
DETERMINATION OF THE CRITICAL BUCKLING LOADS OF EULER COLUMNS USING STODOLA-VIANELLO ITERATION METHOD

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Abstract: The Stodola-Vianello iteration method was implemented in this work to determine the critical buckling load of an Euler column of length l with fixed end ($x = 0$) and pinned end ($x = l$), where the longitudinal axis is the x -direction. The critical buckling loads were found to be variable, depending on the x -coordinate. Integration and the Rayleigh quotients were used to find average buckling coefficients. First iteration gave relative errors of 4% using integration and 2.5% using Rayleigh quotient. Second iteration gave average relative errors less than 1% for both the integration and the Rayleigh quotients. Better estimates of the critical buckling loads were obtained using the Rayleigh quotient in the Stodola-Vianello's iteration.

Keywords: *Stodola –Vianello's iteration method Euler column, critical buckling load, flexural buckling.*

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1.0 Introduction

1.1 Background

Columns are long slender bars that carry axial compressive forces. They are important structural members in civil, mechanical, structural and aeronautics systems. They can be vertical, horizontal or inclined, and are classified as short, intermediate or long columns. Short columns under axially applied compressive forces fail by compressive yielding of their materials shown physically by crushing failure (Rao, 2017; Homepages 2017; Lagace, 2009; Punmia *et al.*, 2002). Long columns under axial compressive force fail by sudden excessive lateral displacement, a phenomenon called flexural buckling (Rao, 2017; Homepages 2017; Lagace, 2009; Punmia *et al.*, 2002). Intermediate columns under axial loads fail by a combination of compressive yielding and flexural buckling.

1.2 Euler’s Theory of Buckling

Euler considered the moment equilibrium of an elemental part of an elastic column of length l with pinned ends $x = 0$ and $x = l$ and subjected to axial compressive force P , together with the moment – curvature equations to obtain the differential equation for the lateral deflection $v(x)$ as the second order differential equation- Equation (1); (Homepages, 2017; Lagace, 2009; Punmia *et al.*, 2002; Jayaram, 2007; Megson, 2005).

$$\frac{d^2v}{dx^2} + \frac{P}{EI} v(x) = 0 \tag{1}$$

where E is the modulus of elasticity of the column material, I is the moment of inertia.

1.3 General Differential Equation for Elastic Column Buckling

A more general differential equation for the elastic column buckling problem employs the formulation principles similar to the flexure of beams, but includes the internal axial forces in the resulting equations. The internal forces and moments acting in an elemental part (Δx) of an elastic column of length under axial compressive forces, and arbitrarily supported at the ends $x = 0$, and $x = l$ are shown in Figure 1.

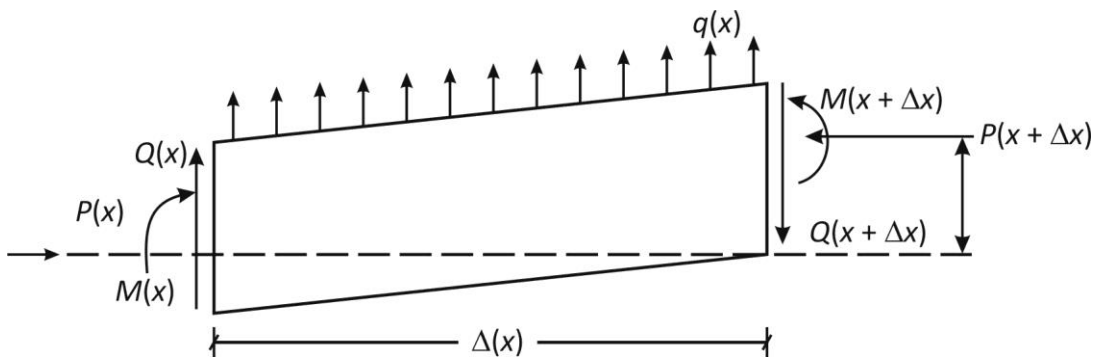


Figure 1: Internal forces and bending moment acting on an elemental column

Consideration of vertical equilibrium yields

$$\frac{dQ}{dx} = q(x) \tag{2}$$

where $Q(x)$ is the shear force distribution, at an arbitrary section, x , and $q(x)$ is the distribution of transverse load.

For horizontal equilibrium,

$$P(x) = P(x + \Delta x) = P(x) + \frac{dP}{dx} dx \quad (3)$$

$P(x)$ is constant

For moment equilibrium,

$$\frac{dM}{dx} + P \frac{dv(x)}{dx} = Q \quad (4)$$

The differential equation for rotational equilibrium of the elemental column has an additional term Pdv/dx which is absent from the corresponding equation for Euler – Bernoulli beam flexure theory.

By differentiation of Equation (4) with respect to x ,

$$\frac{d}{dx} \left(\frac{dM}{dx} + P \frac{dv}{dx} \right) = \frac{dQ}{dx} = q(x) \quad (5)$$

$$\left(\frac{d^2 M}{dx^2} + P \frac{d^2 v}{dx^2} \right) = \frac{dQ}{dx} = q(x) \quad (6)$$

Application of the moment curvature relation yields

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) + P \frac{d^2 v}{dx^2} = q(x) \quad (7)$$

For prismatic, homogeneous columns EI is constant, and the differential equation for the elastic buckling of columns with any type of supports becomes the fourth order equations [6, 7]:

$$EI \frac{d^4 v}{dx^4} + P \frac{d^2 v}{dx^2} = q(x) \quad (8)$$

1.4 Assumptions of the Euler's Theory of Column Buckling

Euler's theory of elastic column buckling is formulated based on the following assumptions (Rao, 2017; Punmia *et al.*, 2002; Jayaram, 2007; Megson, 2005; Lowe, 1971):

- i. The column is straight in the longitudinal direction before the axial load is applied.
- ii. The cross-sectional dimension is uniform along the longitudinal axis
- iii. The material is homogeneous and isotropic
- iv. The self-weight is neglected
- v. The axial compressive load is concentrically applied.
- vi. The axial compression is small and neglected
- vii. The failure is due to flexural buckling

Recent research publications on the elastic buckling of Euler columns of prismatic and non-prismatic cross-sections used various analytical tools such as the variational iteration method, Newmark method, homotopy perturbation method and integral equation method. Some of the recent publications/studies on the elastic buckling of Euler columns are as follows: Coskun and Atay, (2009); Huang Yong and Luo Qi – Zhi (2011); Yayli, (2008); Riley, (2003); Atay, (2009); Arbabi and Li, (1991) and Coskun, (2010).

The research aim is to determine the elastic buckling loads of Euler columns using the Stodola-Vianello's iteration method. The specific objectives are as follows:

- (i) To determine the Stodola-Vianello's iteration for Euler columns with fixed pinned ends at $x = 0$, and $x = l$ respectively
- (ii) To implement the Stodola-Vianello's iteration scheme for Euler column with fixed-pinned ends in order to determine the Euler buckling load and the critical buckling load.

1.5 Issue Related to Elastic Buckling of Euler Column

The research problem is to solve the fourth order ordinary differential equation given as Equation (8) for the problem of elastic buckling of Euler column fixed at $x = 0$, and pinned at $x = l$ with prismatic cross-sections using the Stodola-Vianello iteration method and determine the Euler column elastic buckling loads.

2.0 Theoretical Framework and Methodology

The fourth order ordinary differential equation for the elastic buckling of Euler columns under axial compressive load P when no transverse distributed load $q_z(x)$ is applied is given from Equation (8) by

$$EIv^{iv}(x) + Pv'' = 0 \quad (9)$$

where the primes denote differentiation with respect to, x , the longitudinal coordinate axis.

For prismatic columns EI is constant. For constant loads P is constant. Integration of Equation (9) with respect to x yields:

$$EIv'''(x) + Pv'(x) = c_1 \quad (10)$$

where c_1 is an integration constant

Integration again yields:

$$EIv''(x) + Pv(x) = c_1x + c_2 \quad (11)$$

where c_2 is an integration constant

Rearranging Equation (11) we obtain,

$$v''(x) = -\frac{Pv(x)}{EI} + \frac{c_1x + c_2}{EI} \quad (12)$$

$$v''(x) = -\frac{P}{EI} \left(v(x) - \frac{c_1x}{P} - \frac{c_2}{P} \right) \quad (13)$$

$$v''(x) = \frac{-P}{EI} v(x) + a_1x + a_2 \quad (14)$$

where

$$a_1 = \frac{-c_1}{P} \quad (15)$$

$$a_2 = \frac{-c_2}{P} \tag{16}$$

Integrating Equation (14),

$$v'(x) = \frac{-P}{EI} \left\{ \int_0^x (v(x) + a_1x + a_2) dx + a_3 \right\} \tag{17}$$

where a_3 is an integration constant.

Integrating Equation (17),

$$v(x) = \frac{-P}{EI} \left\{ \int_0^x \int_0^x (v(x) + a_1x + a_2) dx dx + a_3x + a_4 \right\} \tag{18}$$

where a_4 is an integration constant.

The iteration scheme for the $(k - 1)^{th}$ iteration is:

$$v(x)^{k-1} = \frac{-P_{cr}}{EI} \left\{ \int_0^x \int_0^x (v(x)^{k-1} + a_1x + a_2) dx dx + a_3x + a_4 \right\} \tag{19}$$

$$v(x)^k = - \left\{ \int_0^x \int_0^x (v(x)^{k-1} + a_1x + a_2) dx dx + a_3x + a_4 \right\} \tag{20}$$

Then equation (19) becomes

$$v(x)^{k-1} = \frac{P_{cr}}{EI} v^k(x) \tag{21}$$

From Equation (21), the eigenvalue or buckling load is obtained from the iteration:

$$P_{cr} = EI \frac{v(x)^{k-1}}{v(x)^k} \tag{22}$$

where the four integration constants a_i , $i = 1, 2, 3, 4$ are obtained at each iteration step by the application and enforcement of the boundary conditions.

The Stodola-Vianello's iterations scheme given by Equation (22) gives the critical load for P_{cr} in terms of the independent coordinate variables x and the average values of P_{cr} can be obtained for the column, to be the representative buckling load, and thus the buckling coefficients are determined. The average critical buckling load can be obtained by integration to have

$$P_{cr} = \int_0^l EI \frac{v(x)^{k-1}}{v(x)} dx \quad (23)$$

$$P_{cr} = \int_0^l EI \frac{v(x)^{k-1}}{v(x)^k} dx = EI \frac{\int_0^l v(x)^{k-1} dx}{\int_0^l v(x)^k dx} \quad (24)$$

Rayleigh quotient can be introduced as defined for the Euler column elastic buckling problem to have for the k^{th} iterations:

$$P_{cr} = \frac{\int_0^l EI (v''(x))^k dx}{\int_0^l (v'(x))^k dx} \quad (25)$$

3.0 Application of The Stodola-Vianello's Iteration Scheme for Fixed –Pinned Euler Column

3.1 Integration Constants

An Euler column of length l with the longitudinal direction denoted as the x coordinate and the end $A(x = 0)$ fixed and the other end $B(x = l)$ pinned as shown in Figure 2 was considered.

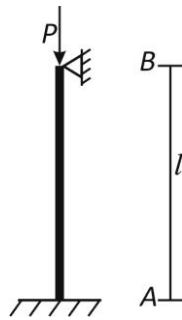


Figure 2: Euler column with fixed pinned ends

The boundary conditions are

$$v_A = \theta_A = 0 \quad (26)$$

$$v_B = M_B = 0 \quad (27)$$

or,

$$v(x = 0) = 0 \quad (28)$$

$$\theta(x = 0) = v(x = 0) = 0 \quad (29)$$

$$v(x = l) = 0 = v(l) \quad (30)$$

$$v''(x = l) = 0 \quad (31)$$

From Equation (19)

$$v(0) = a_4 = 0 \quad (32)$$

From Equation (17)

$$v'(0) = a_3 = 0 \quad (33)$$

From Equation (14)

$$v''(l) = \frac{P}{EI}(v(l) + a_1l + a_2) = 0 \quad (34)$$

$$v(l) + a_1 l + a_2 = 0 \quad (35)$$

From Equation (30)

$$a_1 l + a_2 = 0 \quad (36)$$

$$a_2 = -a_1 l \quad (37)$$

Equation (18) then reduces to:

$$v(x) = \frac{-P}{EI} \left\{ \int_0^x \int_0^x (v(x) + a_1 x + a_2) dx dx \right\} \quad (38)$$

Using Equation (30)

$$v(l) = \frac{-P}{EI} \left\{ \int_0^l \int_0^l (v(x) + a_1 x + a_2) dx dx \right\} = 0 \quad (39)$$

$$\int_0^l \int_0^l (v(x) + a_1 x + a_2) dx dx = 0 \quad (40)$$

$$\int_0^l \int_0^l v(x) dx dx + \int_0^l \int_0^l a_1 x dx dx + \int_0^l \int_0^l a_2 dx dx = 0 \quad (41)$$

$$\int_0^l \int_0^l v dx dx + a_1 \int_0^l \int_0^l x dx dx + a_2 \int_0^l \int_0^l dx dx = 0 \quad (42)$$

$$\int_0^l \int_0^l v dx dx + a_1 \int_0^l \frac{x^2}{2} dx + a_2 \int_0^l x dx = 0 \quad (43)$$

$$\int_0^l \int_0^l v dx dx + a_1 \left[\frac{x^3}{6} \right]_0^l + a_2 \left[\frac{x^2}{2} \right]_0^l = 0 \quad (44)$$

$$\int_0^l \int_0^x v dx dx + \frac{a_1 l^3}{6} + \frac{a_2 l^2}{2} = 0 \tag{45}$$

$$\int_0^l \int_0^x v dx dx + \frac{a_1 l^3}{6} - \frac{a_1 l^3}{2} = 0 \tag{46}$$

$$a_1 \frac{l^3}{3} = \int_0^l \int_0^x v dx dx \tag{47}$$

$$a_1 = \frac{3}{l^3} \int_0^l \int_0^x v(x) dx dx \tag{48}$$

$$a_2 = -a_1 l = \frac{-3}{l^2} \int_0^l \int_0^x v(x) dx dx \tag{49}$$

then,

$$v(x) = \frac{-P}{EI} \left\{ \int_0^x \int_0^x v(x) dx dx + \frac{x^3}{2l^3} \int_0^l \int_0^x v(x) dx dx - \frac{3x^2}{2l^2} \int_0^l \int_0^x v(x) dx dx \right\} \tag{50}$$

3.2 Stodola- Vianello's First Iteration Scheme

A polynomial shape function that satisfies the fixed pinned end conditions at $x = 0$ and $x = l$ is

$$v(x) = a(2x^4 - 5lx^3 + 3l^2x^2) \tag{51}$$

where a is the undetermined or generalized parameter of the deflection field.

From Equation (50),

$$v^1(x) = \frac{-P}{EI} \left\{ \int_0^x \int_0^x v(x)^0 dx dx + \frac{x^3}{2l^3} \int_0^l \int_0^x v(x)^0 dx dx - \frac{3x^2}{2l^2} \int_0^l \int_0^x v(x)^0 dx dx \right\} \tag{52}$$

$$v(x)^1 = a \left\{ \int_0^x \int_0^x a(2x^4 - 5lx^3 + 3l^2x^2) dx dx + \frac{x^3}{2l^3} \int_0^l \int_0^x (2x^4 - 5lx^3 + 3l^2x^2) dx dx - \frac{3x^2}{2l^2} \int_0^l \int_0^x (2x^4 - 5lx^3 + 3l^2x^2) dx dx \right\} \quad (53)$$

$$v(x)^1 = \frac{-a}{60} (4x^6 - 15x^5l + 15x^4l^2 + 2x^3l^3 - 6x^2l^4) \quad (54)$$

The Stodola-Vianello's iteration for the critical load is from Equation (22)

$$P_{cr} = \frac{EIv^{(k-1)}(x)}{v^{(k)}(x)} = EI \frac{v^{(0)}(x)}{v^{(1)}(x)} \quad (55)$$

$$P_{cr} = \frac{60(2x^4 - 5lx^3 + 3l^2x^2)EI}{(4x^6 - 15x^5l + 15x^4l^2 + 2x^3l^3 - 6x^2l^4)} \quad (56)$$

$$P_{cr} = \frac{60(3l - 2x)(l - x)EI}{(6l^3 + 4l^2x - 11lx^2 + 4x^3)(l - x)} \quad (57)$$

$$P_{cr} = \frac{60(3l - 2x)EI}{(6l^3 + 4l^2x - 11lx^2 + 4x^3)} = \frac{K(x)EI}{l^2} \quad (58)$$

where $K(x)$ is the buckling coefficient

The buckling coefficients are tabulated for various points on the column in Table 1.

Table 1: Buckling coefficients for fixed pinned columns

x/l	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$K(x)$	30	26.692	24.406	22.792	21.654	20.87	20.362	20.075	19.962	19.967	20

3.3 Average Buckling Coefficients

The average buckling coefficient is found as:

$$K_{av} = \frac{1}{11} (30 + 26.692 + 24.406 + 22.792 + 21.654 + 20.87 + 20.362 + 20.075 + 19.962 + 19.967 + 20) \quad (59)$$

The average buckling load is found by using integration as:

$$P_{cr} = \frac{EI \int_0^l v(x)^0 dx}{\int_0^l v(x)^1 dx} = \frac{EIa \int_0^l (2x^4 - 5lx^3 + 3l^2x^2) dx}{\frac{a}{60} \int_0^l (4x^6 - 15x^5l + 15x^4l^2 + 2x^3l^3 - 6x^2l^4) dx} \quad (60)$$

$$\int_0^l v(x)^0 dx = a \int_0^l (2x^4 - 5lx^3 + 3l^2x^2) dx \quad (61)$$

$$= a \left[\frac{2x^5}{5} - \frac{5lx^4}{4} + \frac{3l^2x^3}{3} \right]_0^l \quad (62)$$

$$= a \left(\frac{2l^5}{5} - \frac{5l^5}{4} + l^5 \right) = \frac{3}{20} al^5 \quad (63)$$

$$\int_0^l v^{(1)}(x) dx = \frac{a}{60} \int_0^l (4x^6 - 15x^5l + 15l^2x^4 + 2l^3x^3 - 6l^4x^2) dx \quad (64)$$

$$= \frac{al^7}{140} \quad (65)$$

$$P_{cr} = \frac{3 \times 140}{20} \frac{EI}{l^2} = 21 \frac{EI}{l^2} \quad (66)$$

3.4 Use of Rayleigh Quotient

First Stodola –Vianello’s iteration yielded Equation (54). Thus, by differentiation of Equation (54) with respect to x ,

$$\frac{dv}{dx} = \frac{-a}{60}(24x^5 - 75lx^4 + 60l^2x^3 + 6l^3x^2 - 12l^4x) \quad (67)$$

Differentiating again,

$$v(x)'' = \frac{-a}{60}(120x^4 - 300lx^3 + 180l^2x^2 + 12l^3x - 12l^4) \quad (68)$$

$$= \frac{-a}{60}[12(10x^4 - 25lx^3 + 15l^2x^2 + l^3x - l^4)] \quad (69)$$

Then,

$$P_{cr} = \frac{12EI \int_0^l (10x^4 - 25lx^3 + 15l^2x^2 + l^3x - l^4)^2 dx}{\int_0^l (24x^5 - 75lx^4 + 60l^2x^3 + 6l^3x^2 - 12l^4x)^2 dx} \quad (70)$$

$$P_{cr} = 20.243 \frac{EI}{l^2} \quad (71)$$

3.5 Second Stodola-Vianello Iteration Scheme

The second Stodola-Vianello’s iteration scheme is:

$$v(x)^{(2)} = \left\{ \int_0^x \int_0^x v(x)^{(1)} dx dx + \frac{x^3}{2l^3} \int_0^l \int_0^x v(x)^{(1)} dx dx - \frac{3x^3}{2l^2} \int_0^l \int_0^x v(x)^{(1)} dx dx \right\} \quad (72)$$

$$\begin{aligned}
 &= \frac{-a}{60} \left\{ \int_0^x \int_0^x (4x^6 - 15lx^5 + 15l^2x^4 + 2l^3x^3 - 6l^4x^2) dx dx \right. \\
 &+ \frac{x^3}{2l^3} \int_0^l \int_0^x (4x^6 - 15lx^5 + 15l^2x^4 + 2l^3x^3 - 6l^4x^2) dx dx \\
 &\left. - \frac{3x^3}{2l^2} \int_0^l \int_0^x (4x^6 - 15lx^5 + 15l^2x^4 + 2l^3x^3 - 6l^4x^2) dx dx \right\} \quad (73)
 \end{aligned}$$

$$v(x)^{(2)} = \frac{-a}{8400} (10x^8 - 50lx^7 + 70l^2x^6 + 14l^3x^5 - 70l^4x^4 - 13l^5x^3 + 39l^6x^2) \quad (74)$$

The critical buckling load for the second Stodola-Vianello’s iteration is:

$$P_{cr} = \frac{EIv^{(1)}(x)dx}{v^{(2)}(x)dx} \quad (75)$$

$$P_{cr}^{(2)} = \frac{140(4x^4 - 15lx^3 + 15l^2x^2 + 2l^3x - 6l^4)}{(10x^6 - 50lx^5 + 70l^2x^4 + 14l^3x^3 - 70l^4x^2 - 13l^5x + 39l^6)} \frac{EI}{l^2} \quad (76)$$

$$P_{cr}^{(2)} = K^{(2)}(x) \frac{EI}{l^2} \quad (77)$$

where $K^{(2)}(x)$ is the buckling coefficient for the second Stodola-Vianello’s iteration for the Euler column with fixed pinned ends. $K^{(2)}(x)$ is tabulated for various values of x shown in Table 2.

Table 2: Stodola-Vianello’s 2nd iteration buckling coefficient for fixed pinned columns.

x/l	0	0.125	0.25	0.5	0.75	1
$K(x)$	21.54	21.367	21.03	20.40	20.08	20

The average value of the Stodola-Vianello’s second iteration buckling coefficient is

$$K^{(2)} = \frac{1}{6} (21.54 + 21.367 + 21.03 + 20.40 + 20.08 + 20) \quad (78)$$

$$K^{(2)} = 20.78 \quad (79)$$

By the use of integration, $P_{cr}^{(2)}$ is given by:

$$P_{cr}^{(2)} = EI \frac{\int_0^l v(x)^{(1)} dx}{\int_0^l v(x)^{(2)} dx} \quad (80)$$

$$P_{cr}^{(2)} = 20.38 \frac{EI}{l^2} \quad (81)$$

The use of Rayleigh quotient in the second Stodola-Vianello's iteration gives:

$$P_{cr}^{(2)} = 20.196 \frac{EI}{l^2} \quad (82)$$

for fixed pinned Euler columns. The exact solution for P_{cr} is:

$$P_{cr} = 20.1907 \frac{EI}{l^2} \quad (83)$$

4.0 Discussion

The Stodola-Vianello's iteration method has been effectively implemented in this work to determine the critical buckling load of an Euler column under axial compressive force P when the end $x = 0$ is fixed and the other end $x = l$ is pinned; and no transverse distributed load is applied. The Stodola-Vianello iteration scheme was formulated using successive integration to obtain the iteration scheme for buckling as Equation (22). One parameter displacement shape function that satisfies all the end boundary conditions was used to obtain the first and second Stodola – Vianello iterations as Equations (54) and (74), respectively. The buckling coefficients obtained were found to be variable, depending upon the longitudinal coordinate variable x . The buckling coefficients were found for first Stodola – Vianello iteration and were tabulated in Table 2.

The first iteration gave an average critical buckling load given by Equation (59). The second iteration gave an average critical buckling coefficient given by Equation (79). The use of integration gave an average critical buckling load as Equation (66) for the first iteration and Equation (81) for the second. Use of Rayleigh quotient in the Stodola-

Vianello iteration yielded a critical buckling load given by Equation (71) for the first iteration, and Equation (83) in the second iteration.

The relative error of the first iteration was found to be 4% for the method of integration and 2.59% for the Rayleigh quotient. The second Stodola-Vianello iteration yielded relative errors of 0.94% for the method of integration, and 0.026% for the Rayleigh quotient.

5.0 Conclusions

From the work, the following conclusions can be made:

- i. The Stodola-Vianello iteration method reduces the boundary value problem of elastic buckling of Euler columns to an iteration scheme for finding the successive values of $v(x)$ from given functions of $v(x)$.
- ii. The buckling coefficients $K(x)$ were found to be variables, depending upon the longitudinal coordinate variable, x .
- iii. The use of the Rayleigh quotient in the Stodola-Vianello iteration yielded rapidly convergent results for the buckling coefficients, as compared with the simple averaging and integration methods.

References

- Arbabi F. and Li F. (1991). Buckling of variable cross-section columns – integral equation approach. *Journal of Structural Engineering ASCE* Vol 117 No 8 pp 2426 – 2441. Doi:10.1061/(ASCE) 0733-9445 (1991) 117: 8 (2426).
- Atay M.T. (2009). Determination of critical buckling loads for variable stiffness Euler columns using homotopy perturbation method. *Int. Journal Nonlinear Sci and Numer. Simulation* Vol 10 Issue 2 pp 199 – 206.
- Coskun S.B. (2010). Analysis of Tilt – Buckling of Euler Columns with Varying Flexural Stiffness using Homotopy Perturbation Method. *Mathematical Modelling and Analysis* Volume 18 Number 3 pp 275 – 286. Doi:10.3846/1392-6292.2010.15.275-286.
- Coskun S.B. and Atay M.T. (2009). Determination of critical buckling loads for elastic columns of constant and variable cross-sections using variational iteration method. *Computers and Mathematics with Applications* Vol 58 Issues 11 – 12 pp 2260 – 2266. <https://doi.org/10.1016/j.camwa.2009.03.072>
- Homepages. engineering. auckland. ac.nz/.07 Elasticity_ Applications _05_Buckling pdf. Accessed in Feb. 2017.
- Huang Yong and Luo Qi – Zhi (2011). A simple method to determine the critical buckling loads for axially inhomogeneous beams with elastic restraint. *Computers and Mathematics with Applications* Vol 61, pp 2510 – 2517. <http://doi.org/10.1016/j.camwa.2011.02.037>.
- Jayaram M.A (2007): *Mechanism of Materials with Programs in C*, Prentice Hall of India Private Ltd New Delhi.

- Lagace P.A (2009): Unit M.4.7.The Column and Buckling, 16.003/004, Unified Engineering, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology web mit.edu/16/unified/www. SPRING/materials/Lectures/M.4.7 _ Unified 09. pdf. *Accessed in Feb. 2017.*
- Lowe P.G (1971): Classical theory of Structures Based on the Differential Equation, Cambridge University Press, Cambridge.
- Megson T.H. G (2005): Structural and Stress Analysis, Second Edition, Elsevier, Butterworth Heinemann, Amsterdam.
- Punmia B.C, Jain A.K., Jain A.K. (2002): Mechanics of Materials, Laxmi Publications (P) Ltd New Delhi. [http://books.google.com/books? isbn= 8170082153](http://books.google.com/books?isbn=8170082153)
- Rao, P.V. Unit IV, Theory of Columns, [http://www.svce.ac.in/--/Unit%/20111%20\(a\)%20-%20Theory%20of%20Columns%20__Rao.pdf](http://www.svce.ac.in/--/Unit%/20111%20(a)%20-%20Theory%20of%20Columns%20__Rao.pdf) *Accessed in Feb. 2017.*
- Riley C.E. (2003). Elastic Buckling Loads of Slender Columns with Variable Cross-section by the Newmark Method. MSc Thesis Department of Civil Engineering, Colorado State University, Fort Collins Colorado pp 147.
- Yayli M.O. (2018). Buckling analysis of Euler columns embedded in an elastic medium with general elastic boundary conditions. Mechanics Based Design of Structures and Machines Vol 46 Issue 1 pp 110 – 122. [https://doi.org/10.1080/15397734.2017.1292142.](https://doi.org/10.1080/15397734.2017.1292142)