,m}$ and $w_{n,m}$ from eqn. (10). In view of the Kirchoff-Love hypothesis these values are linear across the wall thickness.

The stress values in the k^{th} layer can be obtained by substituting the strain variation through the thickness, into the constitutive relations given in eqn. (1). Thus for any k^{th} layer, the stresses are given by the eqn.:

$$\begin{bmatrix} \sigma_x \\ \sigma_\phi \\ \tau_{x\phi} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \Sigma_x \\ \Sigma_y \\ \gamma_{xy} \end{bmatrix}_k$$

Since the stiffnesses Q_{ij} define the orientation of the individual lamina in a symmetric cross-ply lay-up, the appropriate stiffnesses and strain values must be used when deriving the stress for the particular layer considered. Examples of these are provided in ref. (6).

Concluding comments

The analysis presented provides a solution for the specially orthotropic shell, with through thickness symmetry, when subject to a general loading system. It is contended that the use of the linear elastic shell theory, neglecting transverse shear effects, is valid for moduli ratio confined to the range 1/10 to 10.

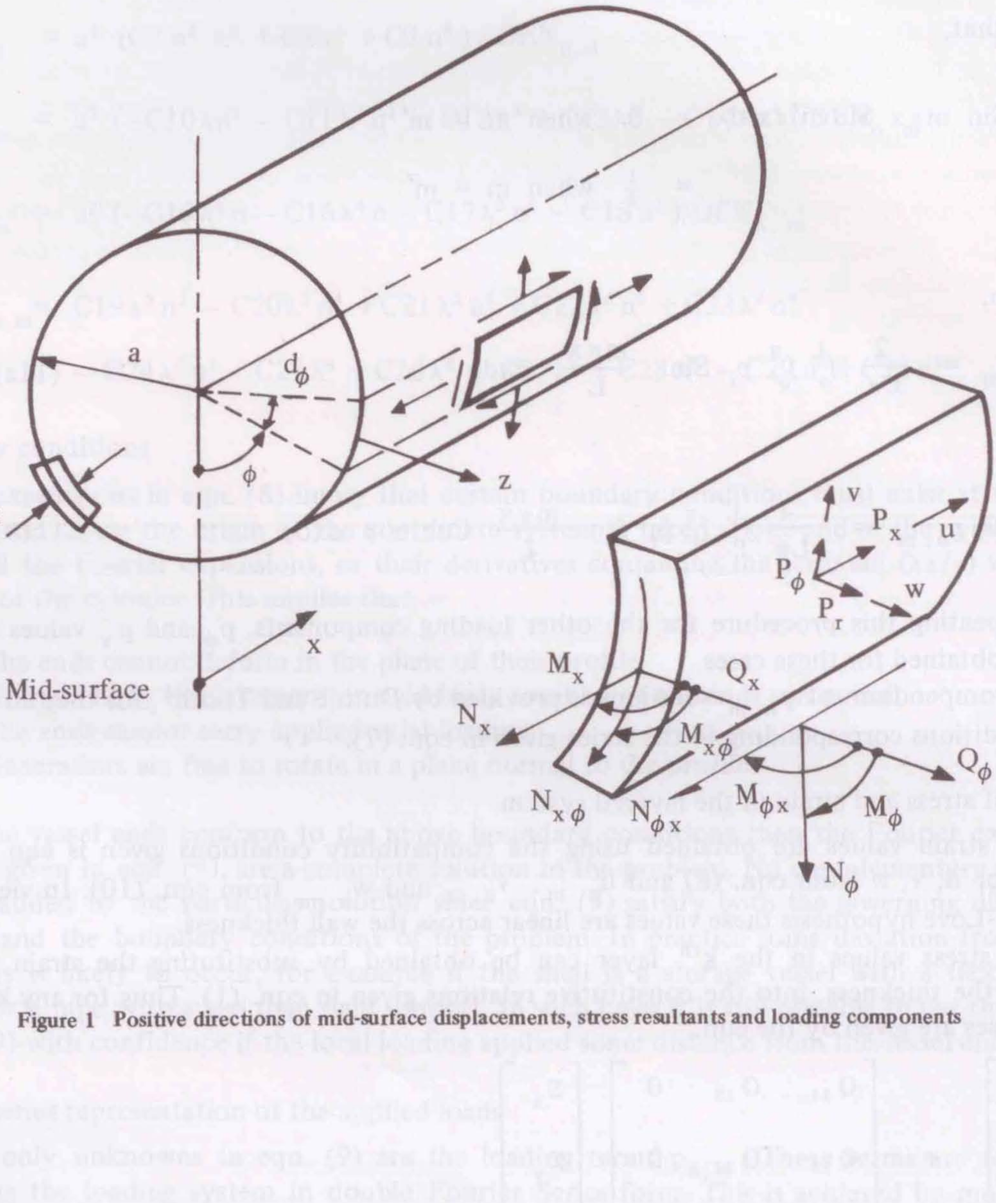

Figure 1 Positive directions of mid-surface displacements, stress resultants and loading components

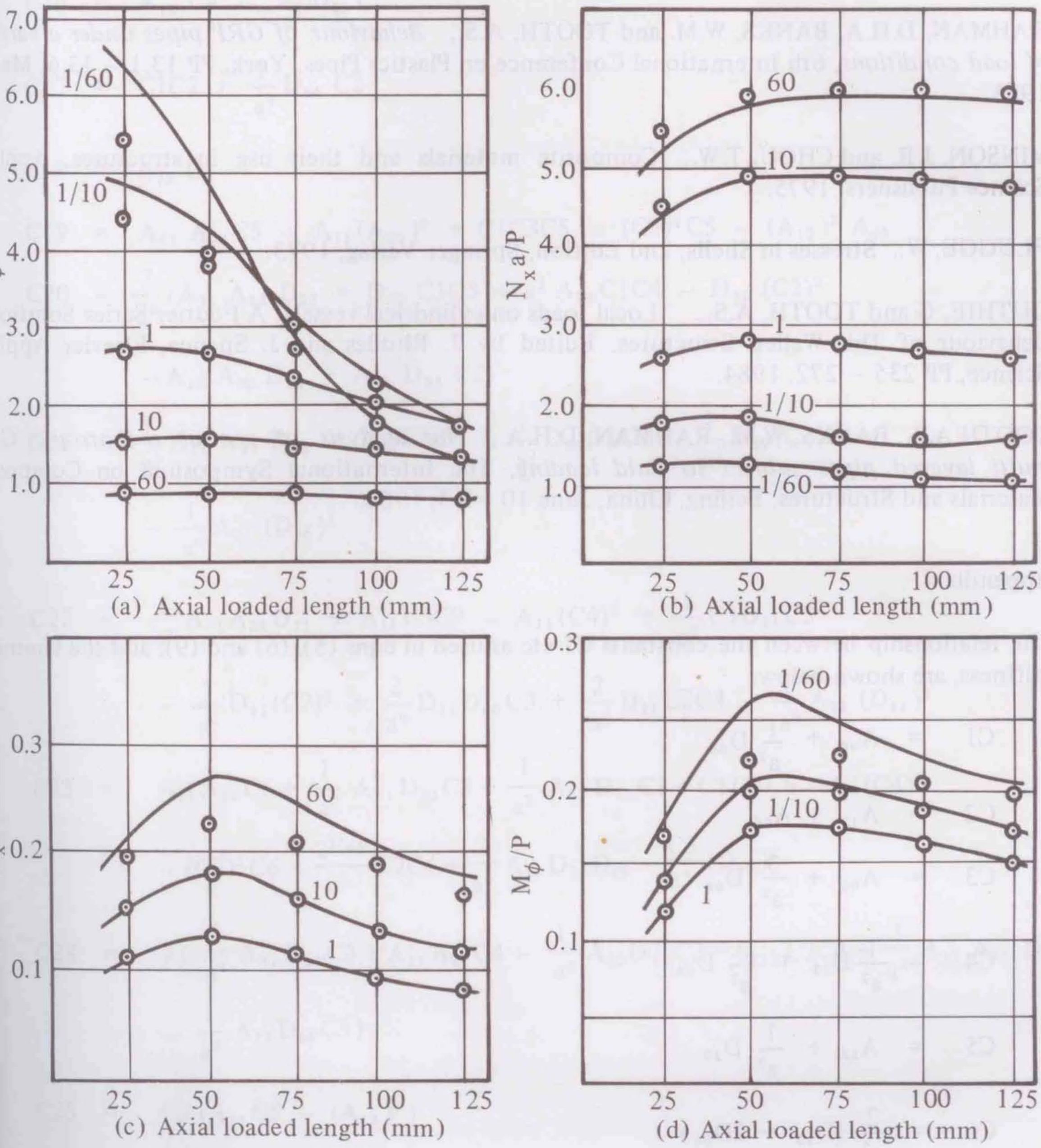

Figure 2 Dimensionless stress resultants, N_ϕ, M_x, M_ϕ, M_x as a function of axial length. The ratio shown are E_{11} / E_{22} with $E/G = 2.6$.

Vinson and Chou[4]

Elastic shell theory, neglecting transverse shear[3].

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Appendix 1

The relationship between the constants C1 etc as used in eqns.(5), (6) and (9), and the laminate stiffness, are shown below:

$$C1 = A_{66} + \frac{1}{a^2} D_{66}$$

$$C2 = A_{12} + A_{66}$$

$$C3 = A_{66} + \frac{3}{a^2} D_{66}$$

$$C4 = \frac{1}{a^2} D_{12} + \frac{3}{a^2} D_{66}$$

$$C5 = A_{22} + \frac{1}{a^2} D_{22}$$

$$C6 = \frac{2}{a^2} (D_{12} + 2D_{66})$$

$$C7 = A_{11} A_{22} + C1\,C3 - (C2)^2$$

$$C8 = A_{11}\,C3$$

$$C9 = A_{22}\,C3$$

$$C10 = A_{22} (A_{12} + A_{66}) - A_{12} A_{22}$$

$$C11 = C2\,C4 - \frac{1}{a^2} D_{11} A_{22} + \frac{1}{a^2} D_{66} C3$$

$$C12 = - A_{12}\,C3$$

$$C13 = \frac{1}{a^2} D_{11} C_6$$

$$C14 = \frac{1}{a^2} A_{22} D_{66}$$

$$C15 = A_{11} A_{22} - A_{12}\, C2$$

$$C16 = A_{11}\, C4 - \frac{1}{a^2} D_{11}\, C2$$

$$C17 = C1C2 + \frac{1}{a^2} D_{66}\, C2$$

$$C18 = A_{22}\, C_1$$

$$C19 = A_{11} A_{22} C5 - A_{11}(A_{22})^2 + C1C3C5 - (C2)^2 C5 - (A_{12})^2 A_{22}$$

$$C20 = \frac{2}{a^2}(A_{11} A_{22} D_{22} + D_{22}\, C1C3 + a^2 A_{22}\, C1C4 - D_{22}\,(C2)^2$$

$$- A_{12} A_{22} D_{66} + A_{22} D_{66}\, C2)$$

$$C21 = \frac{1}{a^2} A_{11} A_{22} D_{22} + A_{22}\, C1C6 + \frac{1}{a^2} D_{22}\, C1C3 - \frac{1}{a^2} D_{22}(C2)^2$$

$$- \frac{1}{a^4} A_{22}\,(D_{66})^2$$

$$C22 = \frac{1}{a^2} A_{11} A_{22} D_{11} + A_{11}\, C3C9 - A_{11}(C4)^2 + \frac{1}{a^2} C1 D_{11}\, C3$$

$$- \frac{1}{a^2} D_{11}(C2)^2 + \frac{2}{a^4} D_{11} D_{66}\, C3 + \frac{2}{a^2} D_{11}\, C2C4 - \frac{1}{a^4} A_{22}\,(D_{11})^2$$

$$C23 = A_{11} A_{22}\, C6 + \frac{1}{a^2} A_{11} D_{22}\, C3 + \frac{1}{a^2} A_{22} D_{11}\, C1 + C1C3C6 - C1(C4)^2$$

$$- (C2)^2 C6 - \frac{2^{D_{66}}}{a^2} C2C4 + \frac{2}{a^4} A_{22} D_{11} D_{66} - \frac{C3}{a^4}(D_{66})^2$$

$$C24 = 2\left(\frac{1}{a^2} A_{11} D_{22}\, C3 + A_{11} A_{22}\, C4 - \frac{1}{a^2} A_{22} D_{11}\, C2 - A_{12}\, C2C4 + \frac{1}{a^2} A_{12} A_{22} D_{11} \right.$$

$$\left. - \frac{1}{a^2} A_{12} D_{66}\, C3 \right)$$

$$C25 = C3\,(A_{11}\, C5 - (A_{12})^2)$$

$$C26 = \frac{1}{a^2} D_{11}\, C3\,(A_{11} - \frac{1}{a^2} D_{11})$$

$$C27 = \frac{2}{a^2} A_{12} D_{11}\, C3$$

$$C28 = C1\,(A_{22}\, C5 - (A_{22})^2)$$

$$C29 = \frac{2}{a^2} A_{22} D_{22}\, C1$$

$$C30 = \frac{1}{a^2} A_{22} D_{22}\, C1 = C29/2.$$

Notation

Q_{ij}	Lamina stiffness matrix
A_{ij}	Matrix of in-plane stifness for laminate, layers $k = 1$ to N $= \sum_{k=1}^{N} (Q_{ij})_k (Z_k - Z_{k-1})$
D_{ij}	Matrix of bending stiffness for limate, layers $k = 1$ to N $= \frac{1}{3} \sum_{k=1}^{N} (Q_{ij})_k (Z_k^3 - Z_{k-1}^3)$
C1, C2, etc	Constants which are functions of A_{ij} and D_{ij} or both, defined in Appendix 1.
x, ϕ, z	Coordinates in the axial, circumferential and radial directions – see figure 1
E_{11}/E_{22}	Ratio of elastic Moduli in the axial and circumferential directions for indivi layers
G_{12}	Shear modulus of layers in the x, ϕ plane.
ν_{ij}	Poisson's ratio for transverse strains in the j-direction when stressed in the i-direc
N_x, N_ϕ M_x, M_ϕ etc	Stress resultants – see figure 1.
n, m	Integer counters, denoting number of terms used in the Fourier series in the circ ferential and axial direction respectively.
$\sigma_x, \sigma_\phi, \tau_{x\phi}$	Normal and shear stresses in the axial and circumferential directions.
$\Sigma_x, \Sigma_\phi, \gamma_{x\phi}$	Normal and shear strains in the axial and circumferential directions.
u, v, w	Mid-surface shell displacements in x, ϕ and radial directions – see figure 1.
p_x, p_ϕ, p_r	Applied loading in the x, ϕ, and radial directions – see figure 1.
$p_{n,m}$	Loading terms employed in Fourier analysis
d, δ	Differentials with respect to x/a and ϕ respectively
a, L, t	Radius of mid-surface, length and thickness of cylindrical shell.
P	Total radial load on the loaded area.