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A STUDY OF STRESS DISTRIBUTION IN ORTHOTROPIC BEAM BY BOUNDARY ELEMENT TECHNIQUE

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ABSTRACT

Boundary element technique is applied to predict the stress distribution in orthotropic beam. The length-depth ratios (L/D) of the beam are chosen within the range of 1.5 and 4.5. The result from orthotropic analysis is compared with isotropic analysis. It shows that the stress distribution predicted by orthotropic analysis is significantly different from isotropic analysis when the 'L/D' ratios less than 4.5.

1 INTRODUCTION

Isotropic analysis is a common method to predict stress in materials. The analysis assumes that the mechanical properties of the material is similar in all aspects. However, not all the materials have isotropic properties. Timber, for example, is an orthotropic material where the mechanical properties are different in three directions [1]. Due to complexity to develop an orthotropic formulation, several analyse done by previous researchers [2,3] only consider isotropic assumption, which is possibly not true for orthotropic materials.

Two methods are available to predict the stress in orthotropic materials, i.e. semi-analytical approach and approximate method (such as finite element technique). In the semi-analytical approach, the stress is approximated by combination of analytical part (such as Saint-Venant or Bernoulli theory) and approximate method such as finite element technique [4,5,6]. The disadvantage of this method is to assume that the bending stress is linearly distributed along the beam's depth. The method is also limited to tip-load cantilever. In finite element technique which is once of the approximate method, the speed of calculation process is significantly slow due to many elements is needed to give a converge solution [4,7,8].

Based on Lekhnitskii's work [9], one infinite plate with a small circular cavity is bent up by a moment applied to each two sides and analyse it with orthotropic and isotropic consideration. His result shows that the stress pattern from orthotropic analysis gives a significant difference compared with isotropic result. In the second case, he considered one cantilever, orthotropic in properties and applied by one concentrated load at the end. In this case, no stress distribution is given but in order to derive the stress expression for that orthotropic cantilever, the plane-sections-remain-plane assumption is used. This assumption however is quite similar to conventional simple beam theory, which is cannot be accepted for the stress solution in a short beam analysis [10,11].

Little information has been found about the stress distribution in orthotropic beam. Hardy and Pipalzadeh [11] shows several patterns of stress distribution along the beam's depth with varying 'L/D' ratios. Timoshenko [12] also shows the stress pattern along the beam's depth. However, all stress patterns shown by previous investigators only consider isotropic properties of the material. To overcome this problems, one type of approximate method, called boundary element technique, is applied in order to observe the stress distribution in orthotropic beam. This type of analysis only discretize the elements at structural boundary and gives faster convergence rate for stress solution compared with finite element method. Furthermore, the problem of the plane-sections-remain-plane assumption used in the conventional orthotropic beam theory (i.e. semi-analytical approach, which is discussed previously) can be avoided.

2 BOUNDARY ELEMENT FORMULATION

Boundary element formulation in elastostatic problems is firstly introduced by Rizzo [13]. This formulation is derived from equilibrium state of forces for every small elements in the structure. If the material is assumed elastic, the displacement at a point 'i' (see Fig. 1) in the beam structure can be related to the displacements and tractions at the boundary, as follow [14]:

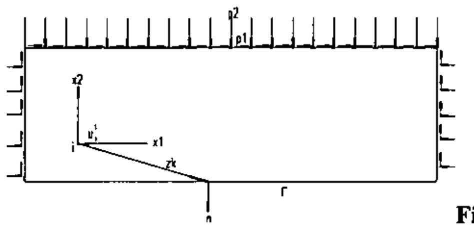


Fig. 1 Beam structure

$$u_l^i = \int_\Gamma u^*_{lk} p_k \, d\Gamma - \int_\Gamma p^*_{lk} u_k \, d\Gamma \quad (1)$$

where:

u_l^i = displacement at a point 'i' in direction x_l.

Γ = Domain of structural boundary.

p_k and u_k are traction and displacement in direction x_k, respectively.

p_{lk}^* and u_{lk}^* are virtual traction and virtual displacement in direction x_k respectively. This traction and displacement are produced from one virtual unit load applied at a point 'i' in direction x_l.

The plane stress orthotropic assumption is used in the boundary element formulation. If x_1 and x_2 are the orthotropic axes of the material (refer to Fig. 1), the stress-strain relationship for one small element in the structure is:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{21} & \beta_{22} & \beta_{26} \\ \beta_{61} & \beta_{62} & \beta_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \quad (2)$$

where:

$\beta_{11} = 1/E_{11}$, $\beta_{12} = -\nu_{12}/E_{11}$, $\beta_{21} = -\nu_{21}/E_{22}$, $\beta_{22} = 1/E_{22}$, $\beta_{66} = 1/G_{12}$, $\beta_{16} = \beta_{26} = \beta_{61} = \beta_{62} = 0$

The parameters ε (and γ), σ (and τ), E, ν and G in the equation (2) indicate strain, stress, elasticity modulus, Poisson's ratio and shear modulus of the material respectively. The subscript index '1' and '2' in these parameters refers to orthotropic axes, x_1 and x_2.

As was proposed by Cruse [15,16], the virtual displacement, u_{lk}^*, in equation (1) to satisfy the condition given by equation (2) is:

$$u_{11}^* = 2\,\mathrm{Re}[p_1 A_{11} \log z_1 + p_2 A_{12} \log z_2] \quad 3(a)$$

$$u_{12}^* = 2\,\mathrm{Re}[q_1 A_{11} \log z_1 + q_2 A_{12} \log z_2] \quad 3(b)$$

where 'Re' denotes real part of complex number. z_k (k=1,2) in the equation (3a) and (3b) (see also Fig. 1) is a complex number that relates the coordinates x_1 and x_2, as follow:

$$z_k = x_1 + \mu_k x_2 \quad (4)$$

where μ_k is the root of characteristic equation of the material. The characteristic equation is given by Lekhnitskii [9], as follow:

$$\beta_{22} - 2\mu\beta_{26} + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{16}\mu^3 + \beta_{11}\mu^4 = 0$$

p_k and q_k (k=1,2) in the equation (3) are the material constants which is given as follow:

$$p_k = \beta_{11}\mu_k^2 + \beta_{12} - \beta_{16}\mu_k$$
$$q_k = \beta_{22}/\mu_k + \beta_{12}\mu_k - \beta_{26}$$

A_{jk} (j=1,2;k=1,2) in the equation (3) is a complex number, which is obtained from solving two following equations, i.e.:

$$\sum_{k=1}^{2}(A_{jk} - \overline{A}_{jk}) = \frac{\delta_{j2}}{2\pi i}$$
$$-\sum_{k=1}^{2}(\mu_k A_{jk} - \overline{\mu}_k \overline{A}_{jk}) = \frac{\delta_{j1}}{2\pi i}$$

The virtual tractions, $p_{lk}{}^*$, in equation (1) to satisfy the condition given by equation (2) is [15,16]:

$$p_{lk}^* = 2\,\mathrm{Re}[Q_{k1}(\mu_1 n_1 - n_2)A_{l1}/z_1 + Q_{k2}(\mu_2 n_1 - n_2)A_{l2}/z_2] \tag{5}$$

where n_1 and n_2 are cosine angle of normal vector, n (see also Fig. 1) with respect to x_1, x_2 axes respectively. The parameter Q_{jk} (j=1,2; k=1,2) in equation (5) is:

$Q_{1k} = \mu_k$, $\quad Q_{2k} = -1 \quad$ (k=1,2)

From equation (3) and (5), all virtual functions are singular if 'z_k' equal to zero. Therefore, by substituting equation (3) and (5) into equation (1), the displacement formulation becomes singular if the point 'i' is located at boundary (i.e. z_k=0). By introducing one virtual hemisphere at a point 'i' on the boundary and by using limiting theory, equation (1) reduces to boundary element formulation, as follow:

$$c_{1k}^i u_k^i + \int_\Gamma p_{1k}^* u_k d\Gamma = \int_\Gamma u_{1k}^* p_k d\Gamma \tag{6}$$

where c_{1k}^i is a parameter that is a function of geometry of the boundary at a point 'i'. The displacement and traction functions, u_k and p_k in the equation (6) are approximated by quadratic expression, which reduce to nodal unknowns at the boundary. These unknown values are determined after assembling process is carried out for each node.

The two integrals in equation (6) (denote by f) are singular. This singularity problem is solved in the assembling process by considering "rigid body motion' proposed by Cruse [17].

For stress formulating at a point 'i' in the structure, the displacement function, u_l^i, in the equation (1) is differentiated, which gives:

$$u_{l,j}^i = -\int_\Gamma u_{lk,j}^* p_k d\Gamma + \int_\Gamma p_{lk,j}^* u_k d\Gamma \qquad (7)$$

The derivative of the displacement function formulated by equation (7) is related to the displacements and tractions at the boundary. The displacement derivative function is singular if the point 'i' is located at the boundary caused by the singularity of the virtual functions, $u_{lk,j}^*$ and $p_{lk,j}^*$. If the point 'i' is located close to boundary, the accuracy of displacement derivatives dependent on the singularity order of virtual functions. The accuracy of displacement derivatives becomes a problem issue if the singularity order of virtual functions is higher. However, by a method proposed by Suhaimi and Zainai [14], the accuracy problems can be solved efficiently if the point 'i' is closely located to the boundary.

After the displacement derivatives have been known, the strain is determined by the strain-displacement relationship as follow:

$$\varepsilon_1 = u_{1,1}$$
$$\varepsilon_2 = u_{2,2}$$
$$\gamma_{12} = u_{1,2} + u_{2,1} \qquad (8)$$

Then, the stress can be found by solving of equation (2).

3 BOUNDARY ELEMENT ANALYSIS AND RESULTS

Orthotropic boundary element technique which is formulated in the section 2 is applied to the beam's structure as shown in Fig. 2. The stress predicted from this technique is studied. X_1 and X_2 axes in Fig. 2 are referred to orthotropic axes. A unit width of beam is used in the analysis and three 'L/D' ratios are studied, that are 1.5, 3.0 and 4.5. The elastic modulus and Poisson's ratio are taken as follow:

E_{11}= 13900 N/mm^2 , E_{22}= 1200 N/mm^2 , V_{12}= 0.3 , G_{12}= 700 N/mm^2

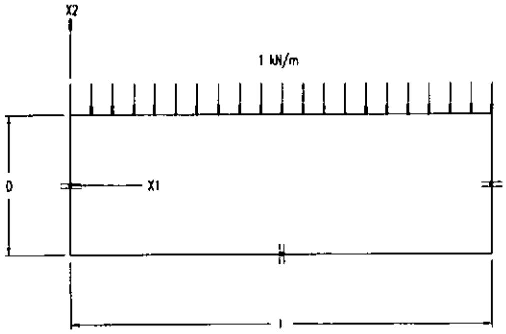


Fig. 2 A beam structure for boundary element analysis

Isotropic boundary element analysis is also applied to the similar beam's structure (see Fig. 2) for stress comparison with orthotropic analysis. The elastic modulus, E, and Poisson's ratio used in the isotropic analysis are similar to E_{11} and V_{12} from orthotropic analysis, i.e. 13900 N/mm^2 and 0.3 respectively.

In order to determine the number of element required for boundary element analysis, isotropic boundary element analysis is applied and the predicting stress is compared to the exact solution. The number of elements is increased until the result is close to exact solution. Timoshenko's isotropic theory [12] is a method used for exact solution determination. For the beam with 'L/D' ratio equal to 1.5, 34 elements were found for isotropic boundary element analysis to give an exact solution, while for 'L/D' ratios equal to 3.0 and 4.5, 30 elements and 26 elements were found respectively. The stress distribution predicted from isotropic boundary element analysis using the above number of elements are shown in Fig. 3, 4 and 5. The stress distribution from Timoshenko's theory is also shown in the same figure for comparison. Good agreement was achieved between isotropic result and Timoshenko's theory. It is believed that the number of elements needed for orthotropic analysis to give a converge solution is achieved if the similar number of elements is used for isotropic analysis. Therefore, the number of elements obtained from isotropic analysis is used once again for orthotropic analysis.

The bending stress distribution predicted from orthotropic analysis is shown in Fig. 3. Nonlinear distribution was found. The stress distribution predicted from isotropic analysis is linear and it shows significantly different from orthotropic stress when 'L/D' ratios equal to 1.5 and 3 (see Fig. 3(a) and 3(b)). The orthotropic stress is higher than isotropic stress at the top and bottom of the beam, and reduces toward the middle of the beam's depth. However, the nonlinearity of orthotropic stress is reduced when 'L/D' ratio is increased and the orthotropic stress is approximately equal to isotropic stress when 'L/D' ratio equal to 4.5 (see Fig. 3(c)).

The shear stress distribution predicted from orthotropic analysis is shown in Fig. 4. It is clearly seen that orthotropic stress is lower than isotropic stress at the middle of the beam's depth and increased toward to the top and bottom of the beam. Significantly different was found for both stresses with 'L/D' ratio 1.5 and 3 (see Fig. 4(a) and 4(b)). However, orthotropic stress approach to isotropic stress when 'L/D' ratios equal to 4.5 (see Fig. 4(c)).

The transverse stress distribution predicted from orthotropic analysis is shown in Fig. 5. Orthotropic and isotropic stress was found to be no different for all 'L/D' ratios. It shows that orthotropic properties of the material do not give much influence for transverse stress in the beam.

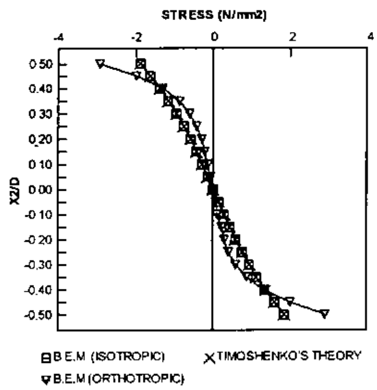
4 CONCLUSION

To conclude this paper, the following remarks are noted:

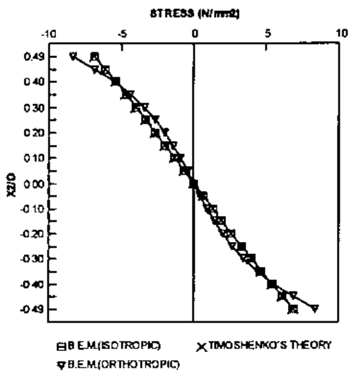
[1] The distribution of stress predicted from orthotropic and isotropic analysis is different. This distribution is different if 'L/D' ratio less than 4.5.

[2] Maximum stress in bending predicted from orthotropic analysis is greater than that from isotropic prediction. However, maximum stress in shear is less than from isotropic prediction.

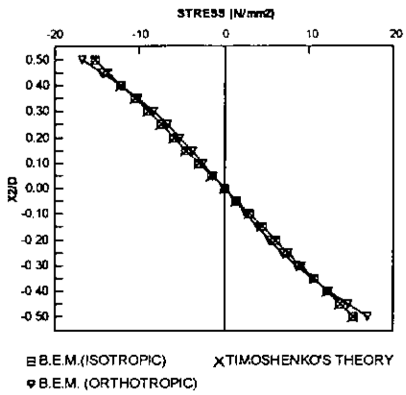
[3] Orthotropic properties do not gives significantly influence for transverse stress in the beam compared with isotropic prediction.



(a) L/D = 1.5



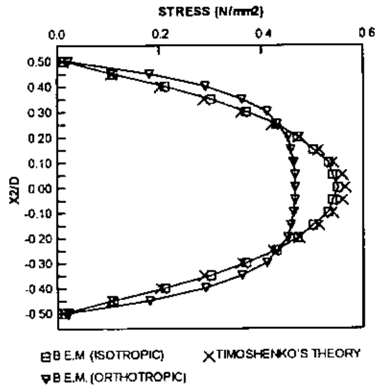
(b) L/D = 3.0



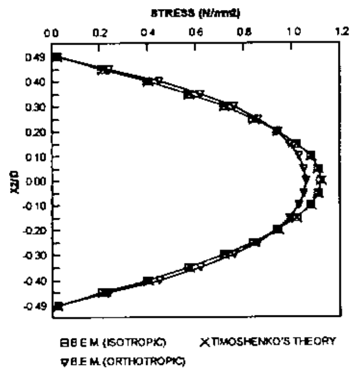
(c) L/D = 4.5

Note: B.E.M. = Boundary element method

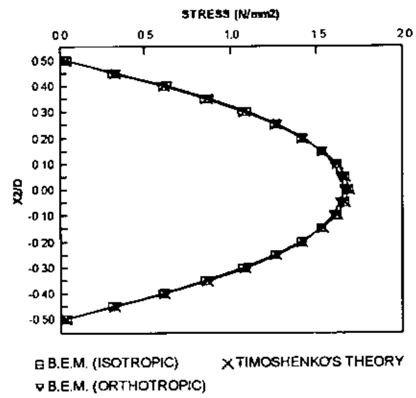
Fig. 3 Bending stress distribution at mid-span of the beam (X_1 =0.5L)



(a) L/D = 1.5



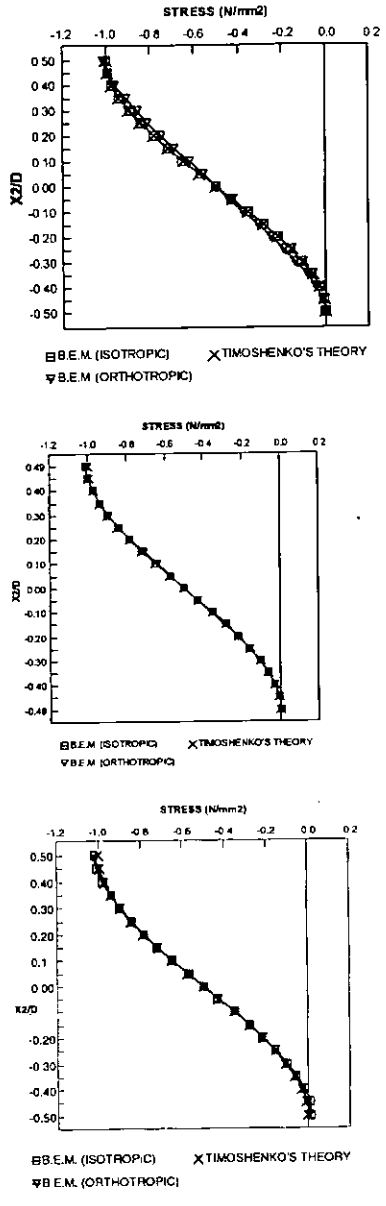
(b) L/D = 3.0



(c) L/D = 4.5

Note: B.E.M. = Boundary element method

Fig. 4 Shear stress distribution at X_1 =0.25L



(a) L/D = 1.5

(b) L/D = 3.0

(c) L/D = 4.5

Note: B.E.M. = Boundary element method

Fig. 5 Transverse stress distribution at the mid-span of the beam (X_1=0.5L)

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