CONCEPT OF DECOMPOSITION AND AGGREGATION METHOD WITH APPLICATION: PART II

Zainol Anuar bin Mohd Sharif, Ph.D.
and
Ng Boon Choong
Faculty of Mechanical Engineering
Universiti Teknologi Malaysia

Synopsis

This paper describes the application of Decomposition and Aggregation method to a model of a practical system so as to analyse its effectiveness. A computer programme (Fortran) have been developed to evaluate and simulate specific parameter of the model. By evaluating the model through simulation the stable condition of the model can be obtained. The procedure for analysing the stability of this system is also shown in detail. The model used in this paper is the overhead crane model and it will be used as an example in understanding the analysis performed by the prescribed method.

Decomposition-Aggregation Method

The application of this method in stability analysis involves the *Liapunov's* second method which utilises the vector *Liapunov* function.

Let's consider a linear system which is not bounded by control.

$$\dot{x} = f(x, t) \quad \dots (1)$$

The basic idea of this method is to analyse whether the system of form (1) is asymptotically stable.

The first step is to consider the stability of the decoupled subsystems. If the decoupled subsystems are unstable, this means that the overall system will be unstable. If the decoupled subsystem are found to be stable, then the test will proceed in checking the stability of the overall system by considering the interconnections. The overall stability will depend on the interconnection between subsystems. There are few important formulae and assumptions which are related to the analysis of dynamic systems. [For further illustration refer to [1]]

1) $V = \| x \|^2 P$

$$x = \left(\sum_{i=1}^{n} x_i^{\,2}\right)^{1/2} = (x^T x)^{1/2} \quad \dots (2)$$

$$= V_i(x_i) = (x_i^{\,T} P_i x_i)^{1/2}$$

2) The estimates of the *Liapunov* function $V_i(x_i)$ are [1, 2]:

$$n_{i1} \| x_i \| \leqslant V_i < n_{i2} \| x_i \|$$

$$\dot{V}_i \leqslant -n_{i3} \| x_i \| \quad \dots (3)$$

$$\| \mathrm{grad}\, V_i \| \leqslant n_{i4}$$

where,

n_{ij} = positive numbers and $x_i = (x_i^{\,T} x_i)^{1/2}$.

V_i = total time derivative of function $V_i(t, x_i)$

From the above assumptions (equation 3), the following relationship may be used:

$$n_{i1} = \lambda^{1/2}(P_i)$$

$$n_{i2} = \Lambda^{1/2}(P_i) \quad \dots (4)$$

$$n_{i3} = 1/2\, \lambda(Q_i)\, \Lambda^{-1/2}(P_i)$$

$$n_{i4} = \lambda^{1/2}(P_i) \; \Lambda(P_i)$$

where;

λ = minimum eigen values of the indicated matrices.

Λ = maximum eigen values of the indicated matrices.

3) Interactions between subsystems follow the constraint below;

$$\| C_i(t, x) \| < \sum_{j=1}^{s} \bar{\xi} \, \| x_j \|$$

where, $\bar{\xi}$ = nonnegative numbers.

$$C_i(t, x) = \sum_{j=1}^{s} J_{ij} \, x_j$$

where, J_{ij} = constant coefficient matrices.

So, $\bar{\xi}_{ij} = [\Lambda(J_{ij}^{T} J_{ij})]^{1/2}$ … (5)

4) The scalar inequalities can be written as:

$$V \leqslant GV \quad \ldots (6)$$

where the sxs matrix G has element g_{ij} specified by

$$g_{ij} = -\delta'_{ij} \, n_{i2}^{-1} n_{i3} + \bar{\xi}_{ij} \, n_{j1}^{-1} n_{i4} \quad \ldots (7)$$

where δ' is a kronecker delta.

Overhead Crane Model

The model used in this analysis, as mentioned earlier, is the overhead crane model. The purpose of using a model is to evaluate and apply the concept and mathematical functions practically. Through the model, various analysis and evaluation can be made based on the method of decomposition and aggregation.

The main objective of this analysis is to obtain the stability limit of overhead crane model with respect to the variation of parameters. Before proceeding any further, a brief introduction to the model is made.

In an overhead crane, a truck of mass m_1 is resting at the centre of a beam with stiffness k_1. The truck is lifting a mass m_2 through a cable stiffness k_2. The system consists of two-degree-of-freedom movements as depicted in (Figure 1)

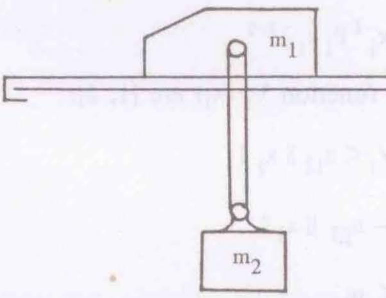


Figure 1 Conceptual model of Overhead crane lifting a mass

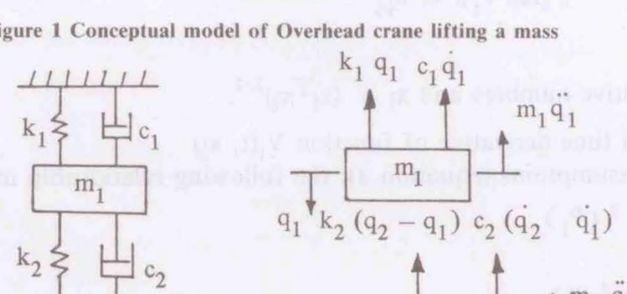


Figure 2 Free-body diagram of the system

The equations of motion can be derived by employing either *Newton's law* or *Lagrange* equations.

$$m_1 \ddot{q}_1 + (c_1 + c_2)\dot{q}_1 + (k_1 + k_2) q_1 - c_2 \dot{q}_2 - k_2 q_2 = 0$$

$$m_2 \ddot{q}_2 + c_2 \dot{q}_2 + k_2 q_2 - c_2 \dot{q}_1 - k_2 q_1 = 0 \qquad \text{... (8)}$$

The values of the masses are normally known and the values of stiffness can be evaluated from the strength of the material depending upon the types of materials used.

For a simply supported beam loaded at the centre within elastic limits, k_1 is obtained as : [3]

$$k_1 = 48E_b I_b / L_b^3 \qquad \text{... (9)}$$

For a cable loaded in tension,

$$k_2 = A_c E_c / L_c \qquad \text{... (10)}$$

where,

E_b = Young's modulus of the material of the beam.
E_c = Young's modulus of the material of the cable.
I_b = Moment of inertia about the neutral axis for the cross section of the beam.
A_c = Cross section area of the cable.
L_b = Length of beam.
L_c = Length of cable.

For a fixed value of m_1, m_2, k_1, k_2, different values of c_1 and c_2 can be simulated to obtained the stability of the overhead crane when the system is free from external forces.

Stability can also be evaluated by simulating the condition where m_2 varies and fixing the other parameters. This will represent a more practical situation in which the load to be lifted by the crane may vary but this type of simulation involves the *Liapunov* method with the application of control. Stability analysis of the overhead crane model will be performed in two stages. The decomposition stage is where the equations of the model will first be decomposed into subsystems. The aggregation method is then employed to test the stability of the decoupled subsystems. Further illustration will be made later in this paper. In the stability analysis of the overhead crane model. the following Physical Characteristic are used.

Parameter	**Description**
m_1	mass of truck
m_2	mass lifted by the truck using a cable
k_1	stiffness coeffcient of the beam supported the truck
k_2	stiffness coefficient of the cable
t	time
$[\dot{\ }] = d/dt$	differentiation with respect to real time t.
c_1	damping coefficient of the beam
c_2	damping coefficient of the cable
q_1	movement of mass m_1
q_2	movement of mass m_2
$\alpha_i = k_i/m_i$	ratio of stiffness coefficient and mass
$\alpha_1 = k_1/m_1$	ratio of stifness coefficient and beam and mass of truck
$\alpha_2 = k_2/m_2$	ratio of stiffness coefficient of cable and lifted mass.
$\alpha_3 = k_1/m_2$	ratio of stiffness coefficient of beam and lifted mass.
$\alpha_4 = k_2/m_1$	ratio of stiffness coefficient of cable and mass of truck.
$\gamma_i = C_i/m_i$	Damping coefficient and mass ratio.
$\gamma_1 = c_1/m_1$	Damping coefficient of beam and mass of truck ratio.

$\gamma_2 = c_2/m_2$ Damping coefficient of cable and lifted mass ratio.

$\gamma_3 = c_1/m_2$ Damping coefficient of beam and lifted mass ratio.

$\gamma_4 = c_2/m_1$ Damping coefficient of cable and mass of truck ratio.

PER1 = ratio of stiffness coefficient $(k_1 + k_2)$ over mass of truck.

PER2 = ratio of damping coefficient $(c_1 + c_2)$ over mass of truck.

Physical Characteristic

The values of the following parameters can be varied in order to obtain the state of stability.

$m_1 = 121.97$ Kg-sec^2/m

$m_2 = 568.78$ Kg-sec^2/m

$k_1 = 0.357 \times 10^5$ Kg/m (Metal beam, steel)

$k_2 = 0.70 \times 10^3$ Kg/m (Wire rope for extra flexible hoisting) (Plow steel)

$c_1 = 2.26$ Kg/sec^2 (Simply supported end)

$c_2 = 9.06$ Kg/sec^2

Decomposition of The Overhead Crane Model

Actual equation:

$$m_1\ddot{q}_1 + (c_1 + c_2)\,\dot{q}_1 + (k_1 + k_2)\,q_1 - c_2\dot{q}_2 - k_2q_2 = 0$$

$$m_2\ddot{q}_2 + c_2\dot{q}_2 + k_2q_2 - c_2\dot{q}_1 - k_2q_1 = 0$$

$$\ddot{q}_1 = -(c_1 + c_2)\,\dot{q}_1/m_1 - (k_1 + k_2)\,q_1/m_1 + c_2\dot{q}_2/m_1 + k_2q_2/m_1$$

$$\ddot{q}_2 = -c_2\dot{q}_2/m_2 - k_2q_2/m_2 + c_2\dot{q}_1/m_2 + k_2q_1/m_2$$

By assuming;

$$x_{11} = q_1\,,\; x_{12} = \dot{q}_1\,,\; x_{21} = q_2\,,\; x_{22} = \dot{q}_2$$

So,

$$\dot{x}_{11} = x_{12}$$

$$\dot{x}_{12} = -(c_1 + c_2)/m_1 \cdot x_{12} - (k_1 + k_2)/m_1 \cdot x_{11} + c_2/m_1 \cdot x_{22} + k_2/m_1 \cdot x_{21}$$

$$\dot{x}_{21} = x_{22}$$

$$\dot{x}_{22} = -c_2/m_2 \cdot x_{22} - k_2/m_2 \cdot x_{21} + c_2/m_2 \cdot x_{12} + k_2/m_2 \cdot x_{11}$$

By putting the above equations into the matrix form:

$$S_1 : \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mathrm{PER1} & -\mathrm{PER\,2} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_4 & \gamma_4 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$S_2 : \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\gamma_2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

So, the free decoupled subsystems become:

$$S_1: \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -PER1 & -PER2 \end{bmatrix}$$

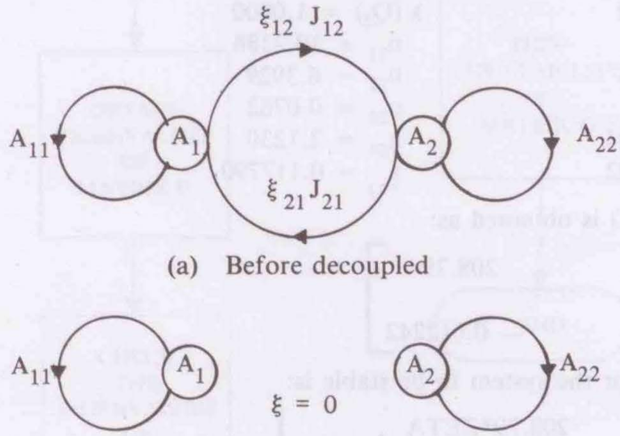
$$S_2: \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\alpha_2 & -\gamma_2 \end{bmatrix}$$

The interconnection matrices can be expressed as:

$$S_1: \quad J_{12} = \begin{bmatrix} 0 & 0 \\ \alpha_4 & \gamma_4 \end{bmatrix}$$

$$S_2: \quad J_{21} = \begin{bmatrix} 0 & 0 \\ \alpha_2 & \gamma_2 \end{bmatrix}$$

The structural configuration of the model system can be expressed by the diagraph shown in Figure 3. This diagraph represents the whole interconnection of the model system. [2].



(a) Before decoupled

(b) After decoupled

Figure 3 Structural Configuration of the System

Aggregation of Overhead Crane Model

The structural configuration of the system S involving the interconnection parameters is as depicted in Figure 3a. The system is decoupled when $\xi = 0$
So to obtain the value of ξ equation (5) in applied:

$$\bar{\xi}_{ij} = [\wedge (J_{ij}{}^T J_{ij})]^{1/2}$$

for i, j = 1, 2 ...

The aggregation stage involved obtaining the matrix P from the equation : [1]

$$A^T P + PA = -Q \quad \dots (11)$$

After obtaining the matrix P, computation of the eigenvalues of P is made. If the eigenvalues of P is positive, the subsystem, S_i is asymptotically stable but if it is negative, the subsystem S_i is not asymptotically stable, thus the whole system is unstable. ⟨Refer to equation (24) [1] for the calculation of the values of P⟩

The matrix Q is first assumed to be the identity matrix.

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The choice of the 2×2 symmetric matrices $Q_i = I$, $i = 1, 2$ as the 2×2 identity matrix, yields the positive definite 2×2 symmetric matrices P_i, $i = 1, 2$.

The computer programme is written for the purpose of calculating and obtaining several values which can assist in determining the overall stability with the consideration of interconnections between the systems. The values calculated by the computer programme are:

1) The value of subsystems matrix A_i.
2) The subsystem *Liapunov* functions $V_i = (x_i^T P x_i)^{1/2}$ for $i = 1, 2$ ⟨refer to equation 13.1 [1]⟩
3) The positive numbers n_{ij} where $i, j = 1, 2$
4) The value of $\bar{\xi}_{ij}$ for $i, j = 1, 2$ [Refer to equation 20.0] [1]
5) The interconnection matrixs J_{ij} for $i, j = 1, 2$
6) The Aggregation matrix G.
7) The solution of the overall stability.

The procedure of obtaining the values mentioned above is shown clearly in the flowchart (Figure 4).

In order to obtain the aggregated matrix G, the following values which have been computed are used.

$\lambda (P_1) = 1611.3800$	$\lambda (P_2 = 3370.5323$
$\Lambda (P_1) = 5.4030$	$\Lambda (P_2) = 40.8452$
$\lambda (Q_1) = 1.0000$	$\lambda (Q_2) = 1.0000$
$n_{11} = 2.3244$	$n_{21} = 19.2388$
$n_{12} = 40.1424$	$n_{22} = 6.3929$
$n_{13} = 0.0125$	$n_{23} = 0.0782$
$n_{14} = 693.2323$	$n_{24} = 2.1230$
$\bar{\xi}_{12} = 5.794392$	$\bar{\xi}_{21} = 0.117790$

The aggregated matrix G is obtained as:

$$G = \begin{bmatrix} -0.000310 & 208.79 \\ 0.11 & -0.012242 \end{bmatrix} \quad \dots (12)$$

The general condition for the system to be stable is:

$$\begin{vmatrix} -0.000310 & 208.79*ZETA \\ 0.11*ZETA & -0.012242 \end{vmatrix} > 0 \quad \dots (13)$$

ZETA = Interconnection factor between the subsystems.

Description on the computer programme

The programme used in the analysis of stability of the overhead crane model is written by using the language FORTRAN 77. The programme consists of a main programme and four subroutiness which execute the part called by the main programme and four subroutines which execute the part called by the main programme. The procedure of solving the problem by using the programme is as follows:

1) Read in all the necessary values.
2) Compute the A matrix.

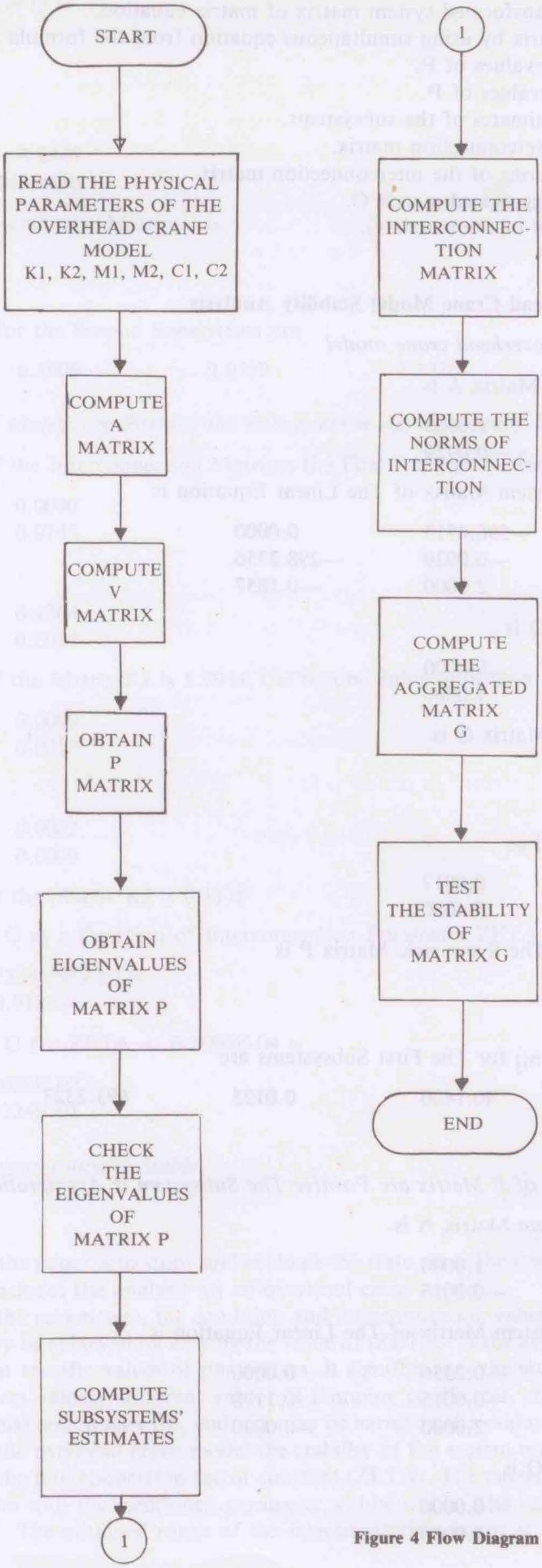


Figure 4 Flow Diagram

3) Compute the transformed system matrix of matrix equation.
4) Solve the P matrix by using simultaneous equation from the formula $A^TP + PA = -Q$.
5) Obtain the eigenvalues of P.
6) Check the eigenvalues of P.
7) Compute the estimates of the subsystems.
8) Compute the interconnection matrix.
9) Compute the norms of the interconnection matrix.
10) Compute the Aggregated matrix G.
11) Test the stability of the matrix G.

Results of the Overhead Crane Model Stability Analysis

Stability analysis of overhead crane model

The First Subsystem Matrix A is

0.0000	1.0000
—298.2356	—0.0929

The Transformed System Matrix of The Linear Equation is

0.0000	—596.4713	0.0000
1.0000	—0.0929	—298.2356
0.0000	2.0000	—0.1857

The Chosen Matrix Q is

1.0000	0.0000
0.0000	1.0000

The Eigenvalues of Matrix Q is

1.0000
1.0000

Matrix P is Obtained as

1611.3800	0.0017
0.0017	5.4030

The Eigenvalues of The Symmetric Matrix P is

1611.3800
5.4030

The Four Estimates n_{ij} for The First Subsystems are

2.3244	40.1420	0.0125	693.2323

All The Eigenvalues of P Matrix are Positive The Subsystem is Asymptotically Stable

The Second Subsystem Matrix A is

0.0000	1.0000
—0.1178	—0.0015

The Transformed System Matrix of The Linear Equation is

0.0000	—0.2356	0.0000
1.0000	—0.0015	—0.1178
0.0000	2.0000	—0.0030

The Chosen Matrix Q is

1.0000	0.0000
0.0000	1.0000

The Eigenvalues of Matrix Q is

1.0000	
1.0000	

Matrix P is Obtained as

370.1388	4.2445
4.2445	3142.0410

The Eigenvalues of the Symmetric Matrix P is

370.1323
40.8432

The Four Estimates n_{ij} for the Second Subsystems are

19.2388	6.3909	0.0782	2.1230

All the Eigenvalues of P Matrix are Positive the Subsystem is Asymptotically Stable

Estimating the Norms of the Interconnection Matrices the First Interconnection Matrix RJ12 is

0.0000	0.0000
5.7949	0.0743

RJTranspose*RJ is

33.5805	0.4304
0.4304	0.0055

The Estimated Norms of the Matrix RJ is 5.7944.The Second Interconnection Matrix RJ21 is

0.0000	0.0000
0.1178	0.0015

RJTranspose*RJ is

0.0139	0.0002
0.0002	0.0000

The Estimated Norms of the Matrix RJ is 0.1178

The Aggregation Matrix G as a Function of Interconnection Parameter ZETA is

–0.00031	208.79*ZETA
0.11*ZETA	–0.01224

The Aggregation Matrix G for ZETA = 0.1000E-04 is

–0.3103E-03	0.2088E-02
0.1076E-05	–0.1224E-01

The Overall System is Asymptotically Stable

Discussion

The main objective of the paper is to study and evaluate the state when the system is stable with certain fixed parameters. This includes the analysis on an overhead crane model.

For specific values of the parameters, the condition and interconnection value (ZETA) for stable state can be obtained. This may be achieved by varying the value of (ZETA). If the asymptotically state cannot be obtained by using that specific values of parameters, it signifies that the stable condition cannot be obtained by the present set values. Different values of damping coefficient, stiffness coefficient (from different types of materials) and masses (m_1 and m_2) may be varied in the evaluation of the stability condition of the system. In the overhead crane model the stability of the system was tested by fixing all the other parameters except the interconnection factor constant (ZETA). The purpose is to find the range of ZETA in which the system with the mentioned parameter will be stable. The value of ZETA is obtained as ZETA $< 4.065 \times 10^{-4}$. The obtained range of the interconnection factor is small due to the conservativeness of the stability procedure. However this value can be increased with different choice of matrice Q_i, i = 1, 2. For example changing the value of matrix Q from Identity matrix to:

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

This produces a bigger interconnection factor, ZETA $< 2.874 \times 10^{-3}$. By performing simulation, various criteria and constraints may be obtained. This clearly shows the advantage of the method were computers are used to carry out the stability analysis thus reducing the memory and computational time.

Conclusion

The decomposition-aggregation method provides a clearer structural properties of the system. However, the system is conservative since approximations and assumptions in obtaining stable condition of the system are involved. The success of the analysis largely depends on the conservativeness of the approximations and assumptions which are outweighed by the feasibility in solving the problem by using computers.

The analysis in this paper involved only dynamic systems where controls are not involved. Future effort should be directed in analysing systems which will involve controls and also the problem involving optimization.

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