CONCEPT OF THE DECOMPOSITION AND AGGREGATION METHOD WITH EXAMPLE: PART 1

Dr. Zainol Anuar Mohd. Sharif
&
Ng Boon Choong
Fakulti Kejuruteraan Jentera
Universiti Teknologi Malaysia

Synopsis

This paper describes the basic concept of the decomposition and aggregation method. It shows the feasibility of the method and its advantages when applied, particularly to large scale systems. This method is extensively used in solving problems related to control engineering, economics, optimization and stability. This paper also illustrates specifically the application of the method of decomposition and aggregation in the analysis of dynamic systems. It is divided into two important parts, namely; the decomposition part which involves breaking up a large system into subsystems and the aggregation part which is obtained through a reformulation of the *Liapunov's* second method (direct method). The relation between the decomposition and the aggregation methods is also shown. The procedure for checking the stability based on this concept is also outlined. For further illustration, an example of a dynamic system has been included. It shows how the system is decomposed and aggregated to suit the requirement for stability analysis.

Introduction

One of the foremost challanges to system theory brought forth by present day technological, environment and societal processes is to overcome the increasing size and complexity of relevant mathematical models especially the dynamic systems. It is found that the analysis of stability, control, optimisation and other characteristics are difficult to solve especially when it involves systems of high dimension. They are either difficult or impossible to solve. It will be very troublesome trying to solve the problems of complex large scale dynamic systems in single step approach.

The engineer with a background in traditional engineering analysis is continually frustrated by the fact that many of his most valuable tools are of limited value in the treatment of complex and high order systems.

Recently, a method of studying the stability, control and optimisation of dynamic systems involving decomposition and aggregation is used. This method involves breaking up large systems into a number of subsystems which are interconnected. The solution of this subproblems are later combined using the interconnection constrains by applying the aggregation principles to solve the original systems.

The discussion in this paper will be limited to stability analysis of dynamic systems which can be modelled by ordinary differential equations.

There are few advantages in using this method when applied to large scale dynamic systems. Putting in point form the *advantages* are:

1. Solve the stability problems *piece by piece* instead of a single step solution.
2. Reduce the liability of errors in the analysis.
3. Provides a clearer structural information of large scale systems which is unobtainable by using conventional method.
4. Easier to solve due to the reduction in dimension.

General Aspects of Dynamic Systems

In analysing the stability of dynamic systems, initially it is appropriate to know what is a dynamic system, the nature of dynamic system and how to analyse the structure of dynamic systems.

Dynamic Systems

It is also called a continuous flow system. Because of the nature of the dynamic systems, the stability analysis of this type of system is essential to ensure that a particular dynamic system is controllable. Structural and stability analysis of dynamic systems is complex because dynamic system, particularly large scale dynamic systems, are usually composed of a number of interconnected subsystems. Under structural perturbations, the interconnected subsystems will face changes, disconnection and various kinds of environmental effects.

Structure

Structure of systems are very important and it is essential to understand the behavior of a particular system before starting to analyze the problem concerning that particular system. The basic structural elements of the formulation are usually described by simple matrices. They are combined as directed graphs and networks to create various structure. An autonomous, or free and stationary system is represented by the following expression;

$$\dot{x} = A\,x \qquad \text{...(1)}$$

where, A = provides information about the structure of the model.

a_{ij} = Elements of the matrix A which specify quantitatively the interconnection or relationship of the agents represented by state $(x_i\, x_j)$.

Example: A pair of interacting agents

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 \qquad \text{...(2)}$$

which can be represented by graph as:

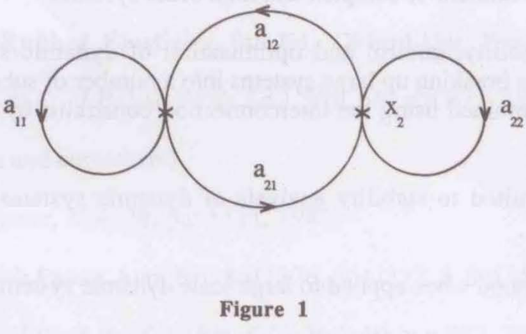


Figure 1

This simultaneous first-order differential equation can be expressed in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{...(3)}$$

The 2 x 2 matrix below is the representation of the adjacency matrix.

$$\text{Representation of adjacency matrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{...(4)}$$

The adjacency matrix can only be expressed in terms of value 1 or 0.
For example:

If $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, there is no indication of interconnection between x_2 and x_1 which mean x_2 does not act on x_1

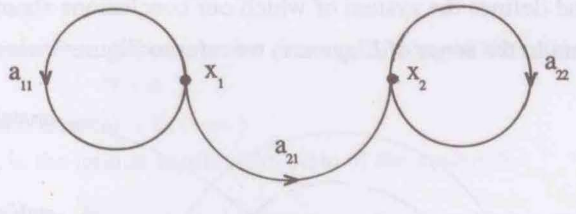


Figure 2

This shows that matrix C represents the basic structure of the system.

Stability Analysis

There are several methods which are applicable for analysing the stability of dynamic systems. For example, the *Routh-Hurwitz* stability criteria which are applicable to linear invariant systems, the generalized *Nyquist* and the *Liapunov* methods which are applicable to both linear and nonlinear systems.

Stability is understood as a situation where the system is in equilibrium and when perturbed, it will return to equilibrium on time. This means that the deviation between the perturbed process and the equilibrium approaches zero as time goes on ($t \rightarrow \infty$) as shown in Figure 3[1]. Again consider the system (free system);

$\dot{x} = A x$
for any initial time $t = t_0$
Initial state $= x_0 = x(t_0)$

It can be stated as : $x(t; t_0, x_0)$. So, the distance between the pertubed process $x(t: t_0, x_0)$ and the equilibrium $x = 0$ is

$$\| x \| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} \qquad \ldots(5)$$

or $\| x \| < R$ (Refer to Figure 3)

So stability is defined mathematically as:

$$x(t; t_0, x_0) \rightarrow 0 \text{ as } t \rightarrow +\infty$$

The most powerful method of solving such problem on stability is by using the second method of *Liapunov* which is also called the direct method of *Liapunov*. This method also gives sufficient condition for:

(i) Asymptotic stability of equilibrium states.

(ii) Designing optimal control systems that are stable.

Consider a system.

$$\dot{x} = f(x, t) \qquad \ldots(6)$$

Equation (5) can be written as $x(t; x_0, t_0)$, where $x = x_0$ at $t = t_0$, where t = observed time.

Equilibrium States

The equation (6) is said to be in equilibrium if there exixts a state x_e where $f(x_e, t) = 0$ for all t.

If the system $\dot{x} = f(x, t)$ is nonlinear, the region where the assumption are valid may be limited to:

$\| x \| < R$ (refer to equation 5)

where, $\| x \| = (x_1^2 + x_2^2 + ...x_n^2)^{1/2}$

R is some finite, real, positive number and defines the system of which our conclusions about the system stability will be valid. In order to define the stability of the system in the sense of *Liapunov*, we refer to Figure 3 where $\delta < \mathcal{E} < R$.

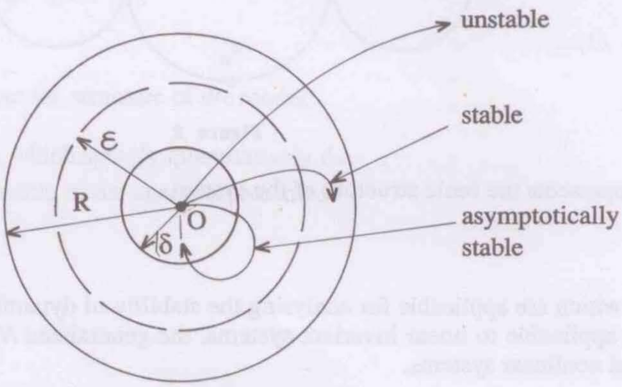


Figure 3 Definition of Stability

Stable: The system is said to be stable if for every radius δ, there exists a radius $\mathcal{E}$ and trajectory which starts inside the radius δ region, will remain in the radius $\mathcal{E}$ region.

Asymptotically Stable : An equilibrium state x_0 of a free dynamic system is asymptotically stable if,

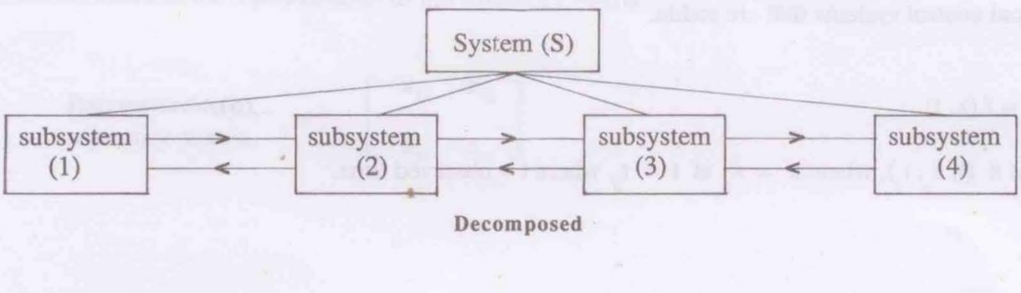
(i) It is stable and,

(ii) Every motion starting sufficiently near x_0 as $t \longrightarrow \infty$. In other words trajectory which starts within the region radius δ in the state space moves to the origin equilibrium as $t \longrightarrow \infty$ [2].

Decomposition

The decomposition principle has been known to be used since 1950 by *Kron* (1963) in the analysis of electrical networks. It was reported that this method is being used since 1843 by *Gerling* concerning predominant principle diagonal [3]. It is extensively used in complete dynamic systems consisting of interacting elements where this complex dynamic system is decomposed into smaller parts which are called subsystems. This simplifies the procedure of solving a complex system just by solving the subsystem which are of lower dimensionality and less complicated. The outcome of the subsystems solutions will later be analysed and combined to obtain a final solution by using the aggregation method for the system.

A basic evaluation by diagram on decomposition is shown below:



Decomposed

subsystem (1) | subsystem (2) | subsystem (3) | subsystem (4)

Decoupled

The system is said to be decoupled if there are no interactions between the subsystems. Mathematically, a continuous dynamic system S described by vector differential equation is:

$$\dot{x} = f(t, x) \qquad \text{...(6)}$$

where, $x(t) \in R^n$ is the state of the system. Assuming that this expression staisfies the condition:

$f(t, 0) = 0, \quad \forall t \in T$

where, T = time interval $(\tau + \infty)$

$x = 0$, is the unique equilibrium state of the system S.

One of the most important factor in stability analysis is the assumption in which a system S is decomposed into subsystem S_i shown as follows:

Interconnected subsystem:

$$\dot{x} = f_i(t, x_i) + C_i(t, x) \; ; \; i = 1, 2, \ldots s \qquad \text{...(7)}$$

The decoupled, free systems are

$$x = f_i(t, x_i); \; i = 1, 2, \ldots s \qquad \text{..(8)}$$

where $x(t) \in R^{ni}$ is the state of S_i.

Consider the general equation (1.0), where;

$$\dot{x} = A x$$

The state vector x can further be decomposed into two components; $x \longrightarrow x_1, x_2$.

Both x_1 and x_2 are state vectors with:

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ \cdot \\ \cdot \\ \cdot \\ x_{1n} \end{bmatrix} \qquad x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ \cdot \\ \cdot \\ \cdot \\ x_{2n} \end{bmatrix}$$

Thus, $\dot{x} = A x$

$$\dot{x} = A[x_1, x_2] \qquad \text{...(9)}$$

where, A is the m x n matrix

x is the vector with n elements.

If the matrix A is a large matrix, it ca be partitioned but the partitioning is related to the numbers in x_{1n} or x_{2n} in the state vector x_1 and x_2 [4].

$$A = \left[\begin{array}{cccccc} a_{11} & a_{12} & \vdots & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{23} & & \cdot \\ \hline a_{31} & a_{32} & \vdots & a_{33} & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{m1} & \cdot & \vdots & \cdots & & a_{mn} \end{array}\right]$$

If the matrix is partitioned into four parts;

$$A = \left[\begin{array}{cccccc} a_{11} & \cdots & \vdots & \cdots & a_{1n} \\ \cdot & & \vdots & & \cdot \\ \hline \cdot & & \vdots & & \cdot \\ a_{m1} & \cdots & \vdots & \cdots & a_{mn} \end{array}\right]$$

So, considering a small matrix, equation (8) becomes

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which can be decomposed into :

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2$$

and can be decoupled into:

$$\dot{x}_1 = A_{11}x_1$$
$$\dot{x}_2 = A_{22}x_2$$

The value of $A_{12}x_2$ and $A_{21}x_1$ are the interconnections between the subsystem.

Aggregation

The concept of aggregation is a very useful method if it is applied to a large system stability analysis. In stability analysis of large scale systems, a straightforward approach by a single *Liapunov* function becomes cumbersome and another level of aggregation is desirable. After a large-scale system is decomposed into a number of interconnected subsystems, a *Liapunov* function is applied to each isolated subsystems. The subsystem *Liapunov* function is later used as components of a vector *Liapunov* function for constructing the aggregate model of the overall system. Stability is then analyse with regard to certain constraints. In forming the aggregate model, it is very important to consider approximations where the conservativeness of the overall result is increased. This aggregation approach will reduce the complexity in the stability analysis of the complex problems. As a result, certain portion of information from the original problem will have to be sacrificed for the purpose of the simplified version. This will certainly affects the accuracy of the result. Researchers and economists however are studying and searching the posibilities for *consistent aggregation* where aggregation is performed but still maintains the accuracy of the result.

Liapunov's Second Method and Aggregation

One of the most important events in the theory of stability analysis of dynamic systems was the publication of *Liapunov's* famous memoire in a Russian journal in 1892. Translated into French in 1947 and reprinted in America in 1947, it is still unappreciated in the west and also many parts of the world [5]. The *Liapunov's* method itself can be viewed as the aggregation process. Let's consider a linear and a system not bounded by control (Equation (6)).

$$\dot{x} = f(x, t)$$

or

$$\dot{x} = f(x)$$

Here, the basic idea of aggregation method described is to analyse whether the system of form (6) is asymptotically stable.

Example:
Assuming equation (6) is decomposed into two subsystems as discussed in the decomposition section;

$$\dot{x}_1 = A_1 x_1 + A_{21} x_2$$

$$\dot{x}_2 = A_2 x_2 + A_{21} x_1$$

The free decoupled subsystem are

$$\dot{x}_1 = A_1 \; x_1$$

$$x_2 = A_2 \; x_2 \qquad ...(11)$$

The aggregated model of the above system may then be formed by combining the individual *Liapunov* function into a vector form (vector *Liapunov* function).

Steps to Analyse Stability of a Particular System

The first step is to consider the stability of the decoupled subsystem as shown in equation (11). If the decoupled subsystem is unstable, this means the overall system is not stable, but if the decoupled subsystems are found to be stable, proceed with the analysis of the interconnected subsystems. The stability will depend on the solution of the interconnected subsystem. There are few important theorems which are related to the analysis of the stability using the *Liapunov* second method.

(a) The equilibrium state x = 0 of the system $x = A_x$ is asymptotically stable if and only if given any positive real symmetric matrix Q, there exists a positive definite real symmetric matrix P such that:

$$A^T P + PA = - Q \qquad ...(12)$$

Assuming the scalar *Liapunov* function of the system (6) is

$$V = (x^T Px)^{1/2} \qquad ...(13.1)$$

or

$$V = \| x \|^2 P \qquad ...(13.2)$$

where

$$x = (\sum_{i=1}^{n} x_i^2)^{1/2} = (x^T x)^{1/2}$$

The values of P in the equation 13.1, namely P_1 and P_2, can be obtained from equation (12). The *Liapunov* scalar function can be stated as:

$$V_1 (x_1) = (x_1^T P_1 x_1)^{1/2}$$

$$V_2 (x_2) = (x_2^T P_2 X_2)^{1/2} \qquad ...(14)$$

(b) The estimates of the *Liapunov* function $V_1(x_1)$ and $V_2(x_2)$ are [6]:

$$n_{i1} \| x \| \leqslant V < n_{i2} \| x \|$$
$$\dot{v}_i \leqslant - n_{i3} \| x_i \|$$
$$\| \text{grad } V_i \| \leqslant n_{i4} \qquad \text{...(15)}$$

where;

n_{ij} = positive numbers and $x = (x^T x)^{1/2}$

$\dot{v}_i$ = total time derivative of the function $V_i(t, x_i)$

From the assumptions given, we can directly obtain by calculations;

$$n_{i1} = \lambda^{1/2}(P_i)$$
$$n_{i2} = \Lambda^{1/2}(P_i) \qquad \text{...(16)}$$
$$n_{i3} = 1/2\, \lambda(Q_i)\, \Lambda^{-1/2}(P_i)$$
$$n_{i4} = \lambda^{1/2}(P_i)\, \Lambda(P_i)$$

where; λ = minimum eigen values of the indicated matrices.
Λ = maximum eigen values of the indicated matrices.

To know whether the decoupled subsystem is in a stable condition as prescribed earlier, check if the eigen values of the matrices P_1 and P_2 are positive definite matrices. Later proceed on analysing the overall stability by considering the interactions between subsystems.

If $\dot{v}_i$ is the total time derivative of the function $V(t, x_i)$ along the motion of the free subsystem $S_i S_o$, when $V = V(x, t) = V(x(t), t)$, thus

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} \cdot \frac{\partial t}{\partial t} + \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial t}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x}$$

$$\dot{v}_i = \frac{\partial V}{\partial t} + (\text{grad } V)^T \dot{x} \qquad \text{...(17)}$$

So, when $\dot{x}_i = f_i(t, x_i) + C_i(t, x)$; $i = 1, 2.....s$

Thus,

$$\dot{\underset{\sim}{V}}_1 = \dot{V}_i + (\text{grad } V_i)^T C_i(t, x) \qquad \text{...(18)}$$

from equation (17).

Analysing the overall stability with the consideration of the interconnection between the subsystems

Assumptions

Assume that the interactions among subsystems follow the constraint:

$$\| C_i\ (t, x)\ \| < \sum_{j=1}^{s} \bar{\xi}\ \| x_j \| \qquad ...(19.1)$$

where $\bar{\xi}$ = non negative numbers.

As example, if the interactions are linear and time invariant

$$C_i\ (t, x) = \sum_{j=1}^{s} \bar{\xi}\ J_{ij} x_j \qquad ...(19.2)$$

where, J_{ij} = constant coefficient matrices.

then, $$\bar{\xi}_{ij} = [\ \Lambda(J_{ij}^{\ T} J_{ij})\ ]^{1/2} \qquad ...(20)$$

(Λ indicates the maximum eigenvalues, as prescribed before)

The scalar inequalities can be written in scalar form as:

$$\dot{v}_i \leqslant Gv \qquad ...(21)$$

where the s x s matrix G has elements g_{ij} specified by;

$$g_{ij} = -\delta'_{ij} n_{i2}^{-1} n_{i3} + \bar{\xi}_{ij} n_{j1}^{-1}\ n_{i4} \qquad ...(22)$$

where δ' is a kronecker delta. The equation (21) is known as the aggregation model of the system S. The matrix G is known as the aggregate matrix. In order to obtain the *aggregate model* of the system S using *Liapunov* functions $V_i\ (t, x_i)$, we apply the estimates of equation (16) and equation (19) into equation (18). So;

$$\dot{\underset{\sim i}{V}} = \dot{V}_i + (\text{grad}\ V_i)^T\ C_i\ (t, x)$$

$$\Rightarrow \quad \dot{\underset{\sim i}{V}} = -n_{i2}^{-1} n_{i3} V_i + n_{i4}\ \bar{\xi}_{ij} V_j n_{ji}^{-1/2} \qquad ...(23)$$

where, i = 1, 2...s

Solution of the Liapunov Matrix Equation

Direct Method Solution

By considering the equation (equation (12));

$$A^T P + PA = -Q$$

for determining the symmetric matrix P, in which A is a real constant n x n matrix and Q is an arbitrary symmetric positive definite matrix. Since P and Q are symmetric, equation (12) can be reduced to a set of 1/2n (n+1) distinct elements of P. An illustration is shown using the simplest case of 2 x 2 matrices.[7]

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

and this matrix can easily reduced to three equations:

$$2a_{11}P_{11} + 2a_{21}P_{12} = -q_{11}$$

$$a_{12}P_{11} + (a_{11} + a_{22})P_{12} + a_{21}P_{22} = -q_{12}$$

$$2a_{12}P_{12} + 2a_{22}P_{22} = -q_{22}$$

It can be expressed as;

$$Bp' = -q' \qquad ...(24)$$

where,

$$B = \begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11} + a_{22} & a_{21} \\ 0 & 2a_{12} & 2a_{22} \end{bmatrix}$$

and $p' = (p_{11}, p_{12}, p_{22})$ $\qquad q' = (q_{11}, q_{12}, q_{22})$

In general, the n x n matrices p and q are given by;

$$p' = (p_{11}, p_{12}, p_{22}, p_{13}, p_{23}, p_{33}...)$$

$$q' = (q_{11}, q_{12}, q_{22}, q_{13}, q_{23}, q_{33}...)$$

and in general, B is formed from A as shown:

$$B = \begin{bmatrix} 2a_{11} & 2a_{21} & 0 & 2a_{31} & 0 & 0 & \\ a_{12} & (a_{11} + a_{22}) & a_{21} & a_{32} & 0 & 0 & ... \\ 0 & 2a_{12} & 2a_{22} & 0 & 2a_{32} & 0 & ... \\ a_{13} & a_{23} & 0 & (a_{11} + a_{33}) & a_{21} & a_{31} & ... \\ 0 & a_{13} & a_{23} & a_{12} & (a_{22} + a_{33}) & 2a_{33} & ... \\ 0 & 0 & 0 & 2a_{13} & 2a_{23} & 2a_{33} & ... \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \\ . & . & . & . & . & . & \end{bmatrix}$$

So since

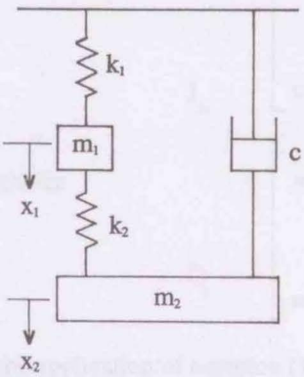
$$Bp = -q$$

$$p = -B^{-1}q$$

However, for large values of n, the direct method is unsuitable due to the impracticability of evaluating determinants of large order even with a computer. A much better method is by using the method known as the skewsymmetric matrix reduction[4].

Example on Decomposition and Aggregation

Consider a free vibrating system



The general equation becomes :

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -C\dot{x}_2 - k_2(x_2 - x_1)$$

Decomposition

$$\ddot{x}_1 = -(k_1 + k_2)/m_1x_1 + k_2/m_1x_2$$

$$\dot{x}_2 = k_2/m_2x_1 - k_2/m_2x_2 - C/m_2\dot{x}_2$$

Assuming $x_{11} = x_1$; $x_{12} = \dot{x}_1$; $x_{21} = x_2$; $x_{22} = \dot{x}_2$

So,

$$\dot{x}_{11} = x_{12}$$

$$\dot{x}_{12} = -(k_1 + k_2)\,x_{11} + k_2/m_1x_{21}$$

$$\dot{x}_{21} = x_{22}$$

$$\dot{x}_{22} = k_2/m_2x_{11} - k_2/m_2x_{21} - C/m_2x_{22}$$

$$S_1 : \begin{bmatrix}\dot{x}_{11}\\ \dot{x}_{12}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -(k_1+k_2)/m_1 & 0\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix} + \begin{bmatrix}0 & 0\\ k_2/m & 0\end{bmatrix}\begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}$$

$$S_2 : \begin{bmatrix}\dot{x}_{21}\\ \dot{x}_{22}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k_2/m_2 & -C/m_2\end{bmatrix}\begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix} + \begin{bmatrix}0 & 0\\ k_2/m_2 & 0\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}$$

The free subsystem is:

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(k_1 + k_2)/m_1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2/m_2 & -C/m_2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

Interactions among the subsystems are represented by the functions:

$$C_1(t, x) = \begin{bmatrix} 0 & 0 \\ k_1/m_1 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$C_2(t, x) = \begin{bmatrix} 0 & 1 \\ k_2/m_2 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

Aggregation

Take the example of the second subsystem:

$$\begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2/m_2 & -C/m_2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

Assuming that $k_2/m_2 = 4 \quad C/m_2 = 5 \quad k_1/m_1 = 2$

So applying the formula $A^T P + PA = -Q$, where Q is the identity matrix.

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} p_{12} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2(0)p_{11} + 2(-4)p_{12} = -1$$
$$p_{11} + (0-5)p_{12} - 4p_{22} = 0$$
$$2(1)p_{12} + 2(-5)p_{22} = -1$$

$$p_{11} = 9/8$$
$$p_{12} = p_{21} = 1/8$$
$$p_{22} = 1/8$$

$$P = \begin{bmatrix} 9/8 & 1/8 \\ 1/8 & 1/8 \end{bmatrix}$$

$C_i(t, x) = J_{ij}X_j$ (From equation 19.2)

and $J_{12} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$J_{21} = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$

Then the choice

$$Q_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = 1,2....s$$

Through the application of equation (16) ;

$$n_{i1} = 0.3310$$
$$n_{i2} = 1.0528$$
$$n_{i3} = 0.4570$$
$$n_{i4} = 0.3669$$

From equation (20) $\bar{\xi}_{ij} = [\Lambda(J_{ij}^T J_{ij})]^{1/2}$

$$\bar{\xi}_{12} = 2$$
$$\bar{\xi}_{21} = 4$$

Using equation (22), elements of G are calculated as;

$$g_{11} = -0.4512$$
$$g_{22} = -0.4512$$
$$g_{12} = 2.2170$$
$$g_{21} = 4.4338$$

and the aggregate model of the equation (21) (V < Gv) is

$$\dot{V} < \begin{bmatrix} -0.4512 & 2.2170 \\ 4.4338 & -0.4512 \end{bmatrix} v$$

For further alanysis of stability, more detail reference will have to be made in stability theory from the aggregated model. This will be described in part II of this topic.

Discussions

The concept of decomposition and aggregation is a very powerful method used in the field of control engineering in analysing stability, optimization, design and in solving problems related to control engineering. This method takes advantage of special structural features of complex systems to reduce the memory and computational requirement when an analysis is carried out by machine calculations. This is due to the feasibility of the decomposition and aggregation method where complex problem can be easily solved by using simulation with computers.

The decomposition-aggregation method produce important structural properties of the system by using the system structure in the decomposition concept. Problems of high dimension are simplified into easier and solvable lower dimension problems.

The *second method* of *Liapunov* which is also considered as the aggregation procedure are most significant in the theory of control systems where:

(a) It provides an abstract tool for studying stability and transient behavior of dynamic systems without solving the differential equations of these systems.

(b) The aggregation method is already available, though unrecognized, in standard results in control theory. These results can be used in other ways to estimate transient response, effect of random perturbations and so on.

However this method is inherently conservative since a series of approximations are involved in establishing conditions for stability.

Conclusions

Refering to the concept of decomposition and aggregation, this theory has been applied to test stability and optimization of models and projects, for example, the spinning *Skylab model* and the *large space telescope* model developed by NASA (National Aeronauticals and Space Administration)[3]. It has been found that the method proved advantageous especially when problems can be solved by doing simulation studies with computers. The decomposition and aggregation method can also be used for nonlinear systems and also to systems which involves control. This will involved another set of equation which is more complicated and which is also derived from the *Liapunov second method*. Since the outlined decomposition and aggregation analysis is completely computerized, proposed improvements can be readily incorporated in more analysis scheme to yield a more powerful method in solving equations related to large scale linear and nonlinear systems.

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