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BUBNOV-GALERKIN METHOD FOR THE ELASTIC BUCKLING OF EULER COLUMNS

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Abstract: In this work the Bubnov-Galerkin variational method was applied to determine the critical buckling load for the elastic buckling of columns with fixed-pinned ends. Coordinate shape functions for Euler column with fixed-pinned ends are used in the Bubnov-Galerkin variational integral equation to obtain the unknown parameters. One parameter and two parameter shape functions were used. In each case, the Bubnov-Galerkin method reduced the boundary value problem to an algebraic eigen-value problem. The solution of the characteristic homogeneous equations yielded the buckling loads. One parameter coordinate shape function yielded relative error of 4% compared with the exact solution. Two parameter coordinate shape function gave a relative error of 0.77%, which is negligible.

Keywords: *Bubnov-Galerkin method, elastic buckling, Euler column buckling, buckling load*

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1.0 Introduction

1.1 Background/Literature Review

Columns are long slender bars under axial compressive forces. They can be horizontal, vertical or inclined. They are classified as short columns, long columns or intermediate columns. When a slender member is subjected to an axial compressive force, it may fail due to buckling (Rao, 2016; homepages, 2016; Lagace, 2009). Buckling is a geometric instability in which the lateral displacement of the axially compressed column can suddenly become very large (Rao, 2016; Lagace, 2009; Punmia *et al.*, 2002; Jayaram, 2007). Short columns fail by crushing or compressive yielding of the material. Long columns fail by buckling or flexural buckling which is a geometric failure or instability. Intermediate columns fail by a combination of crushing and flexural buckling failures.

Intermediate columns fail by both compressive yielding and flexural buckling of the column.

Thin structures subject to compression loads that have not achieved the material strength limits can fail by buckling. (Beeman, 2014). Buckling is characterized by a sudden failure of a structural member subjected to high compressive stress where the actual compressive stress at the imminence of failure is less than the ultimate material compressive strength (Beeman, 2014; Novoselac *et al.*, 2012).

The critical buckling load for an axially compressed column, determined using a linear elastic buckling analysis of an idealized perfect structure does not necessarily correspond to the load at which instability of the real structure takes place (Fernandez, 2013). The calculated critical buckling load does not provide sufficient information about when failure due to the instability of the structure as a whole will occur. This depends on other factors like initial geometrical imperfections, eccentricities of loading, and the nonlinear deflection behaviour of the structure (Yao and Lee, 2011).

Buckling can be analyzed using linear buckling analysis (eigenvalue) or non linear buckling analysis (Eryilmaz *et al.*, 2013). The objective of linear buckling analysis is to determine the buckling load factor and the critical buckling load (Digital Engineering, 2017). Critical buckling loads in linear (eigenvalue) buckling analysis may be determined using any of the following methods:

i. by exact mathematical methods of solving the governing differential equations of equilibrium subject to the boundary conditions. This yields exact values of the critical buckling loads.
ii. by using approximate methods, which may be based on energy principles, variational methods or discrete approximations of the governing differential equations of equilibrium and the boundary conditions.

Mathematically rigourous techniques of solving the boundary value problem of column buckling which consists of solving the governing differential equation of equilibrium on the problem domain subject to the prescribed boundary conditions presents considerable difficulties and can only be achieved for simple buckling problems for structures with low degrees of freedom. Such problems which are difficult to solve in closed analytical form are usually solved using the approximate methods based on discretization of the governing equations, variational methods and energy principles. Approximate methods have been used to solve the column buckling problem by Zdravkovic *et al.*, 2013; Li *et al.*, 2011; Huang and Li, 2011; Kalakowski *et al.*, 2016; Reddy, 2014; Yuan and Wang, 2011; Atay, 2009; and Okay *et al.*, 2010).

Basebuk *et al.*, (2014) used the Hemotopy Analysis Method (HAM) to find the critical buckling load of a column under end load dependent on direction. Eryilmaz *et al.* (2013)

implemented the HAM to determine the buckling loads of Euler columns with a continuous elastic restraint. Atay (2009) determined the critical buckling loads for variable stiffness Euler columns using homotopy perturbation method. Okay *et al.* (2010) used the variational iteration method (VIM) to determine buckling loads and buckling modal shapes of columns. Yuan and Wang (2011) used the differential quadrature method (DQM) to carryout buckling and post buckling analysis of beam-columns. Reddy (2014) implemented buckling analysis of cracked stepped column using the Finite Element Method (FEM). Zdravkovic *et al.* (2013) used the energy method for the efficient estimation of the elastic buckling load of axially compressed three-segment stepped column.

In this paper the Bubnov-Galerkin method is used to determine the critical buckling load of prismatic Euler column of length, l with fixed-pinned ends at $x = 0$, and $x = l$ respectively, where x is the longitudinal coordinate axis of the column.

1.2 Euler Theory of Buckling

Euler considered an elastic column of length l that is pinned at the ends $x = 0$ and $x = l$ and subjected to an axial compressive force P. The column undergoes a lateral deflection denoted by $v(x)$. Moment equilibrium of a section of the deflected column cut at an arbitrary point, x, from the end, $x = 0$, and the application of the moment – curvature equation results in (homepages, 2016; Lagace, 2009; Megson, 2005; Lowe, 1971):

$$-Pv(x) = M(x) = EI\frac{d^2v(x)}{dx^2} \tag{1}$$

Or,

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v(x) = 0 \tag{2}$$

where E is the Young's modulus of elasticity, I is the moment of inertia and $M(x)$ is the bending moment variation along the longitudinal axis of the column, and x denotes the longitudinal axis coordinate of the column.

1.3 General Equation for column buckling

The second order differential Equation (2) applies to columns with simply supported ends x=0 and $x = l$. The more general column buckling equation uses the formulation similar to the bending of a beam, but including the axial forces (Lowe, 1971; homepages, 2016). The forces and moments acting on an elemental part of a column under axial compressive forces are shown in Figure 1.

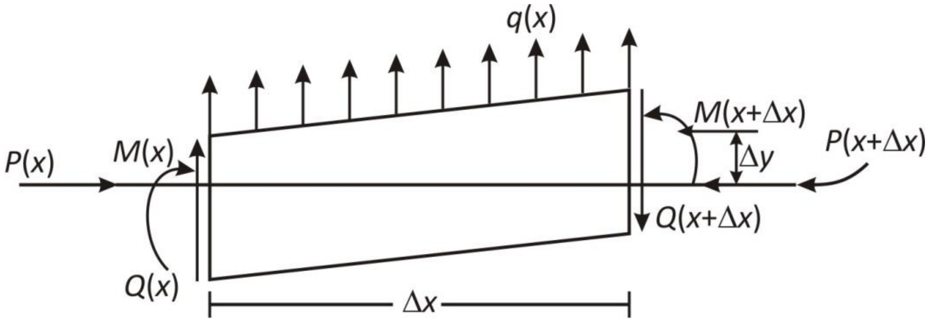

Figure 1: Forces and moments acting on an elemental column

For vertical equilibrium,

$$\frac{dQ}{dx} = q(x) \tag{3}$$

where $Q(x)$ is the shear force.

For horizontal equilibrium

$$P(x) = P(x+\Delta x) = P(x) + \frac{dP}{dx}dx \tag{4}$$

$\therefore$ $P(x)$ is constant

For moment equilibrium,

$$\frac{dM}{dx} + P\frac{dv}{dx} = Q \tag{5}$$

The equation for moment equilibrium contains an extra term $P\frac{dv}{dx}$ which is absent in the equation for Euler Bernoulli beam flexure theory.

By differentiation of Equation (5), with respect to x, we obtain:

$$\frac{d}{dx}\left(\frac{dM}{dx} + P\frac{dv}{dx}\right) = \frac{dQ}{dx} = q(x) \tag{6}$$

Simplifying,

$$\frac{d^2M}{dx^2} + P\frac{d^2v}{dx^2} = q(x) \quad (7)$$

Application of the moment curvature relation in Equation (7) yields:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) + P\frac{d^2v}{dx^2} = q(x) \quad (8)$$

For the prismatic columns, *EI* is constant and the Equation (8) simplifies to (Megson, 2005):

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = q(x) \quad (9)$$

1.4 Assumptions of the Euler's Theory of column buckling

Euler's theory of column buckling is based on the following assumptions (Punmia *et al.*, 2001; Jayaram, 2007; Megson, 2005; Lowe, 1971):

(i) The column is straight in the longitudinal direction before the application of load.
(ii) The column has a uniform cross section throughout its longitudinal axis.
(iii) The column material is isotropic and homogeneous.
(iv) The self-weight of the column material is disregarded.
(v) The line of application of axial compressive load is coincident with the longitudinal axis of the column.
(vi) The reduction in length of the column due to axial compression is very small and disregarded.
(vii) The column fails due to buckling alone.

1.5 Research Aim and Objective

The research aim and objective is to use the Bubnov-Galerkin variational method to determine the elastic buckling loads of Euler columns with fixed pinned ends at x=0, and x=l; respectively.

2.0 Theoretical Framework and Methodology

The governing differential equation for the elastic buckling of prismatic Euler columns under axial compressive load P when transverse loads are absent is given by the fourth order ordinary differential equation (ODE) with constant coefficients given by:

$$EIv^{iv} + Pv'' = 0 \qquad (10)$$

where $v(x)$ is the deflection, E is the Young's Modulus of the column material. I is the moment of inertia.

The fourth order ordinary differential Equation (10) is solved subject to the boundary conditions of the column. The mathematical problem of the column buckling is an eigenvalue problem (Lowe, 1971).

In the Bubnov-Galerkin method, the deflection function $v(x)$ is chosen in terms of shape functions that automatically satisfy the boundary conditions and undetermined parameters called generalized displacement parameters that are sought such that the Bubnov-Galerkin variational integral would vanish. This implies that the weighted error or weighted residual where the shape functions serve as the weighting functions would vanish over the entire column. For one parameter displacement field

$$v(x) = c_1 N_1(x) \qquad (11)$$

where c_1 is the unknown displacement parameter $N_1(x)$ is the displacement shape function that satisfies the boundary conditions. The Bubnov-Galerkin integral for a one parameter displacement field is given by:

$$\int_0^l (EIv^{iv} + Pv'')N_1(x)dx = 0 \qquad (12)$$

$$\int_0^l \left(v^{iv} + \frac{P}{EI}v''\right)N_1(x)dx = 0 \qquad (13)$$

$$\int_0^l \left(c_1 N_1^{iv} N_1 + \frac{P}{EI}c_1 N_1'' N_1\right)dx = 0 \qquad (14)$$

$$c_1 \int_0^l \left(N_1^{iv} N_1 + \frac{P}{EI}N_1'' N_1\right)dx = 0 \qquad (15)$$

$$c_1\left\{\int_0^l N_1^{iv} N_1\, dx + \frac{P}{EI}\int_0^l N_1'' N_1\, dx\right\} = 0 \tag{16}$$

$$c_1\left(k_{11} + \lambda k_{11_g}\right) = 0 \tag{17}$$

where,

$$k_{11} = \int_0^l N_1^{iv} N_1\, dx \tag{18}$$

$$k_{11_g} = \int_0^l N_1'' N_1\, dx \tag{19}$$

and

$$\lambda = \frac{P}{EI} \tag{20}$$

λ is the buckling load factor.

For non-trivial solutions, $c_1 \neq 0$ and the characteristic buckling equation becomes the algebraic eigen-value eigen vector problem

$$\left|k_{11} + \lambda k_{11_g}\right| = 0 \tag{21}$$

Expanding, and solving,

$$\lambda = \frac{-k_{11}}{k_{11_g}} \tag{22}$$

The critical buckling load can then be determined.

For a two parameter displacement field,

$$v(x) = c_1 N_1(x) + c_2 N_2(x) \tag{23}$$

where c_1, c_2 are the two unknown parameter and $N_1(x)$, $N_2(x)$ are the displacement shape functions which are chosen to satisfy the boundary conditions. The Bubnov-Galerkin variational integral is:

$$\int_0^l \left[(c_1 N_1^{iv} + c_2 N_2^{iv}) + \frac{P}{EI}(c_1 N_1'' + c_2 N_2'') \right] N_1 \, dx = 0 \tag{24}$$

$$\int_0^l \left[(c_1 N_1^{iv} + c_2 N_2^{iv}) + \frac{P}{EI}(c_1 N_1'' + c_2 N_2'') \right] N_2 \, dx = 0 \tag{25}$$

$$c_1 \int_0^l (N_1^{iv} + \frac{P}{EI} N_1'') N_1 \, dx + c_2 \int_0^l \left(N_2^{iv} + \frac{P}{EI} N_2'' \right) N_1 \, dx = 0 \tag{26}$$

$$c_1 \int_0^l \left(N_1^{iv} + \frac{P}{EI} N_1'' \right) N_2 \, dx + c_2 \int_0^l \left(N_2^{iv} + \frac{P}{EI} N_2'' \right) N_2 \, dx = 0 \tag{27}$$

Expanding,

$$c_1 \left(\int_0^l N_1^{iv} N_1 \, dx + \frac{P}{EI} \int_0^l N_1'' N_1 \, dx \right) + c_2 \left(\int_0^l N_2^{iv} N_1 \, dx + \frac{P}{EI} \int_0^l N_2'' N_1 \, dx \right) = 0 \tag{28}$$

$$c_1 \left(\int_0^l N_1^{iv} N_2 \, dx + \frac{P}{EI} \int_0^l N_1'' N_2 \, dx \right) + c_2 \left(\int_0^l N_2^{iv} N_2 \, dx + \frac{P}{EI} \int_0^l N_2'' N_2 \, dx \right) = 0 \tag{29}$$

Let
$$k_{12} = \int_0^l N_2^{iv} N_1 \, dx \tag{30}$$

$$k_{12_g} = \int_0^l N_2'' N_1 \, dx \tag{31}$$

$$k_{21} = \int_0^l N_2 N_2^{iv} \, dx \tag{32}$$

$$k_{21_g} = \int_0^l N_1'' N_2 \, dx \tag{33}$$

$$k_{22} = \int_0^l N_2^{iv} N_2 \, dx \tag{34}$$

$$k_{22_g} = \int_0^l N_2'' N_2 \, dx \tag{35}$$

Then, we have:

$$c_1(k_{11} + \lambda k_{11_g}) + c_2(k_{12} + \lambda k_{12_g}) = 0 \tag{36}$$

$$c_1(k_{21} + \lambda k_{21_g}) + c_2(k_{22} + \lambda k_{22_g}) = 0 \tag{37}$$

$$\begin{pmatrix} k_{11} + \lambda k_{11_g} & k_{12} + \lambda k_{12_g} \\ k_{21} + \lambda k_{21_g} & k_{22} + \lambda k_{22_g} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (38)$$

For non-trivial solutions,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The buckling equation becomes:

$$\begin{vmatrix} k_{11} + \lambda k_{11_g} & k_{12} + \lambda k_{12_g} \\ k_{21} + \lambda k_{21_g} & k_{22} + \lambda k_{22_g} \end{vmatrix} = 0 \qquad (39)$$

Expanding,

$$(k_{11} + \lambda k_{11_g})(k_{22} + \lambda k_{22_g}) - (k_{12} + \lambda k_{12_g})(k_{21} + \lambda k_{21_g}) = 0 \qquad (40)$$

This yields a quadratic equation in terms of λ, from which the two roots of λ yield two values of buckling loads.

3.0 Application to the Buckling of fixed - Hinged or Clamped – Pinned Columns

3.1 Problem considered

The elastic buckling of Euler column with fixed-pinned ends at $x=0$ and $x=l$ was considered as shown in Figure 2. The governing differential equation is given by Equation (10).

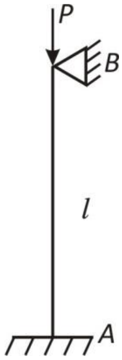

Figure 2: Elastic buckling of Euler column with fixed-pinned ends

The boundary conditions are:

$$v(x=0)=\theta(x=0)=v'(x=0)=v(x=l)=v''(x=l)=0 \tag{41}$$

A suitable displacement coordinate shape function that satisfies the boundary conditions at x=0, and $x = l$ is:

$$v(x)=c_1\left(\left(\frac{x}{l}\right)^4-2.5\left(\frac{x}{l}\right)^3+1.5\left(\frac{x}{l}\right)^2\right) \tag{42}$$

for a one parameter choice of $v(x)$ where $N_1(x)$ is the displacement shape function, c_1 is the generalized displacement parameter:

$$N_1(x)=\left(\frac{x}{l}\right)^4-2.5\left(\frac{x}{l}\right)^3+1.5\left(\frac{x}{l}\right)^2 \tag{43}$$

For a two parameter choice of $v(x)$ Equation (23) is used where $N_1(x)$ is given by Equation (43) and

$$N_2=\left(\frac{x}{l}\right)^5-\frac{7}{3}\left(\frac{x}{l}\right)^4+\frac{4}{3}\left(\frac{x}{l}\right)^3 \tag{44}$$

c_1 and c_2 are the two unknown generalized displacement parameters.

3.2 One term (Parameter) Bubnov-Galerkin Solution

The Bubnov-Galerkin variational integral is given by Equation (15). By differentiation,

$$N_1'' = \frac{12x^2}{l^4} - \frac{15x}{l^3} + \frac{3}{l^2} \tag{45}$$

$$N_1^{iv} = \frac{24}{l^4} \tag{46}$$

Therefore, the Bubnov-Galerkin variational integral becomes:

$$\int_0^l \left\{ EIc_1 \frac{24}{l^4} + Pc_1\left(\frac{12x^2}{l^4} - \frac{15x}{l^3} + \frac{3}{l^2}\right)\right\} * \left(\left(\frac{x}{l}\right)^4 - 2.5\left(\frac{x}{l}\right)^3 + 1.5\left(\frac{x}{l}\right)^2\right) dx = 0 \tag{47}$$

Integrating and simplifying,

$$c_1\left\{\frac{1.8}{l^3} - \frac{0.085714286}{l}\frac{P}{EI}\right\} = 0 \tag{48}$$

The differential equation simplifies to an algebraic eigen-value eigen-vector problem given by Equation (48).

For non-trivial solution, $c_1 \neq 0$

Then the characteristic buckling equation becomes the homogeneous equation

$$\frac{1.8}{l^3} - \frac{P}{EI}\frac{0.085714286}{l} = 0 \tag{49}$$

Solving

$$P_{cr} = 21\frac{EI}{l^2} \tag{50}$$

3.3 Two term (parameter) Bubnov-Galerkin Solution

For the displacement (shape) coordinate functions considered Equations (45) and (44), differentiation yields:

$$N_2'' = \frac{20x^3}{l^5} - \frac{28x^2}{l^4} + \frac{8x}{l^3} \tag{51}$$

$$N_2^{iv} = \frac{120x}{l^5} - \frac{56}{l^4} \tag{52}$$

While N_1'' and N_1^{iv} are given by Equations (45) and (46) respectively

By integration,

$$\int_0^l N_1'' N_1 \, dx = \frac{-0.085714286}{l} \tag{53}$$

$$\int_0^l N_1^{iv} N_1 \, dx = \frac{1.8}{l^3} \tag{54}$$

$$\int_0^l N_2'' N_2 \, dx = \frac{-0.025396824}{l} \tag{55}$$

$$\int_0^l N_2^{iv} N_2 \, dx = \frac{0.609523734}{l^3} \tag{56}$$

$$\int_0^l N_2^{iv} N_1 \, dx = \frac{0.8}{l^3} \tag{57}$$

$$\int_0^l N_2'' N_1 \, dx = \frac{-0.042571429}{l} \tag{58}$$

$$\int_0^l N_1'' N_2 \, dx = \frac{-0.042857142}{l} \tag{59}$$

$$\int_0^l N_1^{iv} N_2 \, dx = \frac{0.8}{l^3} \tag{60}$$

The Galerkin variational integrals then reduce to the system of algebraic equations in terms of the unknown generalized parameters, c_1 c_2 as follows:

$$c_1\left(\frac{1.8}{l^3}-\frac{P}{EI}\frac{0.085714286}{l}\right)+c_2\left(\frac{0.8}{l^3}-\frac{P}{EI}\frac{0.0428571429}{l}\right)=0 \quad (61)$$

$$c_1\left(\frac{0.8}{l^3}-\frac{0.042857142}{l}\frac{P}{EI}\right)+c_2\left(\frac{0.609523734}{l^3}-\frac{0.025396824}{l}\frac{P}{EI}\right)=0 \quad (62)$$

In matrix form,

$$\left\{\begin{pmatrix}1.80 & 0.80\\ 0.80 & 0.609524\end{pmatrix}-\beta\begin{pmatrix}0.085714 & 0.042857\\ 0.042857 & 0.0253968\end{pmatrix}\right\}\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \quad (63)$$

where $\beta=\frac{Pl^2}{EI}$ (64)

$$\begin{pmatrix}(1.80-0.085714\beta) & (0.80-0.042857\beta)\\ (0.80-0.042857\beta) & (0.609524-0.0253968\beta)\end{pmatrix}\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \quad (65)$$

For non-trivial columns

$$\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}\neq 0$$

The characteristic buckling equation then becomes:

$$\left|\begin{pmatrix}1.80-0.085714\beta & 0.80-0.042857\beta\\ 0.80-0.042857\beta & 0.609524-0.0253968\beta\end{pmatrix}\right|=0 \quad (66)$$

By expansion, we obtain after simplification,

$$\beta^2-87.13621\beta+1358.92=0 \quad (67)$$

Solving,

$$\beta=20.34614=\frac{Pl^2}{EI} \quad (68)$$

then,

$$P_{cr} = 20.34614\frac{EI}{l^2} \tag{69}$$

The exact solution for P_{cr} is

$$P_{cr} = 20.1907\frac{EI}{l^2} \tag{70}$$

4.0 Discussion

The Bubnov-Galerkin variational method has been successfully applied in this work to determine the critical buckling load of Euler columns under axial compressive load P when the ends are fixed at $x = 0$ and pinned at $x = l$. Coordinate shape functions that satisfy all the boundary conditions at the fixed-pinned ends were employed in a one parameter and a two parameter displacement shape functions as Equations (43) and (44). The Galerkin variational integral statement for a one-parameter solution was obtained as Equation (15). For two parameter solution, the Galerkin variational integral statement was obtained as the system of Equations (61) and (62) upon evaluation of the integrals. For a one-parameter Bubnov-Galerkin method, the problem reduced to an algebraic-eigen value problem expressed by Equation (48). The characteristic buckling equation was obtained as Equation (49) and the solution gave the critical buckling load for one parameter Bubnov-Galerkin solution as Equation (50). Similarly, the two parameter Bubnov-Galerkin solution reduced to the algebraic eigen-value problem expressed by Equation (63) or (65). The corresponding characteristic buckling equation was found as Equation (66). Equation (66) yielded a quadratic equation in terms of β, which was solved to obtain the critical buckling load for a two parameter Bubnov-Galerkin solution as Equation (69). Comparison of the one term and two term Bubnov-Galerkin solutions with the exact solutions given as Equation (70) shows the one term solution has a relative error of 4% while the two terms Bubnov- Galerkin solution has a relative error of 0.77%. This shows the effectiveness of the Bubnov-Galerkin method in the elastic buckling analysis of Euler columns with fixed-pinned ends.

5.0 Conclusions

From the study, the following conclusions can be made:

i. The Bubnov-Galerkin variational method simplifies the boundary value problem of elastic buckling of Euler-column to an algebraic eigenvalue problem.

ii. As the number of undetermined generalized displacement parameters increase, the accuracy of the Bubnov-Galerkin method increases provided the coordinate (shape) functions satisfy all the boundary conditions at the ends.

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