OPTIMAL CONTROL OF A BILINEAR SYSTEM SUBJECTED TO A QUADRATIC COST FUNCTION USING LIE ALGEBRA

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Synopsis

Optimal control for a Biliner System subjected to a quadratic cost functional was derived by applying Lie Algerbra. Interesting results were obtained when the system matrice commute and when the Lie sub-algebra generated by the system matrices is nilpotent.

Introduction

Classical approach of calculus of variation and the Pontryagin Maximum Principle give an explicit expression for the optimal system of a linear quadratic regulator problem. An associated gain matrix is governed by a computable matrix Riccati equation. The existence and uniqueness of an optimal control to this problem is well known (1,5). On the other hand, relatively little has been reported in literature concerning determination of optimal controls of bilinear system. Notable exception includes work by Mohler[6]. Jacobson[4] and a few others. In this paper, results obtained by Banks and Yew[2] are applied to a Bilinear System.

Problem Formulation

The optimal control problem can be stated as follows: Given

i) a set of a first order differential equation which represent a time-invariant control system, known as the state equation, and are represented by the following compact vector form

$$x(t) = f(x,u,t), \qquad x(t_0) = x_0 \qquad \ldots(1)$$

where the vector function f(x,u,t) may be a bilinear function in the form

$$f(x,u,t) = (A + \sum_{i=1}^{m} u_i B_i)\, x(t) \qquad \ldots(2)$$

ii) a quadratic form scalar function known as the performance index or the cost functional, usually denoted by J(u),

$$J(u) = \int_{t_0}^{t_f} f_0(x,u)\, dt \qquad \ldots(3)$$

So, the optimal control problem is to find an optimal control function u*(t) which minimise the performance index J(u) by considering the Lie Algebra generated by the system matrices.

Optimal Control

Let us consider a Bilinear System

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad x(0) = x_o \qquad \ldots(4)$$

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where $x \in R^n$ and u_i's are scalar controls which minimise the simple cost functional

$$J(u) = \int_{t_o}^{t_f} \sum_{i=1}^{m} u_i^2(t)\,dt + x^T(t_f)\,F\,x(t_f) \quad \text{.... (5)}$$

The Hamiltonian for the optimal control problem subjected to (. . 4) is

$$H = \sum_{i=1}^{m} u_i^2 + \lambda^T (Ax + \sum_{i=1}^{m} u_i B_i) \quad \text{.... (6)}$$

By Maximum Principle, we obtain the equation

$$\frac{\partial H}{\partial x} = \lambda^T A + \lambda^T \sum_{i=1}^{m} u_i B_i$$

Therefore

$$\dot{\lambda}^T = -(\lambda^T A + \lambda^T \sum_{i=1}^{m} u_i B_i) \quad \text{.... (7)}$$

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x \quad \text{.... (8)}$$

and hence the optimal control

$$u_j = -\frac{1}{2} \lambda^T B_j x,\ 1 \leqslant j \leqslant m \quad \text{.... (9)}$$

where u_j is the j'th control.

Taking the derivative of (9)

$$-\dot{u}_j = \lambda^T [B_j, A]\,x + \lambda^T \sum_{i=1}^{m} u_i [B_j, B_i]\,x \quad \text{.... (10)}$$

Proposition 4

If $[B_j, B_i] = [B_j, A] = 0$, (i.e. B_j, B_i and A commute), then the optimal control u^* is a constant.

Proof:

From (7) and (8)

$$\frac{d}{dx}(\lambda^T B_j x) = \dot{\lambda}^T B_j x + \lambda^T B_j \dot{x}$$

$$= -(\lambda^T A + \lambda^T \sum_{i=1}^{m} u_i B_i) B_j x + \lambda^T B_j (Ax + \sum_{i=1}^{m} u_i B_i x)$$

$$= \lambda^T [B_j, A]\,x + \lambda^T \sum_{i=1}^{m} [B_j, B_i]\,x \quad \text{.... (11)}$$

and so

$$\dot{u} = 0$$

if $[B_j, A] = [B_j, B_i] = 0$

Proposition 5

Under the assumption of proposition 4, the constant optimal controls, uj* are the solution of the equation

$$2u + x_o^T \exp\left(\left(AT + \sum_{i=1}^{m} u_i B_i\right) T\right) (B_j TF + FB_j) \exp\left(\left(A + \sum_{i=1}^{m} u_i B_i\right) T\right) x_o = 0 \quad \text{.... (12)}$$

Proof:

Since u_j*'s are constant, we have

$$J(u^*) = \sum_{i=1}^{m} u_i^{*2} T + x^T(t_f) F x(t_f)$$

However

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i B_i\right) x$$

and so

$$x(T) = \exp\left(\left(A + \sum_{i=1}^{m} u_i^* B_i\right) T\right) x_o$$

Hence

$$J(u^*) = \sum_{i=1}^{m} u_i^{*2} T + x_o^T \left(\exp\left(\left(AT + \sum_{i=1}^{m} u_i^* B_i^T\right) T\right)\right) F.$$

$$\left(\exp\left(\left(A + \sum_{i=1}^{m} u_i^* B_i\right) T\right)\right) x_o$$

$$\frac{\partial J(u^*)}{\partial u_j} = 2u_j^* T + Tx^T \left(\exp\left(\left(A^T + \sum_{i=1}^{m} u_i B_i^T\right) T\right)\right).$$

$$(B_j^T F + FB_j)\left(\exp\left(\left(A + \sum_{i=1}^{m} u_i B_i\right) T\right)\right) x_o$$

Since A and B_j's commute.

It is very clear that the condition $[B_j, A] = [B_j, B_i] = 0$ is very strong. Naturally we seek similar conditions on the control to those above when $[B_j, A] \neq 0$ and $[B_j, B_i] \neq 0$. Of course u_j's will no longer be constants.

Lemma 6

For any (n by n) matrix X we have

$$\frac{d(\lambda^T X x)}{dt} = \lambda^T \left[X, A + \sum_{i=1}^{m} u_i B_i\right] x$$

$$= \lambda^T [X, A] x + \sum_{i=1}^{m} u_i \lambda^T [X, B_i] x \quad \text{.... (13)}$$

Proof:

From (7) and (8), we have

$$\dot{\lambda}^T X x = -\lambda^T \left(A + \sum_{i=1}^{m} u_i B_i\right) X x$$

$$\lambda^T X\dot{x} = \lambda^T X \left(A + \sum_{i=1}^{m} u_i B_i \right) x$$

Hence

$$\therefore \frac{d(\lambda^T X x)}{dt} = \dot{\lambda}^T X x + \lambda^T X \dot{x}$$

$$= -\lambda^T \left(A + \sum_{i=1}^{m} u_i B_i \right) X x + \lambda^T X \left(A + \sum_{i=1}^{m} u_i B_i \right) x$$

$$= \lambda^T [X, A]\, x + \sum_{i=1}^{m} u_i \lambda^T [X, B_i]\, X$$

Now consider the Lie Algebra M of all n by n matrices and let $M(A, B_1, \ldots\ldots, B_m)$ denote the Lie subalgebra generated by $A, B_1, \ldots\ldots, B_m$. Thus $M(A, B_1, \ldots\ldots, B_m)$ consists of $A, B_1, \ldots, B_m$ and their combination. Since M is a finite-dimensional Lie Algebra (of dimension n^2), $M(A, B_1, \ldots, B_m)$ must be finite dimensional with dimension $m < n^2$. Let $x_1, \ldots, x_n$ be a basis of $M(A, B_1, \ldots, B_m)$, and write

$$v_1 = 2u_1 = \lambda^T B_1 x$$

$$\vdots$$

$$v_k = 2u_k = \lambda^T B_k x, \quad 1 \leq k \leq m \qquad \text{.... (14)}$$

$$v_{m+1} = \lambda^T X_1 x$$

$$\vdots$$

$$v_n = \lambda^T X_{n-m} x$$

$$\vdots$$

$$v_{n+m} = \lambda^T X_n x$$

Then

$$\dot{v}_1 = \lambda^T [B_1, A]\, x + \lambda^t \sum_{i=2}^{m} u_i [B_1, B_i]\, x$$

$$\vdots$$

$$\dot{v}_k = \lambda^T [B_k, A] + \lambda^T \sum_{\substack{i=1 \\ i \neq k}}^{m} u_i [B_k, B_i]\, x, \quad 1 \leqslant k \leqslant m$$

$$\vdots$$

$$\dot{v}_{m+1} = \lambda^T [X_1, A]\, x + \lambda^T \sum_{i=1}^{m} u_i [X_1, B_i]\, x \qquad \text{.... (15)}$$

$$\dot{v}_n = \lambda^T [X_{n-m}, A] x + \lambda^T \sum_{i=1}^{m} u_i [X_{n-m}, B_i] x$$

$$\dot{v}_{n+m} = \lambda^T [X_n, A] x + \lambda^T \sum_{i=1}^{m} u_i [X_n, B_i] x$$

If $[X_i, A]$, $[X_i, B_k]$, $[B_k, B_i]$, $[B_k, A]$ belongs to $M(A, B_1, \ldots, B_m)$ we may write

$$[X_i, A] = \sum_{j=1}^{m} \alpha_{ij} X_j$$

$$[X_i, B] = \sum_{j=1}^{m} \beta_{kij} X_j \quad \text{.... (16)}$$

$$[B_k, A] = \sum_{j=1}^{m} b_{ki} X_j$$

$$[B_k, B_i] = \sum_{j=1}^{m} b_{kij} X_j \quad \text{.... (17)}$$

for some constants α_{ij}, β_{kij}, b_{kj}, b_{kij}. Substituting (17) and (16) into (15), we have

$$\dot{v}_k = \sum_{j=1}^{m} b_{kj} v_{m+j} + \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{1}{2} v_i \sum_{j=1}^{m} b_{kij} v_{m+j} \quad 1 \leq k \leq m \quad \text{.... (18)}$$

$$\dot{v}_1 = \sum_{j=1}^{m} \alpha_{1j} v_{j+m} + \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{1}{2} v_i \sum_{j=1}^{m} \beta_{i1j} v_{j+m}$$

These equations (18) may be written in the form

$$\dot{v} = f(v) \quad \text{.... (19)}$$

for some (non-linear) function of v. We may solve equation (19) numerically for $u_1, u_2, \ldots, u_m$ in terms of t_o, t_f and some initial value v_o, so we may write

$$u_j(t) = U_j(t: t_o, t_f, v_o), \quad 1 \leq j \leq m \quad \text{.... (20)}$$

where

$$v_o = (v_1(t_o), v_2(t_o), \ldots, v_{m+n}(t_o))$$

Substituting (20) into (4) we have

$$\dot{x} = A_x + \sum_{j=1}^{m} U_j B_j x, \; x(t_o) = x_o \quad \ldots (21)$$

Solving (21) numerically, then we have

$$x = \xi(t; t_o, t_f, v_o) \quad \ldots (22)$$

Substituting (22) and (20) into cost functional (5), we have

$$J(v) = \int_{t_o}^{t_f} \sum_{j=1}^{m} U_j^2 (t; t_o, t_f, v_o)\, dt + \xi^T (t; t_o, t_f, v_o) F \xi (t; t_o, t_f, v_o) \quad \ldots (23)$$

and we minimise J with respect to v_o to obtain control initial value $1/2\, u_j^* (t_o)$.

Proposition 9

If $X \in (\mathrm{Ad}\, M(A, B_1, \ldots, B_m)^1 B_1$ then

$$\frac{d(\lambda^T X x)}{dt} = \lambda^T Y x + \sum_{i=1}^{m} u_i \lambda^T Z_i x$$

where $Y, Z_1, \ldots, Z_m \in (\mathrm{Ad}\, M(A, B_1, \ldots, B_m)^{1+1} B_1$

Proof: This follows from Lemma 6

Corollory 11

If $M(A, B_1, \ldots, B_m)$ is nilpotent and $(\mathrm{Ad}\, M(A, B_1, \ldots B_m))^{k=0}$, then

$$\lambda^T X x = 0$$

for any $X \in (\mathrm{Ad}\, M(A, B_1, \ldots, B_m))^{k-1}$

Applying the descending central series, the basis $X_1, X_2, \ldots, X_m$ can be chosen[2]. Using this basis of $M(A, B_1, \ldots, B_m)$ it is easy to see that (18) takes the form

$$\dot{v}_k = b_k^T v + \sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{1}{2} v_i b_{ki}^T v, \quad 1 \leqslant k \leqslant m \quad \ldots (24)$$

$$\dot{v} = \tau v + \sum_{i=1}^{m} \frac{1}{2} v_{i \triangle i} v \quad \ldots (25)$$

where $\tau, {}_{\triangle i}$ are nilpotent matrices, and

$$b_k = (b_{k1}, \ldots, b_{km})$$
$$b_{ki} = (b_{1ki}, \ldots, b_{mki})$$

Conclusion

In this paper, we have considered the minimum fuel problem for a Bilinear System with m-controls and have shown how to obtain the optimal control by considering the Lie Algebra generated by the system matrices. We have shown that if A and B_i's commute, then the optimal controls are constant and if $M(A, B_1, \ldots, B_m)$ is a nilpotent Lie Algebra then the optimal controls can be solved numerically.

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